# INTEGRAL EQUATIONS AND SEMIGROUPS[1]

BY

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Consider an $n \times n$ matrix $\phi$ of absolutely continuous functions on an interval $S$ of real numbers. If each of $G$ and $H$ is also such a matrix, then, as is well known (cf. [2, p. 352] for comments and references), the differential requirement that

$$\left.\begin{aligned} G'(s) - G(s)\phi'(s) &= 0 \\ H'(s) + \phi'(s)H(s) &= 0 \end{aligned}\right\} \text{ almost everywhere on } S,$$

where 0 is the $n \times n$ zero matrix, is equivalent to the (Stieltjes) integral requirement that if $c$ is in $S$ then for all $x$ and $y$ in $S$

$$G(y) = G(c) + \int_c^y G\cdot d\phi \quad \text{and} \quad H(x) = H(c) + \int_x^c d\phi\cdot H;$$

moreover, there is a fundamental matrix $W$ of continuous functions on $S \times S$ which satisfies—without exception—

$$\text{(i)} \qquad W(x, y) = 1 + \int_x^y W(x, \ )\cdot d\phi = 1 + \int_x^y d\phi\cdot W( \ , y)$$

where 1 is the $n \times n$ unit matrix, and provides $G$ and $H$ in the form

$$G(y) = G(c)W(c, y) \quad \text{and} \quad H(x) = W(x, c)H(c).$$

The relationship (i) has been extended by H. S. Wall [9], [10], with the condition of absolute continuity on $\phi$ replaced by that of continuity and bounded variation, the intrinsic nature of the *harmonic matrices* $W$ so obtained being determined explicitly; the reciprocal formulas (involving sum- and product-integrals)

$$\text{(ii)} \quad \phi(y) - \phi(x) = \int_x^y W( \ , c)\cdot dW(c, \ ), \qquad W(x, y) = {}_x\prod^y[1 + d\phi]$$

were discovered, respectively, by Wall [10] and this author [4]. The continuity condition on $\phi$ has been relaxed, in two different directions, by T. H. Hildebrandt [2] and by the present author [5], [6].

This paper is concerned with connections between additive and multiplicative integration processes, where the integration is directed along intervals in some linearly ordered system and the functions involved satisfy various conditions of boundedness, having their values in a normed algebraic ring which is complete as a metric space.

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For a linear ($\leqq$) ordering $\mathcal{O}$ of a nondegenerate set $S$, and a complete normed ring $N$ (with unit 1 such that $|1| = 1$), we determine classes $\mathcal{O}\mathcal{A}$ and $\mathcal{O}\mathcal{M}$ of functions $V$ and $W$ (from $S \times S$ to $N$) such that the integral-like formulas

$$\text{(iii)} \qquad V(a, b) = {}_a\sum^b [W - 1] \quad \text{and} \quad W(a, b) = {}_a\prod^b [1 + V]$$

are mutually reciprocal; the notation indicates the limit, through refinement of $\mathcal{O}$-subdivisions $\{t_p\}_0^m$ of $\{a, b\}$, of finitely continued sums $\sum [W(t_{p-1}, t_p) - 1]$ and products $\prod [1 + V(t_{p-1}, t_p)]$.

This determination leads to an integral-equation theory (of Cauchy-left and Cauchy-right integrals) which, in case $S$ is the real line, properly extends our earlier results [5], [6] and complements the theory developed by Hildebrandt [2]. In Section 10 we give a detailed description of all these interrelated results.

If $\mathcal{O}$ arises canonically from a semigroup operation $\sigma$ on $S$ ($\{x, z\}$ being in $\mathcal{O}$ only in case $\sigma(x, y) = z$ for some $y$ in $S$, and $\sigma(\sigma(x, a), b) = \sigma(x, c)$ only in case $\sigma(a, b) = c$), we obtain some connections between additive and multiplicative homomorphisms from this semigroup into the ring $N$. Some fundamental facts about such semigroups are obtained in Section 8, and Section 9 is devoted to the aforesaid connections, these being corollary to the theory developed in Sections 1 through 7.

## 1. Continuously continued sums and products

Suppose $S$ is a nondegenerate set and $\mathcal{O}$ is a *linear ordering* of $S$, i.e., a subset of $S \times S$ with the following properties:

(i) if each of $\{x, y\}$ and $\{y, z\}$ is in $\mathcal{O}$, then $\{x, z\}$ is in $\mathcal{O}$,

(ii) if $\{x, y\}$ is in $\mathcal{O}$ and $\{y, x\}$ is in $\mathcal{O}$, then $y$ is $x$, and

(iii) if $\{x, y\}$ is in $S \times S$, then $\{x, y\}$ or $\{y, x\}$ is in $\mathcal{O}$.

A function $f$, from $S \times S$ to any (algebraic) ring, is *$\mathcal{O}$-additive* provided that, if each of $\{x, y\}$ and $\{y, z\}$ is in $\mathcal{O}$, then

$$f(x, y) + f(y, z) = f(x, z) \quad \text{and} \quad f(z, y) + f(y, x) = f(z, x),$$

and is *$\mathcal{O}$-multiplicative* provided that for all $\{x, y\}$ and $\{y, z\}$ in $\mathcal{O}$

$$f(x, y)f(y, z) = f(x, z) \quad \text{and} \quad f(z, y)f(y, x) = f(z, x).$$

If $g$ is a function from $S$ to a ring, then $dg$ denotes a function $f$ from $S \times S$ such that $f(x, y) = g(y) - g(x)$ for all $\{x, y\}$ in $S \times S$.

Note that if $N$ is a ring, each of $g$ and $h$ is a function from $S$ to $N$, and $f$ is a function from $S \times S$ such that $f(x, y) = g(y) - g(x)$ for $\{x, y\}$ in $\mathcal{O}$ and $f(x, y) = h(y) - h(x)$ for $\{y, x\}$ in $\mathcal{O}$, then $f$ is $\mathcal{O}$-additive; conversely, each $\mathcal{O}$-additive function from $S \times S$ to $N$ arises in this way. On the other hand, if $g$ is a function from $S$ to a ring, then $dg$ is *fully additive*, in the sense that if $f = dg$, then $f(x, y) + f(y, z) = f(x, z)$ for *all* $x$, $y$, and $z$ in $S$.

An $\mathfrak{O}$-*subdivision* of a member $\{x, y\}$ of $S \times S$ is a sequence $\{t_p\}_0^n$ such that $\{t_0, t_n\}$ is $\{x, y\}$ and

(i) if $\{x, y\}$ is in $\mathfrak{O}$, then $\{t_{p-1}, t_p\}$ is in $\mathfrak{O}$ $(p = 1, \cdots, n)$, and

(ii) if $\{y, x\}$ is in $\mathfrak{O}$, then $\{t_p, t_{p-1}\}$ is in $\mathfrak{O}$ $(p = 1, \cdots, n)$.

A *refinement* of the $\mathfrak{O}$-subdivision $t$ of the member $\{x, y\}$ of $S \times S$ is an $\mathfrak{O}$-subdivision of $\{x, y\}$ of which $t$ is a subsequence.

Suppose, finally, that $N$ is a *ring*, with additive identity element denoted by 0 and multiplicative identity element denoted by 1, and that $|\cdot|$ is a *norm* for $N$, with respect to which $N$ is *complete*, such that $|1|$ is the number 1. If $h$ is a function from $S \times S$ to $N$ and $\{a, b\}$ is a member of $S \times S$,

(i) ${}_a\sum^b h$ denotes a member $Z$ of $N$ with the property that, for each positive number $c$, there is an $\mathfrak{O}$-subdivision $s$ of $\{a, b\}$ such that if $\{t_p\}_0^n$ is a refinement of $s$ then $|Z - \sum_t h| < c$, where $\sum_t h$ denotes the continued sum (in the ring $N$)

$$\sum_1^n h(t_{p-1}, t_p) = h(t_0, t_1) + \cdots + h(t_{n-1}, t_n).$$

(ii) ${}_a\prod^b h$ denotes a member $Z$ of $N$ with the property that, for each positive number $c$, there is an $\mathfrak{O}$-subdivision $s$ of $\{a, b\}$ such that if $\{t_p\}_0^n$ is a refinement of $s$, then $|Z - \prod_t h| < c$, where $\prod_t h$ denotes the continued product (in the ring $N$)

$$\prod_1^n h(t_{p-1}, t_p) = h(t_0, t_1) \cdots h(t_{n-1}, t_n).$$

We assume tacit definitions of corresponding ideas involving functions from $S \times S$ to the set of real numbers. It should be noted, however, that $N$ is *not* assumed to be an *algebra* over the real or complex numbers[2] and is *not* assumed to be *commutative*.

## 2. The numerical case

Let $\mathfrak{O}\mathfrak{A}^+$ denote the set of all $\mathfrak{O}$-additive functions from $S \times S$ to the set of nonnegative real numbers, and let $\mathfrak{O}\mathfrak{M}^+$ denote the set of all $\mathfrak{O}$-multiplicative functions from $S \times S$ to the set of real numbers not less than 1.

LEMMA 2.1. *If $\alpha$ is in $\mathfrak{O}\mathfrak{A}^+$ and $\{a, b\}$ is in $S \times S$, then*

$${}_a\prod^b [1 + \alpha] = \text{L.U.B.} \prod_t [1 + \alpha] \qquad \textit{for all } \mathfrak{O}\textit{-subdivisions } t \textit{ of } \{a, b\}.$$

*Indication of proof.* $[1 + v_1 + v_2] \leqq [1 + v_1][1 + v_2] \leqq \text{Exp}\{v_1 + v_2\}$ for all nonnegative real numbers $v_1$ and $v_2$.

LEMMA 2.2. *If $\mu$ is in $\mathfrak{O}\mathfrak{M}^+$ and $\{a, b\}$ is in $S \times S$, then*

$${}_a\sum^b [\mu - 1] = \text{G.L.B.} \sum_t [\mu - 1] \qquad \textit{for all } \mathfrak{O}\textit{-subdivisions } t \textit{ of } \{a, b\}.$$

[2] This is contrary to the convention adopted by Masani [7].

*Indication of proof.* If $\{x, z\}$ is in $S \times S$, then

$$0 \leqq [\mu(x, y) - 1][\mu(y, z) - 1] = [\mu(x, z) - 1] - \{[\mu(x, y) - 1] + [\mu(y, z) - 1]\}$$

for all $y$ in $S$ such that $\{x, y, z\}$ is an $\mathcal{O}$-subdivision of $\{x, z\}$.

THEOREM 2.1. *If $\alpha$ belongs to $\mathcal{OA}^+$, then the conditions*

$$\mu(a, b) = {}_a\prod^b [1 + \alpha] \qquad \textit{for each } \{a, b\} \textit{ in } S \times S$$

*define a member $\mu$ of $\mathcal{OM}^+$; conversely, for $\mu$ in $\mathcal{OM}^+$, the conditions*

$$\alpha(a, b) = {}_a\sum^b [\mu - 1] \qquad \textit{for each } \{a, b\} \textit{ in } S \times S$$

*define a member $\alpha$ of $\mathcal{OA}^+$.*

THEOREM 2.2. *There is a reversible function $\mathcal{E}^+$, from $\mathcal{OA}^+$ onto $\mathcal{OM}^+$, such that each of the following is a necessary and sufficient condition for the member $\{\alpha, \mu\}$ of $\mathcal{OA}^+ \times \mathcal{OM}^+$ to belong to $\mathcal{E}^+$:*

(i) $\mu(a, b) = {}_a\prod^b [1 + \alpha]$ *for each $\{a, b\}$ in $S \times S$.*

(ii) $\alpha(a, b) = {}_a\sum^b [\mu - 1]$ *for each $\{a, b\}$ in $S \times S$.*

*Proof.* If $\alpha$ is in $\mathcal{OA}^+$ and (i) is true, then $\mu \geqq 1 + \alpha$, and, for each $\{x, y\}$ in $S \times S$ and each $\mathcal{O}$-subdivision $\{t_p\}_0^n$ of $\{x, y\}$,

$$\begin{aligned}
0 &\leqq \textstyle\sum_t [\mu - 1] - \alpha(x, y) = \sum_t [\mu - 1 - \alpha] \\
&= \textstyle\sum_1^n [\mu(t_{p-1}, t_p) - 1 - \alpha(t_{p-1}, t_p)] \\
&\leqq \textstyle\sum_1^n \{\prod_1^{p-1} [1 + \alpha(t_{q-1}, t_q)]\} [\mu(t_{p-1}, t_p) - 1 - \alpha(t_{p-1}, t_p)] \\
&\qquad \cdot \{\textstyle\prod_{p+1}^n \mu(t_{r-1}, t_r)\} \\
&= \textstyle\prod_1^n \mu(t_{p-1}, t_p) - \prod_1^n [1 + \alpha(t_{p-1}, t_p)] = \mu(x, y) - \prod_t [1 + \alpha],
\end{aligned}$$

whence (ii) is true. Conversely, if $\mu$ is in $\mathcal{OM}^+$ and (ii) is true, then $\alpha \leqq \mu - 1$, and, for each such $\{x, y\}$ and $t$,

$$\begin{aligned}
0 &\leqq \mu(x, y) - \textstyle\prod_t [1 + \alpha] = \prod_1^n \mu(t_{p-1}, t_p) - \prod_1^n [1 + \alpha(t_{p-1}, t_p)] \\
&= \textstyle\sum_1^n \{\prod_1^{p-1} [1 + \alpha(t_{q-1}, t_q)]\} [\mu(t_{p-1}, t_p) - 1 - \alpha(t_{p-1}, t_p)] \\
&\qquad \cdot \{\textstyle\prod_{p+1}^n \mu(t_{r-1}, t_r)\} \\
&\leqq \textstyle\sum_1^n \{\prod_1^{p-1} \mu(t_{q-1}, t_q)\} [\mu(t_{p-1}, t_p) - 1 - \alpha(t_{p-1}, t_p)] \\
&\qquad \cdot \{\textstyle\prod_{p+1}^n \mu(t_{r-1}, t_r)\} \\
&\leqq \{\textstyle\prod_1^n \mu(t_{q-1}, t_q)\} \sum_1^n [\mu(t_{p-1}, t_p) - 1 - \alpha(t_{p-1}, t_p)] \\
&= \mu(x, y) \{\textstyle\sum_t [\mu - 1] - \alpha(x, y)\},
\end{aligned}$$

whence (i) is true.

There emerges from the preceding argument a fact which will be useful in the subsequent development, and which we now record.

THEOREM 2.3. *If $\alpha$ is in $\mathcal{OA}^+$ and $\mu = \mathcal{E}^+(\alpha)$, then, for each $\{x, y\}$ in $S \times S$ and each $\mathcal{O}$-subdivision $t$ of $\{x, y\}$,*

$$\textstyle\sum_t [\mu - 1] - \alpha(x, y) \leqq \mu(x, y) - \prod_t [1 + \alpha].$$

## 3. The fundamental correspondence

Let $\mathcal{OA}$ denote the set of all $\mathcal{O}$-additive functions $V$ from $S \times S$ to the (complete normed) ring $N$ such that, if $\{a, b\}$ is in $S \times S$, there is a number $u$ such that if $\{t_p\}_0^n$ is an $\mathcal{O}$-subdivision of $\{a, b\}$, then

$$\textstyle\sum_t | V | = \sum_1^n | V(t_{p-1}, t_p)| \leqq u.$$

Let $\mathcal{OM}$ denote the set of all $\mathcal{O}$-multiplicative functions $W$ from $S \times S$ to $N$ such that, if $\{a, b\}$ is in $S \times S$, there is a number $u$ such that, if $\{t_p\}_0^n$ is an $\mathcal{O}$-subdivision of $\{a, b\}$, then[3] $\sum_t | W - 1 | = \sum_1^n | W(t_{p-1}, t_p) - 1 | \leqq u$.

LEMMA 3.1. *If $V$ is an $\mathcal{O}$-additive function from $S \times S$ to $N$, then the condition that $V$ belong to $\mathcal{OA}$ is equivalent to the requirement that there be a member $\alpha$ of $\mathcal{OA}^+$ such that $| V | \leqq \alpha$.*

*Indication of proof.* The requirement is clearly sufficient. If $V$ belongs to $\mathcal{OA}$, then, for each $\{x, y\}$ in $S \times S$, let $\alpha(x, y)$ be

$${}_x\textstyle\sum^y | V | = \text{L.U.B.} \sum_t | V | \qquad \text{for all } \mathcal{O}\text{-subdivisions } t \text{ of } \{x, y\}.$$

LEMMA 3.2. *If $W$ is an $\mathcal{O}$-multiplicative function from $S \times S$ to $N$, then the condition that $W$ belong to $\mathcal{OM}$ is equivalent to the requirement that there be a member $\mu$ of $\mathcal{OM}^+$ such that $| W - 1 | \leqq \mu - 1$.*

*Indication of proof.* The requirement is clearly sufficient. If $W$ belongs to $\mathcal{OM}$, then, for each $\{x, y\}$ in $S \times S$, let

$$h(x, y) = \text{L.U.B.} \textstyle\sum_t | W - 1 | \qquad \text{for all } \mathcal{O}\text{-subdivisions } t \text{ of } \{x, y\};$$

clearly $h(x, y) + h(y, z) \leqq h(x, z)$ for all $x$, $y$, and $z$ in $S$ such that $\{x, y, z\}$ is an $\mathcal{O}$-subdivision of $\{x, z\}$; let $e$ be a member of $S$, and $\alpha$ a member of $\mathcal{OA}^+$ such that, if $\{x, y\}$ is in $S \times S$, then

$$\begin{aligned} \alpha(x, y) &= h(x, e) - h(y, e) && \text{if } \{x, y, e\} \text{ is an } \mathcal{O}\text{-subdivision of } \{x, e\}, \\ \alpha(x, y) &= h(x, e) + h(e, y) && \text{if } \{x, e, y\} \text{ is an } \mathcal{O}\text{-subdivision of } \{x, y\}, \\ \alpha(x, y) &= h(e, y) - h(e, x) && \text{if } \{e, x, y\} \text{ is an } \mathcal{O}\text{-subdivision of } \{e, y\}; \end{aligned}$$

let $\mu = \mathcal{E}^+(\alpha)$; if $\{x, y\}$ is in $S \times S$ and either $\{x, y, e\}$ is an $\mathcal{O}$-subdivision of $\{x, e\}$ or $\{e, x, y\}$ is an $\mathcal{O}$-subdivision of $\{e, y\}$,

$$| W(x, y) - 1 | \leqq h(x, y) \leqq \alpha(x, y) \leqq \mu(x, y) - 1,$$

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[3] Observe that $W(x, x) = 1$, since $\{x, x\}$ is in $\mathcal{O}$, and so there is a number $u$ such that $n| W(x, x) - 1 | \leqq u$ $(n = 1, 2, \cdots)$.

whereas, if $\{x, e, y\}$ is an $\mathcal{O}$-subdivision of $\{x, y\}$, then

$$
\begin{aligned}
|W(x, y) - 1| &= |W(x, e)W(e, y) - 1| \\
&= |[W(x, e) - 1][W(e, y) - 1] \\
&\qquad + [W(x, e) - 1] + [W(e, y) - 1]| \\
&\leqq \alpha(x, e)\alpha(e, y) + \alpha(x, e) + \alpha(e, y) \\
&= [1 + \alpha(x, e)][1 + \alpha(e, y)] - 1 \leqq \mu(x, y) - 1.
\end{aligned}
$$

THEOREM 3.1. *If $\alpha$ is in $\mathcal{OA}^+$ and $V$ is a member of $\mathcal{OA}$ such that $|V| \leqq \alpha$, then, for each $\{a, b\}$ in $S \times S$, ${}_a\prod^b [1 + V]$ exists, and*

$$|{}_a\prod^b [1 + V] - [1 + V(a, b)]| \leqq {}_a\prod^b [1 + \alpha] - [1 + \alpha(a, b)].$$

*Indication of proof.* If $\{t_p\}_0^{n+1}$ is an $\mathcal{O}$-subdivision of $\{x, y\}$, then, for $1 \leqq q \leqq p \leqq n + 1$, we see that

$$
\begin{aligned}
&\prod_1^p [1 + V(t_{r-1}, t_r)] - \prod_1^{p-1} [1 + V(t_{r-1}, t_r)] \\
&\qquad = \left\{\prod_1^{p-1} [1 + V(t_{r-1}, t_r)]\right\} V(t_{p-1}, t_p), \\
&\prod_q^p [1 + V(t_{r-1}, t_r)] - \prod_{q+1}^p [1 + V(t_{r-1}, t_r)] \\
&\qquad = V(t_{q-1}, t_q)\left\{\prod_{q+1}^p [1 + V(t_{r-1}, t_r)]\right\},
\end{aligned}
$$

where $\prod_1^0 = \prod_{p+1}^p = 1$, and

$$
\begin{aligned}
&\prod_1^{n+1} [1 + V(t_{r-1}, t_r)] - [1 + V(x, y)] \\
&\quad = \sum_{p=1}^{n+1} \left\{\prod_1^{p-1} [1 + V(t_{r-1}, t_r)] - 1\right\} V(t_{p-1}, t_p) \\
&\quad = \sum_{p=1}^{n} \left\{\prod_1^{p} [1 + V(t_{r-1}, t_r)] - 1\right\} V(t_p, t_{p+1}) \\
&\quad = \sum_{p=1}^{n} \sum_{q=1}^{p} V(t_{q-1}, t_q)\left\{\prod_{q+1}^p [1 + V(t_{r-1}, t_r)]\right\} V(t_p, t_{p+1}),
\end{aligned}
$$

whence it follows that

$$|\prod_t [1 + V] - [1 + V(x, y)]| \leqq \prod_t [1 + \alpha] - [1 + \alpha(x, y)].$$

From these considerations, and from the identity

$$\prod_1^m B_q - \prod_1^m A_q = \sum_{q=1}^m \left\{\prod_{p=1}^{q-1} A_p\right\} (B_q - A_q)\left\{\prod_{r=q+1}^m B_r\right\},$$

if $s$ is an $\mathcal{O}$-subdivision of $\{a, b\}$ and $t$ is a refinement of $s$, then

$$|\prod_t [1 + V] - \prod_s [1 + V]| \leqq \prod_t [1 + \alpha] - \prod_s [1 + \alpha].$$

THEOREM 3.2. *If $\mu$ is in $\mathcal{OM}^+$ and $W$ is a member of $\mathcal{OM}$ such that*

$$|W - 1| \leqq \mu - 1,$$

*then, for each $\{a, b\}$ in $S \times S$, ${}_a\sum^b [W - 1]$ exists, and*

$$|[W(a, b) - 1] - {}_a\sum^b [W - 1]| \leqq [\mu(a, b) - 1] - {}_a\sum^b [\mu - 1].$$

*Indication of proof.* If $\{t_p\}_0^{n+1}$ is an $\mathcal{O}$-subdivision of $\{x, y\}$, then, for $1 \leqq q \leqq p \leqq n + 1$, we see that

$$W(x, t_p) - W(x, t_{p-1}) = W(x, t_{p-1})[W(t_{p-1}, t_p) - 1],$$

$$W(t_{q-1}, t_p) - W(t_q, t_p) = [W(t_{q-1}, t_q) - 1]W(t_q, t_p),$$

and, recalling that $W(x, x) = W(t_p, t_p) = 1$,

$$\begin{aligned}[W(x, y) - 1] &- \textstyle\sum_{p=1}^{n+1} [W(t_{p-1}, t_p) - 1] \\ &= \textstyle\sum_{p=1}^{n+1} [W(x, t_{p-1}) - 1][W(t_{p-1}, t_p) - 1] \\ &= \textstyle\sum_{p=1}^{n} [W(x, t_p) - 1][W(t_p, t_{p+1}) - 1] \\ &= \textstyle\sum_{p=1}^{n} \sum_{q=1}^{p} [W(t_{q-1}, t_q) - 1]W(t_q, t_p)[W(t_p, t_{p+1}) - 1],\end{aligned}$$

whence it follows that

$$|[W(x, y) - 1] - \textstyle\sum_t [W - 1]| \leqq [\mu(x, y) - 1] - \sum_t [\mu - 1].$$

Therefore, if $s$ is an $\mathcal{O}$-subdivision of $\{a, b\}$ and $t$ is a refinement of $s$, then

$$|\textstyle\sum_s [W - 1] - \sum_t [W - 1]| \leqq \sum_s [\mu - 1] - \sum_t [\mu - 1].$$

THEOREM 3.3. *There is a reversible function $\mathcal{E}$, from $\mathcal{O}\mathcal{A}$ onto $\mathcal{O}\mathcal{M}$, such that each of the following is a necessary and sufficient condition for the member $\{V, W\}$ of $\mathcal{O}\mathcal{A} \times \mathcal{O}\mathcal{M}$ to belong to $\mathcal{E}$:*

(i) $W(a, b) = {}_a\prod^b [1 + V]$ *for each* $\{a, b\}$ *in* $S \times S$.

(ii) $V(a, b) = {}_a\sum^b [W - 1]$ *for each* $\{a, b\}$ *in* $S \times S$.

(iii) *There is a member* $\{\alpha, \mu\}$ *of* $\mathcal{E}^+$ *such that*

$$| W(x, y) - 1 - V(x, y)| \leqq \mu(x, y) - 1 - \alpha(x, y)$$

*for each* $\{x, y\}$ *in* $S \times S$.

COROLLARY 3.1. *If $\{V, W\}$ is in $\mathcal{E}$, then these are equivalent:*

(1) $V(y, x) = -V(x, y)$ *for each* $\{x, y\}$ *in* $S \times S$.

(2) *There is a function $\phi$ from $S$ to $N$ such that $V = d\phi$.*

(3) *If $c$ is a positive number and $s$ is an $\mathcal{O}$-subdivision of the member $\{x, y\}$ of $S \times S$, there is a refinement $\{t_p\}_0^n$ of $s$ such that*

$$|\textstyle\sum_1^n \{[W(t_{p-1}, t_p) - 1] + [W(t_p, t_{p-1}) - 1]\}| < c.$$

(4) *There is a member $\{\alpha, \mu\}$ of $\mathcal{E}^+$ such that if $\{x, y\}$ is in $S \times S$, then*

$$|[W(x, y) - 1] + [W(y, x) - 1]| \leqq 2\{\mu(x, y) - 1 - \alpha(x, y)\}.$$

COROLLARY 3.2. *If $W$ belongs to $\mathcal{O}\mathcal{M}$, $e$ is in $S$, each of $F$ and $G$ is a function from $S$ to $N$, and $F(x) = W(x, e)$ and $G(x) = W(e, x)$ for all $x$ in $S$, then each of $dF$ and $dG$ belongs to $\mathcal{O}\mathcal{A}$.*

COROLLARY 3.3. *If* $\{\alpha, \mu\}$ *belongs to* $\mathcal{E}^+$ *and* $\{V, W\}$ *is a member of* $\mathcal{E}$ *such that* $|V| \leqq \alpha$, *then, for each* $\{x, y\}$ *in* $S \times S$ *and each* $\mathfrak{O}$*-subdivision* $\{t_p\}_0^n$ *of* $\{x, y\}$, *each of*

$$W(x, y) - 1 - \sum_1^n W(x, t_{p-1})V(t_{p-1}, t_p)$$

$$\text{and} \quad W(x, y) - 1 - \sum_1^n V(t_{p-1}, t_p)W(t_p, y)$$

*has norm not exceeding* $\mu(x, y)\{\sum_t [\mu - 1] - \alpha(x, y)\}$.

*Indication of proof.* Apply Theorem 3.3(iii) in comparing the indicated sums with

$$W(x, y) - 1 = \sum_1^n W(x, t_{p-1})[W(t_{p-1}, t_p) - 1]$$
$$= \sum_1^n [W(t_{p-1}, t_p) - 1]W(t_p, y).$$

The following theorem is a corollary result of a somewhat different nature, which includes the observation that if $\{\alpha, \mu\}$ is in $\mathcal{E}^+$ and either $\alpha$ or $\mu$ is symmetric, then so is the other.

THEOREM 3.4. *Suppose that* $b > 0$ *and* $J$ *is a function from* $N$ *to* $N$ *such that* $J(1) = 1$ *and, for all* $X$, $Y$, *and* $Z$ *in* $N$,

$$J(X + Y) = J(X) + J(Y), \qquad J(XY) = J(Y)J(X),$$
$$\text{and} \quad |J(Z)| \leqq b|Z|.$$

*If* $J_1$ *is the function to which* $\{g, h\}$ *belongs only in case each of* $g$ *and* $h$ *is a function from* $S \times S$ *to* $N$ *and* $h(x, y) = J(g(y, x))$ *for all* $\{x, y\}$ *in* $S \times S$, *then* $J_1$ *commutes with* $\mathcal{E}$, *viz., if* $\{V, W\}$ *is in* $\mathcal{E}$, *then* $\{J_1(V), J_1(W)\}$ *is in* $\mathcal{E}$.

## 4. The homogeneous integral equations

Let $\mathfrak{OB}$ denote the set of all functions $\phi$ from $S$ to $N$ such that $d\phi$ belongs to $\mathfrak{OA}$. If $F$ is a function from $S$ to $N$ and $V$ is a function from $S \times S$ to $N$ (*or* each is a function to the set of real numbers), the statement that

$$Z = (\text{L}) \int_a^b F \cdot V$$

means that $\{a, b\}$ is in $S \times S$ and $Z = {}_a\sum^b h$, where $h$ is the function defined by

$$h(x, y) = F(x)V(x, y) \qquad \text{for each } \{x, y\} \text{ in } S \times S.$$

Similarly, $(\text{R}) \int_a^b V \cdot G = {}_a\sum^b h$ for $h$ defined by $h(x, y) = V(x, y)G(y)$, and $(\text{L, R}) \int_a^b F \cdot V \cdot G = {}_a\sum^b h$ for $h(x, y) = F(x)V(x, y)G(y)$.

If $W$ is a function from $S \times S$ and $e$ is in $S$, then $W(\ , e)$ denotes the function consisting of all ordered pairs of the form $\{x, W(x, e)\}$ for $x$ in $S$, and $W(e, \ )$ denotes the function consisting of all ordered pairs of the form

$\{x, W(e, x)\}$ for $x$ in $S$. The following type of notation is also used when convenient:

$$(\mathrm{L}) \sum_t F \cdot V = \sum_1^n F(t_{p-1}) V(t_{p-1}, t_p) \qquad \text{for } t = \{t_p\}_0^n .$$

LEMMA 4.1 (Integration-by-parts). *If each of $F$ and $G$ is a function from $S$ to $N$ and either of* $(\mathrm{L}) \int_a^b F \cdot dG$ *and* $(\mathrm{R}) \int_a^b dF \cdot G$ *exists, then the other exists, and*

$$(\mathrm{L}) \int_a^b F \cdot dG - F(a)\, dG(a, b) = dF(a, b) G(b) - (\mathrm{R}) \int_a^b dF \cdot G.$$

LEMMA 4.2. *If $\alpha$ is in $\mathfrak{O}\mathfrak{A}^+$, then, for each $\{x, y\}$ in $S \times S$, the integral* $(\mathrm{L}) \int_x^y \alpha(x, \ ) \cdot \alpha$ *exists and is*[4]

$$\text{L.U.B. } (\mathrm{L}) \sum_t \alpha(x, \ ) \cdot \alpha \qquad \textit{for all } \mathfrak{O}\textit{-subdivisions } t \textit{ of } \{x, y\}.$$

LEMMA 4.3. *If each of $F$ and $G$ is in $\mathfrak{O}\mathfrak{B}$ and $\alpha$ is a member of $\mathfrak{O}\mathfrak{A}^+$ such that $|dF| \leqq \alpha$ and $|dG| \leqq \alpha$, then, for each $\{a, b\}$ in $S \times S$,* $(\mathrm{L}) \int_a^b F \cdot dG$ *exists and—if $e$ is a member of $S$ such that $\{e, a, b\}$ is an $\mathfrak{O}$-subdivision of $\{e, b\}$—*

$$\left| (\mathrm{L}) \int_a^b F \cdot dG - F(a)\, dG(a, b) \right| \leqq (\mathrm{L}) \int_a^b \alpha(e, \ ) \cdot \alpha - \alpha(e, a) \alpha(a, b).$$

*Indication of proof.* If $\{e, x, y\}$ is an $\mathfrak{O}$-subdivision of $\{e, y\}$ and $\{t_p\}_0^n$ is an $\mathfrak{O}$-subdivision of $\{x, y\}$, then

$$|(\mathrm{L}) \sum_t F \cdot dG - F(x)\, dG(x, y)| = |\sum_1^n dF(x, t_{p-1})\, dG(t_{p-1}, t_p)|$$
$$\leqq \sum_1^n \alpha(x, t_{p-1}) \alpha(t_{p-1}, t_p) = (\mathrm{L}) \sum_t \alpha(e, \ ) \cdot \alpha - \alpha(e, x) \alpha(x, y).$$

Hence, if $\{e, a, b\}$ is an $\mathfrak{O}$-subdivision of $\{e, b\}$ and $s$ is an $\mathfrak{O}$-subdivision of $\{a, b\}$ and $t$ is a refinement of $s$, then

$$|(\mathrm{L}) \sum_t F \cdot dG - (\mathrm{L}) \sum_s F \cdot dG| \leqq (\mathrm{L}) \sum_t \alpha(e, \ ) \cdot \alpha - (\mathrm{L}) \sum_s \alpha(e, \ ) \cdot \alpha.$$

LEMMA 4.4 (Integration-by-substitution). *If $\{a, b\}$ is in $S \times S$ and $V$ is a member of $\mathfrak{O}\mathfrak{A}$, each of $F$ and $G$ is in $\mathfrak{O}\mathfrak{B}$, and $F_1$ and $G_1$ are members of $\mathfrak{O}\mathfrak{B}$ such that*

$$dF_1(x, y) = (\mathrm{L}) \int_x^y F \cdot V \quad \textit{and} \quad dG_1(x, y) = (\mathrm{R}) \int_x^y V \cdot G$$

*for each $\mathfrak{O}$-subdivision $\{a, x, y, b\}$ of $\{a, b\}$, then*

$$(\mathrm{R}) \int_a^b dF_1 \cdot G = (\mathrm{L}, \mathrm{R}) \int_a^b F \cdot V \cdot G = (\mathrm{L}) \int_a^b F \cdot dG_1 .$$

---

[4] Obviously a more general lemma involving two functions, as well as one involving R-integrals, could be stated; we desist, since the sequel does not require it.

*Indication of proof.* Let $\alpha$ be a member of $\mathcal{O}\mathcal{a}^+$ such that $|V| \leqq \alpha$ and $|dF| \leqq \alpha$ and $|dG| \leqq \alpha$, $C = |F(a)| + \alpha(a, b) + |G(b)|$, and let $t$ be an $\mathcal{O}$-subdivision of $\{a, b\}$; we have these estimates:

$$|(\mathrm{R}) \textstyle\sum_t dF_1 \cdot G - (\mathrm{L}, \mathrm{R}) \sum_t F \cdot V \cdot G|$$
$$\leqq \left\{(\mathrm{L}) \int_a^b \alpha(a, \ ) \cdot \alpha - (\mathrm{L}) \textstyle\sum_t \alpha(a, \ ) \cdot \alpha\right\} C;$$

$$|(\mathrm{L}, \mathrm{R}) \textstyle\sum_t F \cdot V \cdot G - (\mathrm{L}) \sum_t F \cdot dG_1|$$
$$\leqq \left\{(\mathrm{L}) \int_a^b \alpha(a, \ ) \cdot \alpha - (\mathrm{L}) \textstyle\sum_t \alpha(a, \ ) \cdot \alpha\right\} C.$$

THEOREM 4.1. *If $e$ is in $S$ and $\{V, W\}$ belongs to $\mathcal{E}$ and $U$ is a function from $S$ to $N$, the following statements are equivalent:*

(i) *$U$ is a member of $\mathcal{O}\mathcal{B}$ such that, for each $z$ in $S$,*

$$U(z) = U(e) + (\mathrm{L}) \int_e^z U \cdot V.$$

(ii) $U(z) = U(e) W(e, z)$ *for each $z$ in $S$.*

*Indication of proof.* From Corollaries 3.2 and 3.3, (ii) implies (i), and for each $\{a, b\}$ in $S \times S$ we have

$$W(a, b) = 1 + (\mathrm{R}) \int_a^b V \cdot W( \ , b).$$

Supposing (i) to be true and $z$ to be a member of $S$, we have

$$dU(x, y) = (\mathrm{L}) \int_x^y U \cdot V \quad \text{and} \quad W(y, z) - W(x, z) = -(\mathrm{R}) \int_x^y V \cdot W( \ , z)$$

for each $\{x, y\}$ in $S \times S$ such that $\{e, x, y, z\}$ is an $\mathcal{O}$-subdivision of $\{e, z\}$; hence, by Lemmas 4.1 and 4.4,

$$\begin{aligned} U(z) &- U(e) W(e, z) \\ &= (\mathrm{L}) \int_e^z U \cdot dW( \ , z) + (\mathrm{R}) \int_e^z dU \cdot W( \ , z) \\ &= -(\mathrm{L}, \mathrm{R}) \int_e^z U \cdot V \cdot W( \ , z) + (\mathrm{L}, \mathrm{R}) \int_e^z U \cdot V \cdot W( \ , z). \end{aligned}$$

THEOREM 4.2. *If $e$ is in $S$ and $\{V, W\}$ belongs to $\mathcal{E}$ and $U$ is a function from $S$ to $N$, the following statements are equivalent:*

(i) *$U$ is a member of $\mathcal{O}\mathcal{B}$ such that, for each $z$ in $S$,*

$$U(z) = U(e) + (\mathrm{R}) \int_z^e V \cdot U.$$

(ii) $U(z) = W(z, e) U(e)$ *for each $z$ in $S$.*

*Indication of proof.* From Corollaries 3.2 and 3.3, (ii) implies (i), and for each $\{a, b\}$ in $S \times S$ we have

$$W(a, b) = 1 + (\mathrm{L}) \int_a^b W(a, \ )\cdot V.$$

Supposing (i) to be true and $z$ to be a member of $S$, we have

$$dU(x, y) = -(\mathrm{R}) \int_x^y V\cdot U \quad \text{and} \quad W(z, y) - W(z, x) = (\mathrm{L}) \int_x^y W(z, \ )\cdot V$$

for each $\{x, y\}$ in $S \times S$ such that $\{z, x, y, e\}$ is an $\mathfrak{O}$-subdivision of $\{z, e\}$; now apply Lemmas 4.1 and 4.4 to $W(z, e)U(e) - U(z)$ along the lines indicated in the preceding argument.

THEOREM 4.3. *If $V$ is in $\mathfrak{O}\mathfrak{A}$ and $W$ is a function from $S \times S$ to $N$, each of the following is a necessary and sufficient condition for $W$ to be the member $\mathcal{E}(V)$ of $\mathfrak{O}\mathfrak{M}$:*

(i) *If $\{a, b\}$ is in $S \times S$, then $W(a, \ )$ is in $\mathfrak{O}\mathfrak{B}$, and*

$$W(a, b) = 1 + (\mathrm{L}) \int_a^b W(a, \ )\cdot V.$$

(ii) *If $\{a, b\}$ is in $S \times S$, then $W( \ , b)$ is in $\mathfrak{O}\mathfrak{B}$, and*

$$W(a, b) = 1 + (\mathrm{R}) \int_a^b V\cdot W( \ , b).$$

## 5. The nonhomogeneous equations

We treat the nonhomogeneous versions of Theorems 4.1 and 4.2 by considering the (appropriately normed) rings $N'$ and $N''$ of 2-by-2 matrices with respective forms

$$\begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} \qquad (X, Y, \text{and } Z \text{ in } N),$$

the corresponding function-classes $\mathfrak{O}\mathfrak{A}'$, $\mathfrak{O}\mathfrak{M}'$, $\mathfrak{O}\mathfrak{A}''$, and $\mathfrak{O}\mathfrak{M}''$, and the corresponding mappings $\mathcal{E}'$ and $\mathcal{E}''$.

If each of $V$ and $K$ belongs to $\mathfrak{O}\mathfrak{A}$, then it is easy, by using Theorem 4.3, to see that

$$\mathcal{E}'\begin{pmatrix} V & 0 \\ K & 0 \end{pmatrix} = \begin{pmatrix} W & 0 \\ G & 1 \end{pmatrix},$$

where $W = \mathcal{E}(V)$ and for each $\{a, b\}$ in $S \times S$

$$G(a, b) = (\mathrm{L}) \int_a^b G(a, \ )\cdot V + K(a, b) = (\mathrm{R}) \int_a^b K\cdot W( \ , b);$$

moreover, it is similarly easy to see that

$$\mathcal{E}''\begin{pmatrix} V & K \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} W & H \\ 0 & 1 \end{pmatrix},$$

where $W = \mathcal{E}(V)$ and for each $\{a, b\}$ in $S \times S$

$$H(a, b) = (\mathrm{L}) \int_a^b W(a, \ )\cdot K = (\mathrm{R}) \int_a^b V\cdot H( \ , b) + K(a, b).$$

With the theorems of Section 4, these two observations lead respectively to the following two theorems.

THEOREM 5.1. *If $e$ is in $S$ and each of $V$ and $K$ is in $\mathcal{OA}$ and $W = \mathcal{E}(V)$ and $U$ is a function from $S$ to $N$, then the following statements are equivalent:*

(i) *$U$ is a member of $\mathcal{OB}$ such that, for each $z$ in $S$,*

$$U(z) = U(e) + (\mathrm{L}) \int_e^z U\cdot V + K(e, z).$$

(ii) $U(z) = U(e)W(e, z) + (\mathrm{R}) \int_e^z K\cdot W( \ , z)$ *for each $z$ in $S$.*

THEOREM 5.2. *If $e$ is in $S$ and each of $V$ and $K$ is in $\mathcal{OA}$ and $W = \mathcal{E}(V)$ and $U$ is a function from $S$ to $N$, then the following statements are equivalent:*

(i) *$U$ is a member of $\mathcal{OB}$ such that, for each $z$ in $S$,*

$$U(z) = U(e) + (\mathrm{R}) \int_z^e V\cdot U + K(z, e).$$

(ii) $U(z) = W(z, e)U(e) + (\mathrm{L}) \int_z^e W(z, \ )\cdot K$ *for each $z$ in $S$.*

We state two immediate corollaries which, in effect, complete the analysis of the mappings $\mathcal{E}'$ and $\mathcal{E}''$ in terms of $\mathcal{E}$.

COROLLARY 5.1. *If each of $V_1$, $V_2$, and $K$ is in $\mathcal{OA}$, then*

$$\mathcal{E}'\begin{pmatrix} V_1 & 0 \\ K & V_2 \end{pmatrix} = \begin{pmatrix} W_1 & 0 \\ G & W_2 \end{pmatrix},$$

*where $W_1 = \mathcal{E}(V_1)$, $W_2 = \mathcal{E}(V_2)$, and for each $\{a, b\}$ in $S \times S$*

$$G(a, b) = (\mathrm{L}, \mathrm{R}) \int_a^b W_2(a, \ )\cdot K\cdot W_1( \ , b).$$

COROLLARY 5.2. *If each of $V_1$, $V_2$, and $K$ is in $\mathcal{OA}$, then*

$$\mathcal{E}''\begin{pmatrix} V_1 & K \\ 0 & V_2 \end{pmatrix} = \begin{pmatrix} W_1 & H \\ 0 & W_2 \end{pmatrix},$$

*where $W_1 = \mathcal{E}(V_1)$, $W_2 = \mathcal{E}(V_2)$, and for each $\{a, b\}$ in $S \times S$*

$$H(a, b) = (\mathrm{L}, \mathrm{R}) \int_a^b W_1(a, \ )\cdot K\cdot W_2( \ , b).$$

## 6. The Peano series

THEOREM 6.1. *If $V$ is in $\mathcal{OA}$ and $W = \mathcal{E}(V)$, then*

$$W(x, y) = \sum_{p=0}^{\infty} G_p(x, y) \qquad \textit{for each } \{x, y\} \textit{ in } S \times S,$$

*where*

(i) $G_0(x, y) = 1, \quad G_p(x, y) = (\mathrm{L}) \int_x^y G_{p-1}(x, \ ) \cdot V \ (p = 1, 2, \cdots)$, *and*

(ii) *for each $\{x, b\}$ in $S \times S$ the convergence is uniform over the set of all $y$ such that $\{x, y, b\}$ is an $\mathcal{O}$-subdivision of $\{x, b\}$.*

THEOREM 6.2. *If $V$ is in $\mathcal{OA}$ and $W = \mathcal{E}(V)$, then*

$$W(x, y) = \sum_{p=0}^{\infty} H_p(x, y) \qquad \textit{for each } \{x, y\} \textit{ in } S \times S,$$

*where*

(i) $H_0(x, y) = 1, \quad H_p(x, y) = (\mathrm{R}) \int_x^y V \cdot H_{p-1}( \ , y) \ (p = 1, 2, \cdots)$,

*and*

(ii) *for each $\{a, y\}$ in $S \times S$ the convergence is uniform over the set of all $x$ such that $\{a, x, y\}$ is an $\mathcal{O}$-subdivision of $\{a, y\}$.*

*Indication of proofs.* Supposing that $W = \mathcal{E}(V)$, and that the infinite sequences $G$ and $H$ are defined as indicated, we use results from Sections 4 and 5 to justify the following computations:

$$\begin{aligned} W(x, y) &- \textstyle\sum_{p=0}^{n} G_p(x, y) \\ &= (\mathrm{L}) \int_x^y \{W(x, \ ) - \sum_0^n G_p(x, \ )\} \cdot V + G_{n+1}(x, y) \\ &= (\mathrm{R}) \int_x^y dG_{n+1}(x, \ ) \cdot W( \ , y) = (\mathrm{L}, \mathrm{R}) \int_x^y G_n(x, \ ) \cdot V \cdot W( \ , y); \end{aligned}$$

$$\begin{aligned} W(x, y) &- \textstyle\sum_{p=0}^{n} H_p(x, y) \\ &= (\mathrm{R}) \int_x^y V \cdot \{W( \ , y) - \sum_0^n H_p( \ , y)\} + H_{n+1}(x, y) \\ &= -(\mathrm{L}) \int_x^y W(x, \ ) \cdot dH_{n+1}( \ , y) = (\mathrm{L}, \mathrm{R}) \int_x^y W(x, \ ) \cdot V \cdot H_n( \ , y). \end{aligned}$$

Now let $\{\alpha, \mu\}$ be a member of $\mathcal{E}^+$ such that $|V| \leqq \alpha$, let $g$ and $h$ be infinite sequences obtained from $\alpha$ just as $G$ and $H$ are obtained from $V$, and let $\{a, b\}$ be a member of $S \times S$. Considering $\{x, y\}$ such that $\{a, x, y, b\}$ is an $\mathcal{O}$-subdivision of $\{a, b\}$, we have

$$\mu(x, y) - \sum_0^n g_p(x, y) = (\mathrm{L}, \mathrm{R}) \int_x^y g_n(x, \ ) \cdot \alpha \cdot \mu( \ , y) \geqq 0, \quad \text{and}$$

$$\mu(x, y) - \sum_0^n h_p(x, y) = (\mathrm{L}, \mathrm{R}) \int_x^y \mu(x, \ ) \cdot \alpha \cdot h_n( \ , y) \geqq 0;$$

thus we see that $g$ and $h$ have the limit 0; the following estimates are established inductively (under the same hypotheses):

$$| G_n(x, y) | \leqq g_n(x, y) \leqq g_n(x, b), \qquad | H_n(x, y) | \leqq h_n(x, y) \leqq h_n(a, y),$$

$$\Big| W(x, y) - \sum_0^n G_p(x, y) \Big| \leqq (\mathrm{R}) \int_x^y dg_{n+1}(x, \ )\cdot\mu( \ , y) \leqq g_{n+1}(x, b)\mu(x, b),$$

$$\Big| W(x, y) - \sum_0^n H_p(x, y) \Big| \leqq -(\mathrm{L}) \int_x^y \mu(x, \ )\cdot dh_{n+1}( \ , y) \leqq \mu(a, y)h_{n+1}(a, y).$$

The various assertions of the two theorems now follow.

## 7. The fully multiplicative case

In Corollary 3.1 we characterized those members $\{V, W\}$ of $\mathcal{E}$ such that $V$ is fully additive. In this section we characterize those members $\{V, W\}$ of $\mathcal{E}$ such that $W$ is *fully multiplicative*:

$$W(x, y)W(y, z) = W(x, z) \qquad \text{for } all\ x,\ y, \text{ and } z \text{ in } S;$$

since $W(x, x) = 1$ for each $W$ in $\mathcal{OM}$ and $x$ in $S$, the condition that the member $W$ of $\mathcal{OM}$ be fully multiplicative is equivalent to the requirement that $W(x, y)W(y, x) = 1$ for all $\{x, y\}$ in $S \times S$. In connection with Corollary 3.2, we record the following fact:

THEOREM 7.1. *If $W$ is a fully multiplicative function from $S \times S$ to $N$, then in order that $W$ belong to $\mathcal{OM}$ it is necessary and sufficient that, for each $e$ in $S$, $W(e, e) = 1$ and each of $W( \ , e)$ and $W(e, \ )$ belong to $\mathcal{OB}$.*

THEOREM 7.2. *If $\{V, W\}$ is in $\mathcal{E}$ and $h$ is a function such that*

$$h(x, y) = [1 + V(x, y)][1 + V(y, x)] - 1 \qquad \textit{for each } \{x, y\} \textit{ in } S \times S,$$

*then the following statements are equivalent:*

(1) $W(x, y)W(y, x) = 1$ *for all* $\{x, y\}$ *in* $S \times S$.

(2) ${}_x\sum^y | h | = 0$ *for all* $\{x, y\}$ *in* $S \times S$.

*Indication of proof.* Let $\{\alpha, \mu\}$ be a member of $\mathcal{E}^+$ such that $|V| \leqq \alpha$. Since for each $\{x, y\}$ in $S \times S$

$$h(x, y) = [1 + V(x, y)][1 + V(y, x) - W(y, x)] + [1 + V(x, y) - W(x, y)]W(y, x) + [W(x, y)W(y, x) - 1],$$

it follows from Theorem 3.1 that, if (1) is true, for each $\{x, y\}$ in $S \times S$ and each $\mathcal{O}$-subdivision $\{t_p\}_0^n$ of $\{x, y\}$,

$$\textstyle\sum_t | h | \leqq \mu(x, y)\left\{ \sum_1^n [\mu(t_{n+1-p}, t_{n-p}) - 1] - \alpha(y, x) \right\} + \left\{ \sum_t [\mu - 1] - \alpha(x, y) \right\}\mu(y, x),$$

so that, by Theorem 2.2, (2) is true. On the other hand, for each $\{x, y\}$ in

$S \times S$ and each $\mathfrak{O}$-subdivision $\{t_p\}_0^n$ of $\{x, y\}$,

$$\prod_1^n [1 + V(t_{q-1}, t_q)] \cdot \prod_1^n [1 + V(t_{n+1-r}, t_{n-r})] - 1$$
$$= \sum_1^n \{ \prod_1^p [1 + V(t_{q-1}, t_q)] \cdot \prod_{n+1-p}^n [1 + V(t_{n+1-r}, t_{n-r})]$$
$$- \prod_1^{p-1} [1 + V(t_{q-1}, t_q)] \cdot \prod_{n+2-p}^n [1 + V(t_{n+1-r}, t_{n-r})] \}$$
$$= \sum_1^n \{ \prod_1^{p-1} [1 + V(t_{q-1}, t_q)] \} h(t_{p-1}, t_p) \{ \prod_{n+2-p}^n [1 + V(t_{n+1-r}, t_{n-r})] \},$$

whence we have

$$| \prod_t [1 + V] \cdot \prod_1^n [1 + V(t_{n+1-r}, t_{n-r})] - 1 | \leqq \mu(x, y)\mu(y, x) \sum_t | h |;$$

therefore (2) implies (1).

A combination of Corollary 3.1 and Theorem 7.2 yields the following result.

THEOREM 7.3. *If $\phi$ is in $\mathfrak{OB}$ and $W = \mathcal{E}(d\phi)$, then the following statements are equivalent:*

(1) $W(x, y)W(y, x) = 1$ *for all $\{x, y\}$ in $S \times S$.*

(2) ${}_x\sum^y | [d\phi]^2 | = 0$ *for all $\{x, y\}$ in $S \times S$.*

The statement that $Z = (\mathrm{M}) \int_a^b F \cdot dG$ means that each of $F$ and $G$ is a function from $S$ to $N$ (or each is a function from $S$ to the set of real numbers) and $\{a, b\}$ is in $S \times S$ and $2Z = {}_a\sum^b h$, where $h$ is the function defined by

$$h(x, y) = [F(x) + F(y)][G(y) - G(x)] \qquad \text{for each } \{x, y\} \text{ in } S \times S.$$

Similarly, $Z = (\mathrm{M}) \int_a^b dG \cdot H$ means that $2Z = {}_a\sum^b h$ for $h$ defined by $h(x, y) = [G(y) - G(x)][H(x) + H(y)]$, while $Z = (\mathrm{M}) \int_a^b F \cdot dG \cdot H$ means that $4Z = {}_a\sum^b h$ for $h$ defined by

$$h(x, y) = [F(x) + F(y)][G(y) - G(x)][H(x) + H(y)].$$

To utilize this mean integral concept, we assume *for the remainder of this section* that the ring $N$ does not have characteristic 2, viz., that if $Z$ is in $N$ and $2Z = 0$, then $Z = 0$.

*Remark* 1. *If either of* $(\mathrm{M}) \int_a^b F \cdot dG$ *and* $(\mathrm{M}) \int_a^b dF \cdot G$ *exists, then the other exists, and*

$$(\mathrm{M}) \int_a^b F \cdot dG + (\mathrm{M}) \int_a^b dF \cdot G = F(b)G(b) - F(a)G(a).$$

*Remark* 2. *If each of $F$ and $G$ belongs to $\mathfrak{OB}$ and $\{x, y\}$ is in $S \times S$ and ${}_x\sum^y dF \cdot dG = 0$, then*

$$(\mathrm{M}) \int_x^y F \cdot dG = (\mathrm{L}) \int_x^y F \cdot dG.$$

THEOREM 7.4. *If $e$ is a member of $S$, each of $\phi$ and $F$ belongs to $\mathfrak{O}\mathfrak{B}$, ${}_e\sum^x |[d\phi]^2| = 0$ for each $x$ in $S$, and $W = \mathcal{E}(d\phi)$, then the following two statements are equivalent:*

(i) $F(x) = F(e) + (\mathrm{M})\int_e^x F\cdot d\phi$ *for each $x$ in $S$;*

(ii) $F(x) = F(e)W(e, x)$ *for each $x$ in $S$;*

*moreover, the following two statements are also equivalent:*

(iii) $F(x) = F(e) + (\mathrm{M})\int_x^e d\phi\cdot F$ *for each $x$ in $S$;*

(iv) $F(x) = W(x, e)F(e)$ *for each $x$ in $S$.*

*Indication of proof.* To see that (ii) implies (i), note that if $\{t_p\}_0^n$ is an $\mathfrak{O}$-subdivision of $\{a, b\}$, then

$$\begin{aligned}\textstyle\sum_1^n [W(a, t_p) - W(a, t_{p-1})][\phi(t_p) - \phi(t_{p-1})] \\ = \textstyle\sum_1^n W(a, t_{p-1})[d\phi(t_{p-1}, t_p)]^2 \\ + \textstyle\sum_1^n W(a, t_{p-1})[W(t_{p-1}, t_p) - 1 - d\phi(t_{p-1}, t_p)]\, d\phi(t_{p-1}, t_p),\end{aligned}$$

from which it follows, by Theorems 4.1 and 3.3, that

$$W(a, b) = 1 + (\mathrm{L})\int_a^b W(a, \ )\cdot d\phi = 1 + (\mathrm{M})\int_a^b W(a, \ )\cdot d\phi;$$

the implication of (iii) by (iv) is a consequence of Theorems 4.2 and 3.3 by a similar line of reasoning. Regarding the converse implications, suppose, for example, that (i) is true. For $x$ in $S$ we carry out the following computations:

$$2F(x) = 2F(e) + (\mathrm{L})\int_e^x F\cdot d\phi + (R)\int_e^x F\cdot d\phi;$$

$$\begin{aligned}2(\mathrm{L})\int_e^x dF\cdot W(\ , x) &= (\mathrm{L, L})\int_e^x F\cdot d\phi\cdot W(\ , x) + (\mathrm{R, L})\int_e^x F\cdot d\phi\cdot W(\ , x) \\ &= (\mathrm{L, R})\int_e^x F\cdot d\phi\cdot W(\ , x) + (\mathrm{R, R})\int_e^x F\cdot d\phi\cdot W(\ , x) \\ &= 2(\mathrm{R})\int_e^x dF\cdot W(\ , x);\end{aligned}$$

$$\begin{aligned}2(\mathrm{R})\int_e^x dF\cdot W(\ , x) &= -(\mathrm{L})\int_e^x F\cdot dW(\ , x) - (\mathrm{R})\int_e^x F\cdot dW(\ , x) \\ &= 2F(e)W(e, x) - 2F(x) + (\mathrm{R})\int_e^x dF\cdot W(\ , x) + (\mathrm{L})\int_e^x dF\cdot W(\ , x);\end{aligned}$$

$$2F(e)W(e, x) - 2F(x) = (\mathrm{R})\int_e^x dF\cdot W(\ , x) - (\mathrm{L})\int_e^x dF\cdot W(\ , x);$$

(ii) follows, since $N$ does not have characteristic 2. Implication of (iv) by (iii) follows from entirely analogous computation.

In the final theorem of this section, we will find convenient the hypothesis that the ring $N$ is torsion-free, i.e., that if $Z$ is in $N$ and $n$ is a positive integer and $nZ = 0$, then $Z = 0$.

THEOREM 7.5. *If $\phi$ is in $\mathcal{OB}$ and $W = \mathcal{E}(d\phi)$ and, for each $\{x, y\}$ in $S \times S$, $\phi(x)\phi(y) = \phi(y)\phi(x)$ and $W(x, y)W(y, x) = 1$, and the ring $N$ is torsion-free, then*

$$W(x, y) = \operatorname{Exp}\{\phi(y) - \phi(x)\} \qquad \textit{for all } \{x, y\} \textit{ in } S \times S.$$

*Indication of proof.* With reference to the infinite sequence $G$ indicated in Theorem 6.1, it is sufficient to show that for each $\{x, y\}$ in $S \times S$,

$$(p!)G_p(x, y) = [d\phi(x, y)]^p \qquad \text{for } p = 1, 2, \cdots .$$

Note that $G_1 = d\phi$, let $\{x, y\}$ be in $S \times S$, and let $F = d\phi(x, \ )$. Suppose that $p$ is a positive integer such that, for each $z$ in $S$,

$$(n!)G_n(x, z) = F(z)^n \qquad \text{for } n = 1, \cdots, p;$$

the following three steps are justified, respectively, by Lemma 4.1, by Theorem 7.3 and the special hypotheses concerning $\phi$, and by the inductive hypothesis together with Lemma 4.4:

$$\begin{aligned}(\mathrm{L}) \int_x^y F^p \cdot dF &= F(y)^{p+1} - (\mathrm{R}) \int_x^y dF^p \cdot F \\ &= F(y)^{p+1} - (\mathrm{L}) \int_x^y F \cdot dF^p \\ &= F(y)^{p+1} - p\ (\mathrm{L}) \int_x^y F^p \cdot dF;\end{aligned}$$

hence we have

$$F(y)^{p+1} = (p + 1) \cdot (\mathrm{L}) \int_x^y F^p \cdot dF = (p + 1)(p!) \cdot (\mathrm{L}) \int_x^y G_p(x, \ ) \cdot d\phi.$$

*Remark.* By Theorem 3.3, if $\{V, W\}$ is in $\mathcal{E}$, then the condition that $V$ have (multiplicatively) commuting values is *equivalent* to the condition that $W$ have the same property; thus we see that, with $N$ torsion-free, Theorem 7.5 provides a *complete characterization* of the fully multiplicative members $W$ of $\mathcal{OM}$ such that $W$ has commuting values and satisfies the conditions of Corollary 3.1.

## 8. Canonically ordered semigroups

A *canonically ordered semigroup* is an ordered pair $\{S, \sigma\}$ such that $S$ is a nondegenerate set and $\sigma$ is a function from $S \times S$ to $S$ with the following properties:

(1) The subset of $S \times S$, to which $\{x, z\}$ belongs only in case there is a member $y$ of $S$ such that $\sigma(x, y) = z$, is a linear ordering of $S$, viz.,

(i) if $x$ is in $S$ and $\{a, b\}$ is in $S \times S$, then there is a member $c$ of $S$ such that $\sigma(\sigma(x, a), b) = \sigma(x, c)$,

(ii) if $\sigma(\sigma(x, a), b) = x$, then $\sigma(x, a) = x$, and

(iii) if $\{x, y\}$ is in $S \times S$, then there is a member $\{a, b\}$ of $S \times S$ such that $\sigma(x, a) = y$ or $\sigma(y, b) = x$.

(2) If $x$ is in $S$ and $\{a, b\}$ is in $S \times S$, then $\sigma(a, b)$ is the only member $c$ of $S$ such that $\sigma(\sigma(x, a), b) = \sigma(x, c)$.

Suppose, now, that $\{S, \sigma\}$ is a canonically ordered semigroup and that $\mathcal{O}$ is the linear ordering of $S$ *determined by* $\sigma$, i.e., $\mathcal{O}$ is the subset of $S \times S$ to which $\{x, z\}$ belongs only in case there is a member $y$ of $S$ such that $\sigma(x, y) = z$.

THEOREM 8.1. *The requirement that*

$$\sigma(x, \delta(x, z)) = z \qquad \textit{for each } \{x, z\} \textit{ in } \mathcal{O}$$

*defines a function* $\delta$ *from* $\mathcal{O}$ *to* $S$ *with the following properties*:

(1) *If* $x$ *is in* $S$ *and* $\{a, b\}$ *is in* $S \times S$, *then* $\{\sigma(x, a), \sigma(x, b)\}$ *belongs to* $\mathcal{O}$ *only in case* $\{a, b\}$ *belongs to* $\mathcal{O}$, *and in this case*

$$\delta(\sigma(x, a), \sigma(x, b)) = \delta(a, b).$$

(2) *If each of* $\{x, y\}$ *and* $\{y, z\}$ *is in* $\mathcal{O}$, *then*

$$\sigma(\delta(x, y), \delta(y, z)) = \delta(x, z).$$

*Indication of proof.* Suppose $x$ is in $S$ and $\{a, c\}$ is in $S \times S$ and $\sigma(x, a) = \sigma(x, c)$; let $b$ be a member of $S$ such that $\sigma(a, b) = a$; then

$$\sigma(x, c) = \sigma(x, a) = \sigma(x, \sigma(a, b)) = \sigma(\sigma(x, a), b),$$

so $c = \sigma(a, b)$. Thus, the stated requirement does define $\delta$ to be a function on $\mathcal{O}$ to $S$; the asserted properties of $\delta$ follow from similarly simple arguments.

THEOREM 8.2. *There is a member* $e$ *of* $S$ *such that, for each* $x$ *in* $S$, $\sigma(e, x) = \sigma(x, e) = x$.

*Proof.* Suppose $z$ is in $S$ and $y$ is a member of $S$ such that $\{y, \delta(z, z)\}$ is in $\mathcal{O}$; then, by Theorem 8.1 (1), $\{\sigma(z, y), z\}$ is in $\mathcal{O}$; but $\{z, \sigma(z, y)\}$ is also in $\mathcal{O}$, so that $\sigma(z, y) = z$, and, therefore, $y = \delta(z, z)$. Hence, if $z$ is in $S$, then $\delta(z, z)$ is a member $e$ of $S$ such that $\{e, x\}$ is in $\mathcal{O}$ for each $x$ in $S$; $e$ is therefore independent of $z$, and $\sigma(x, e) = x$ for each $x$ in $S$. If $x$ is in $S$, then, since $x = \delta(x, \sigma(x, x))$, it follows from Theorem 8.1 (2) that

$$\sigma(e, x) = \sigma(\delta(x, x), \delta(x, \sigma(x, x))) = \delta(x, \sigma(x, x)) = x.$$

The following theorem follows almost immediately from 8.1 and 8.2, and provides means for applying the results from earlier sections to this specialized setting (as in the next section).

THEOREM 8.3. *If* $\{x, y\}$ *is a member of* $\mathcal{O}$, *then*

(1) *if* $\{s_p\}_0^n$ *is an* $\mathcal{O}$*-subdivision of* $\{e, \delta(x, y)\}$, *then* $\{\sigma(x, s_p)\}_0^n$ *is an* $\mathcal{O}$*-subdivision* $t$ *of* $\{x, y\}$ *such that* $\delta(t_{p-1}, t_p) = \delta(s_{p-1}, s_p)$ *for* $p = 1, \cdots, n$, *and—conversely—*

(2) *if* $\{t_p\}_0^n$ *is an* $\mathcal{O}$*-subdivision of* $\{x, y\}$, *then* $\{\delta(x, t_p)\}_0^n$ *is that* $\mathcal{O}$*-subdivision* $s$ *of* $\{e, \delta(x, y)\}$ *such that* $t = \{\sigma(x, s_p)\}_0^n$.

*Remark* 1. There exists a set $S$ and two functions $\sigma_1$ and $\sigma_2$ from $S \times S$ to $S$ such that all the following are true:

(i) $\{S, \sigma_1\}$ and $\{S, \sigma_2\}$ are canonically ordered semigroups,

(ii) $\sigma_1$ is symmetric and $\sigma_2$ is not symmetric, and

(iii) the linear ordering of $S$ determined by $\sigma_1$ is the linear ordering of $S$ determined by $\sigma_2$.

For example, let $S$ be the set to which $x$ belongs only in case $x$ is a complex number and either $\operatorname{Re} x = 0$ and $\operatorname{Im} x \geqq 0$ or $\operatorname{Re} x > 0$; for each $\{x, y\}$ in $S \times S$, let

$$\sigma_1(x, y) = x + y \quad \text{and} \quad \sigma_2(x, y) = x + (1 + \operatorname{Re} x)y.$$

*Remark* 2. If, in the preceding example, $S_c$ is the subset of $S$ to which $x$ belongs only in case there is an ordered pair $\{m, n\}$ of nonnegative integers such that $\operatorname{Re} x = 2^m - 1$ and $2^n(\operatorname{Im} x)$ is an integer, and $\sigma_c$ is the contraction of $\sigma_2$ to $S_c \times S_c$, $\{S_c, \sigma_c\}$ is a canonically ordered noncommutative semigroup, and there exists an order-preserving mapping from $S_c$ into the set of nonnegative rational numbers.

## 9. Representations of semigroups

In this section we suppose $\{S, \sigma\}$ is a canonically ordered semigroup, $\mathcal{O}$ is the linear ordering of $S$ determined by $\sigma$, $\delta$ is the function from $\mathcal{O}$ to $S$ determined as in Theorem 8.1, and $e$ is the member of $S$ determined as in Theorem 8.2. A function $f$, from $S$ to any ring, is *$\sigma$-additive* provided that $f(\sigma(x, y)) = f(x) + f(y)$ for all $\{x, y\}$ in $S \times S$, and is *$\sigma$-multiplicative* provided that $f(\sigma(x, y)) = f(x)f(y)$ for all $\{x, y\}$ in $S \times S$. We note that

(i) if $f$ is $\sigma$-additive on $S$, then there is an $\mathcal{O}$-additive $V$ on $S \times S$ such that $V(x, y)$ is $f(\delta(x, y))$ or 0 according as $\{x, y\}$ is in $\mathcal{O}$ or not, and

(ii) if $u$ is $\sigma$-multiplicative on $S$, there is an $\mathcal{O}$-multiplicative $W$ on $S \times S$ such that $W(x, y)$ is $u(\delta(x, y))$ or $u(e)$ according as $\{x, y\}$ is in $\mathcal{O}$ or not.

Let $\mathfrak{A}_\sigma$ denote the set of all $\sigma$-additive functions $F$ from $S$ to $N$ such that, if $x$ is in $S$, there is a number $b$ such that, if $\{s_p\}_0^n$ is an $\mathcal{O}$-subdivision of $\{e, x\}$, then $\sum_s | F(\delta) | = \sum_1^n | F(\delta(s_{p-1}, s_p)) | \leqq b$. Let $\mathfrak{M}_\sigma$ denote the set of all $\sigma$-multiplicative functions $U$ from $S$ to $N$ such that, if $x$ is in $S$, there is a number $b$ such that, if $\{s_p\}_0^n$ is an $\mathcal{O}$-subdivision of $\{e, x\}$, then[5]

$$\textstyle\sum_s | U(\delta) - 1 | = \sum_1^n | U(\delta(s_{p-1}, s_p)) - 1 | \leqq b.$$

[5] Observe that $U(e) = 1$ since $\delta(e, e) = e$; see footnote 2.

Now, all the following results are direct applications of the corresponding results from Sections 2 through 7. Supporting proofs, which we omit, are readily constructed by use of Theorem 8.3, with the observations in the first paragraph of this section.

THEOREM 9.1. *There is a reversible function $\mathcal{E}_\sigma^+$, from the class $\mathcal{A}_\sigma^+$ of all $\sigma$-additive functions from $S$ to the set of nonnegative real numbers, onto the class $\mathcal{M}_\sigma^+$ of all $\sigma$-multiplicative functions from $S$ to the set of real numbers not less than 1, such that each of the following is a necessary and sufficient condition for the member $\{f, u\}$ of $\mathcal{A}_\sigma^+ \times \mathcal{M}_\sigma^+$ to belong to $\mathcal{E}_\sigma^+$:*

(1) $u(x) = {}_e\prod^x [1 + f(\delta)]$ *for each $x$ in $S$.*

(2) $f(x) = {}_e\sum^x [u(\delta) - 1]$ *for each $x$ in $S$.*

(3) $u(x) = 1 + (\mathrm{L}) \int_e^x u \cdot df$ *for each $x$ in $S$.*

THEOREM 9.2. *If $F$ is a $\sigma$-additive function from $S$ to $N$ and $U$ is a $\sigma$-multiplicative function from $S$ to $N$, then*

(1) *$F$ belongs to $\mathcal{A}_\sigma$ only in case there exists a member $f$ of $\mathcal{A}_\sigma^+$ such that $|F(x)| \leqq f(x)$ for each $x$ in $S$.*

(2) *$U$ belongs to $\mathcal{M}_\sigma$ only in case there exists a member $u$ of $\mathcal{M}_\sigma^+$ such that $|U(x) - 1| \leqq u(x) - 1$ for each $x$ in $S$.*

THEOREM 9.3. *There is a reversible function $\mathcal{E}_\sigma$, from $\mathcal{A}_\sigma$ onto $\mathcal{M}_\sigma$, such that each of the following is a necessary and sufficient condition for the member $\{F, U\}$ of $\mathcal{A}_\sigma \times \mathcal{M}_\sigma$ to belong to $\mathcal{E}_\sigma$:*

(1) $U(x) = {}_e\prod^x [1 + F(\delta)]$ *for each $x$ in $S$.*

(2) $F(x) = {}_e\sum^x [U(\delta) - 1]$ *for each $x$ in $S$.*

(3) *There is a member $\{f, u\}$ of $\mathcal{E}_\sigma^+$ such that*

$$|U(x) - 1 - F(x)| \leqq u(x) - 1 - f(x) \quad \text{for each } x \text{ in } S.$$

(4) $U(x) = 1 + (\mathrm{L}) \int_e^x U \cdot dF$ *for each $x$ in $S$.*

THEOREM 9.4. *If $F$ belongs to $\mathcal{A}_\sigma$ and $\sigma$ is symmetric, then*

$$F(x)F(y) = F(y)F(x) \qquad \text{for all } \{x, y\} \text{ in } S \times S.$$

THEOREM 9.5. *If $\{F, U\}$ belongs to $\mathcal{E}_\sigma$, then $U$ has the convergent series expansion*

$$U(x) = \textstyle\sum_{p=0}^\infty K_p(x) \qquad \text{for all } x \text{ in } S,$$

*where*

(i) $K_0(x) = 1, \quad K_p(x) = (\mathrm{L}) \int_e^x K_{p-1} \cdot dF \ (p = 1, 2, \cdots)$, *and*

(ii) *for each $b$ in $S$ the convergence is uniform over the set of all $x$ in $S$ such that $\{x, b\}$ belongs to $\mathcal{O}$.*

THEOREM 9.6. *If* $\{F, U\}$ *is in* $\mathcal{E}_\sigma$ *and* $W = \mathcal{E}(dF)$, *then, for each* $\{x, y\}$ *in* $\mathcal{O}$,

$$W(x, y) = W(e, \delta(x, y)) = U(\delta(x, y)) \quad and \quad W(y, x) = W(\delta(x, y), e);$$

*moreover, the following two statements are equivalent*:

(1) $W(z, e)U(z) = U(z)W(z, e) = 1$ *for each* $z$ *in* $S$.

(2) ${}_e\sum^z | F(\delta)^2 | = 0$ *for each* $z$ *in* $S$.

THEOREM 9.7 (The Exponential Case). *If the ring* $N$ *is torsion-free and* $\{F, U\}$ *is a member of* $\mathcal{E}_\sigma$ *such that*

(1) $F(x)F(y) = F(y)F(x)$ *for each* $\{x, y\}$ *in* $S \times S$, *and*

(2) ${}_e\sum^z | F(\delta)^2 | = \Theta$ *for each* $z$ *in* $S$,

*then*

$$U(z) = \operatorname{Exp}\{F(z)\} \qquad \textit{for each } z \textit{ in } S.$$

*Remark* 1. In case $N$ is an algebra over the real numbers, $\{S, \sigma\}$ is the additive semigroup of nonnegative real numbers, and $A$ is a member of $N$ such that $F(z) = zA$ for each $z$ in $S$, Theorem 9.7 is contained in the Hille-Phillips [3] representation theorem for a "semigroup $U$ with bounded infinitesimal generator $A$."

*Remark* 2. By analogy with terminology in ordinary differential equations, the analysis in this section is the "constant coefficient case" of the preceding theory—at least in case $\sigma$ is symmetric.

*Remark* 3. The analysis in this section can be phrased in terms of the semigroup of translations of $S$ of the form $\sigma(x, \ )$, and in this aspect is a study of those members $\{V, W\}$ of $\mathcal{E}$ such that $V$ is fully additive and invariant under these translations:

$$V(\sigma(x, a), \sigma(x, b)) = V(a, b) \qquad \text{for all } x, a, \text{ and } b \text{ in } S.$$

## 10. Development on the real line

In this section we suppose that $S$ is the real line and $\mathcal{O}$ is the usual ordering of $S$, i.e., $\{x, z\}$ belongs to $\mathcal{O}$ only in case $x \leqq z$. We recognize $\mathcal{OB}$ as the class of functions $\phi$ from $S$ to $N$ such that, on each interval, $\phi$ is of bounded variation with respect to the norm on the ring $N$. We let $\mathcal{OB}_Q$ denote the set of all $\phi$ in $\mathcal{OB}$ such that, if $x$ is in $S$, then $d\phi(x-, x)^2 = d\phi(x, x+)^2 = 0$, and we let $\mathcal{OB}_C$ denote the set of all continuous members of $\mathcal{OB}$.[6]

Let $\mathcal{H}$ be the class of all fully multiplicative functions $W$ from $S \times S$ to $N$ such that, if $x$ is in $S$, then $W(x, x) = 1$ and each of $W(\ , x)$ and $W(x, \ )$ is in $\mathcal{OB}$. By Theorem 7.1, we recognize $\mathcal{H}$ as the class of fully multiplicative members of $\mathcal{OM}$. We let $\mathcal{H}_Q$ denote the set of all $W$ in $\mathcal{H}$ such that, if $x$ is

[6] In case there is a nonzero member $Z$ of $N$ such that $Z^2 = 0$, the class $\mathcal{OB}_Q$ properly includes $\mathcal{OB}_C$.

in $S$, then

$$W(x-, x) + W(x, x-) = W(x+, x) + W(x, x+) = 2,$$

and we let $\mathcal{H}_C$ denote the set of all $W$ in $\mathcal{H}$ such that if $x$ is in $S$ then each of $W(\ , x)$ and $W(x,\ )$ is continuous.

By Theorem 3.2 of [5], the requirement that the member $\phi$ of $\mathcal{OB}$ belong to $\mathcal{OB}_Q$ implies condition (2) of Theorem 7.3, and the requirement that the member $W$ of $\mathcal{H}$ belong to $\mathcal{H}_Q$ implies condition (3) of Corollary 3.1; the converse implications are easily established. It follows that $\mathcal{H}_Q$ is precisely the set of all $W$ of the form $\mathcal{E}(d\phi)$ for $\phi$ in $\mathcal{OB}_Q$. It follows from Theorem 3.3 that $\mathcal{H}_C$ is precisely the set of all $W$ of the form $\mathcal{E}(d\phi)$ for $\phi$ in $\mathcal{OB}_C$. We now give a brief history of these ideas.

For the case that $N$ is a finite-dimensional matrix algebra over the real or complex numbers, H. S. Wall [9], [10] has obtained the following results, using ordinary Stieltjes integrals throughout. For $\phi$ in $\mathcal{OB}_C$ and $V = d\phi$, the equations in Theorem 4.3 are found to be equivalent and have a unique solution $W$ in $\mathcal{H}_C$, the solution being provided by the series in Theorem 6.2, and providing solutions of nonhomogeneous equations as given here in Theorem 5.2. The set of all solutions $W$ for $\phi$ in $\mathcal{OB}_C$ fills up $\mathcal{H}_C$, and the $\phi$ is recovered from the $W$ by the formula

$$\phi(b) - \phi(a) = \int_a^b W(\ , r)\cdot dW(r,\ ),$$

which is independent of $r$ and is the first version of formula (ii) in Theorem 3.3 of the present paper. There were some extensions to the case of the normed algebra of continuous linear transformations in a Hilbert space. Important applications were made to continuous continued fractions and related nonlinear equations (complemented by the present author [4], [5], and extended by Neuberger [8]).

Extension of Wall's theory, to the case that $N$ is the normed algebra of continuous linear transformations in a complete normed linear space, was carried out in [4], There, the formula (i) of Theorem 3.3 was obtained for $\phi$ in $\mathcal{OB}_C$ and $V = d\phi$, and solutions were found to the nonhomogeneous equations as given in Theorems 5.1 and 5.2, with discontinuities allowed for the $K$.

For the same algebra $N$, extension of all the preceding was made [5], [6]—using Stieltjes-mean integrals—to the fully multiplicative case as summarized in Theorem 7.4, thus relaxing the continuity conditions theretofore imposed. The classes $\mathcal{OB}_Q$ and $\mathcal{H}_Q$ were found to correspond under the mapping $W = \mathcal{E}(d\phi)$. Those results are all included in the present treatment.

Further relaxation of continuity requirements was effected by T. H. Hildebrandt [2], using a version of the Lebesgue-Stieltjes integral suggested by W. H. Young [11] (also, Hildebrandt [1]). The solution space for the homogeneous equations was found to be the whole class $\mathcal{H}$. The analysis of Hildebrandt's results with respect to the present treatment needs clarification, which we now provide.

Using the notation (Y) $\int_a^b$ to denote the integral, with the convention (Y) $\int_b^a = -$ (Y) $\int_a^b$ as used by Hildebrandt, we first note certain estimates; these can be obtained as corollary results to the existence, but can also be obtained by examination of relevant sums—thus netting an existence theorem as in Lemma 4.3. This lemma also leads directly to formulas for integration-by-parts and integration-by-substitution.

LEMMA 10.1. *If each of F and G is in* $\mathcal{OB}$, $\alpha$ *is a member of* $\mathcal{OA}^+$ *such that* $|dF| \leqq \alpha$ *and* $|dG| \leqq \alpha$, *and* $x < y < z$, *then each of*

$$(\mathrm{Y}) \int_x^z F \cdot dG - \{F(x)\, dG(x, x+) + F(y)\, dG(x+, z-) + F(z)\, dG(z-, z)\}$$

*and*

$$(\mathrm{Y}) \int_x^z dF \cdot G - \{dF(x, x+)G(x) + dF(x+, z-)G(y) + dF(z-, z)G(z)\}$$

*has norm not exceeding* $\alpha(x+, z-)^2$.

We distinguish, with Hildebrandt, two subsets of $\mathcal{OB}$ as follows: the member $\phi$ of $\mathcal{OB}$ belongs to $\mathcal{OB}_1$ only in case each of $[1 - d\phi(z-, z)]^{-1}$ and $[1 - d\phi(z+, z)]^{-1}$ is in $N$ for each $z$ in $S$, and belongs to $\mathcal{OB}_2$ only in case each of $[1 - d\phi(z, z-)]^{-1}$ and $[1 - d\phi(z, z+)]^{-1}$ is in $N$ for each $z$ in $S$. For $\phi$ in the appropriate subclass of $\mathcal{OB}$, we define certain functions from $S \times S$ to $N$ as follows:

$$C_1(x, z) = \begin{cases} [1 + d\phi(x, x+)][1 + d\phi(x+, z-)][1 - d\phi(z-, z)]^{-1} & \text{if } x < z, \\ [1 + d\phi(x, x-)][1 + d\phi(x-, z+)][1 - d\phi(z+, z)]^{-1} & \text{if } x > z; \end{cases}$$

$$C_2(z, x) = \begin{cases} [1 - d\phi(z, z+)]^{-1}[1 + d\phi(z+, x-)][1 + d\phi(x-, x)] & \text{if } z < x, \\ [1 - d\phi(z, z-)]^{-1}[1 + d\phi(z-, x+)][1 + d\phi(x+, x)] & \text{if } z > x; \end{cases}$$

$$D_1(x, z) = \begin{cases} 1 + d\phi(x, z) + [1 - d\phi(z-, z)]^{-1} d\phi(z-, z)^2 & \text{if } x < z, \\ 1 + d\phi(x, z) + [1 - d\phi(z+, z)]^{-1} d\phi(z+, z)^2 & \text{if } x > z; \end{cases}$$

$$D_2(z, x) = \begin{cases} 1 + d\phi(z, z+)^2[1 - d\phi(z, z+)]^{-1} + d\phi(z, x) & \text{if } z < x, \\ 1 + d\phi(z, z-)^2[1 - d\phi(z, z-)]^{-1} + d\phi(z, x) & \text{if } z > x; \end{cases}$$

$$C_1(x, x) = D_1(x, x) = 1, \qquad V_1(x, z) = {}_x\textstyle\sum^z [D_1 - 1];$$

$$C_2(x, x) = D_2(x, x) = 1, \qquad V_2(z, x) = {}_z\textstyle\sum^x [D_2 - 1].$$

For $\phi$ in $\mathcal{OB}_1$, Hildebrandt finds the solution $U$ of the equation

$$U(z) = U(e) + (\mathrm{Y}) \int_e^z U \cdot d\phi \qquad \text{for all } z \text{ in } S$$

in the form $U(z) = U(e)\{{}_e\prod^z C_1\}$, and (for $\phi$ in $\mathcal{OB}_2$) the solution of

$$U(z) = U(e) + (\mathrm{Y}) \int_z^e d\phi \cdot U \qquad \text{for all } z \text{ in } S$$

in the form $U(z) = \{{}_z\prod^e C_2\} U(e)$.

Straightforward (but somewhat tedious) computation leads to the following three lemmas.

LEMMA 10.2. *If $\phi$ is in $\mathcal{OB}_1$, then $V_1$ is in $\mathcal{OA}$, $\mathcal{E}(V_1)$ is in $\mathcal{K}$, and, for each $\{x, z\}$ in $S \times S$, ${}_x\prod^z [1 + V_1] = {}_x\prod^z D_1 = {}_x\prod^z C_1$; if $\phi$ is in $\mathcal{OB}_2$, then $V_2$ is in $\mathcal{OA}$, $\mathcal{E}(V_2)$ is in $\mathcal{K}$, and, for each $\{z, x\}$ in $S \times S$, ${}_z\prod^x [1 + V_2] = {}_z\prod^x D_2 = {}_z\prod^x C_2$.*

LEMMA 10.3. *If $e$ is in $S$, $\phi$ is in $\mathcal{OB}_1$, $U$ is in $\mathcal{OB}$, and*

*either*
$$U(z) = U(e) + (\mathrm{Y}) \int_e^z U \cdot d\phi \qquad \text{for all } z \text{ in } S$$

*or*
$$U(z) = U(e) + (\mathrm{L}) \int_e^z U \cdot V \qquad \text{for all } z \text{ in } S,$$

*then $U(z) = U(z-)[1 - d\phi(z-, z)]^{-1} = U(z+)[1 - d\phi(z+, z)]^{-1}$ for all $z$.*

LEMMA 10.4. *If $e$ is in $S$, $\phi$ is in $\mathcal{OB}_2$, $U$ is in $\mathcal{OB}$, and*

*either*
$$U(z) = U(e) + (\mathrm{Y}) \int_z^e d\phi \cdot U \qquad \text{for all } z \text{ in } S$$

*or*
$$U(z) = U(e) + (\mathrm{R}) \int_z^e V_2 \cdot U \qquad \text{for all } z \text{ in } S,$$

*then $U(z) = [1 - d\phi(z, z-)]^{-1} U(z-) = [1 - d\phi(z, z+)]^{-1} U(z+)$ for all $z$.*

Application of these lemmas leads to two theorems which, with Theorems 4.1 and 4.2, yield Hildebrandt's principal results for the homogeneous equations. Corresponding results can then be obtained for the nonhomogeneous cases by techniques which we have employed in Section 5; for example, consideration of the matrices

$$\Phi = \begin{pmatrix} \phi & \theta \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad [1 - d\Phi]^{-1} = \begin{pmatrix} [1 - d\phi]^{-1} & [1 - d\phi]^{-1}\, d\theta \\ 0 & 1 \end{pmatrix}$$

leads to solutions $U$ in $\mathcal{OB}$ of the system

$$U(z) = U(e) + (\mathrm{Y}) \int_z^e d\phi \cdot U + \theta(e) - \theta(z)$$

for $\phi$ in $\mathcal{OB}_2$ and $\theta$ in $\mathcal{OB}$.

THEOREM 10.1. *If $e$ is in $S$ and $U$ is in $\mathcal{OB}$, then for $\phi$ in $\mathcal{OB}_1$ the following two statements are equivalent*:

$$(1)\quad U(z) = U(e) + (\mathrm{Y}) \int_e^z U\cdot d\phi \quad \textit{for all } z \textit{ in } S,$$

$$(2)\quad U(z) = U(e) + (\mathrm{L}) \int_e^z U\cdot V_1 \quad \textit{for all } z \textit{ in } S;$$

*whereas, for $\phi$ in $\mathcal{OB}_2$ the following are equivalent*:

$$(3)\quad U(z) = U(e) + (\mathrm{Y}) \int_z^e d\phi\cdot U \quad \textit{for all } z \textit{ in } S,$$

$$(4)\quad U(z) = U(e) + (\mathrm{R}) \int_z^e V_2\cdot U \quad \textit{for all } z \textit{ in } S.$$

THEOREM 10.2. *If $W$ belongs to $\mathcal{H}$, then the formulas*

$$(\mathrm{Y}) \int_x^z W(\ \ , r)\cdot dW(r,\ \ ) = d\phi_1(x, z),$$

$$(\mathrm{Y}) \int_z^x dW(\ \ , r)\cdot W(r,\ \ ) = d\phi_2(x, z)$$

*yield (independently of $r$) a member $\phi_1$ of $\mathcal{OB}_1$ and a member $\phi_2$ of $\mathcal{OB}_2$* .

*Remark.* Using the integration-by-substitution theorem for the Y-integrals, this last theorem yields, for each $W$ in $\mathcal{H}$, members $\phi_1$ of $\mathcal{OB}_1$ and $\phi_2$ of $\mathcal{OB}_2$ such that

$$W(a, b) = 1 + (\mathrm{Y}) \int_a^b W(a,\ \ )\cdot d\phi_1 = 1 + (\mathrm{Y}) \int_a^b d\phi_2\cdot W(\ \ , b)$$

for all $\{a, b\}$ in $S \times S$. Thus one sees, as indicated earlier in this section, that the solution space for these Y-integral systems is the whole class $\mathcal{H}$.

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