THE EQUIVALENCE OF TWO AREAS FOR NONPARAMETRIC DISCONTINUOUS SURFACES

BY

J. H. MICHAEL[1]

1. Introduction

Casper Goffman has made an extensive study of the area of a discontinuous nonparametric surface. He had defined this area by starting with a class of surfaces for which an area is already known, choosing a suitable metric for the class, showing that the area is lower semicontinuous on this class and satisfies a requirement that he calls property A, and then extending the area to the completion of the class by the Fréchet process. By taking different classes and different metrics he obtains various "areas", and he has shown that most of these areas are the same.

In [3] he considers several of these areas, one of which is the following. Let S be the space of polyhedral functions on the unit square $[0, 1] \times [0, 1]$, metrized by the $\mathfrak{L}_1$ metric

$$\delta(p, q) = \iint_I | p(x, y) - q(x, y) | \, dx \, dy$$

and with elementary area functional E. The completion T of S consists of all equivalence classes of summable functions, and the Fréchet process extends E to a functional Φ on T. Thus

$$\Phi(f) = \inf [\liminf_{r\to\infty} E(p_r)],$$

where the infimum is taken over all sequences $\{p_r\}$ of polyhedral functions such that $\delta(f, p) \to 0$ as $r \to \infty$. He carries out an exhaustive investigation of the functional Φ and shows that it forms a very satisfactory area.

In [3] Goffman also considers the following area. Let C be the class of continuous functions on I with metric

$$d(f, g)$$

defined as the measure of the set where $f \neq g$. He shows that the Lebesgue area functional A is lower semicontinuous on C and has property A, and hence it extends by the Fréchet process to a functional Ψ on the completion of C, which by Lusin's theorem is the set of equivalence classes of measurable functions.

It has been conjectured by Goffman that

$$\Phi(f) = \Psi(f) \tag{1}$$

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for every summable f, and it is the purpose of this paper to prove this conjecture. In fact, we prove a little more than this. We take the subclass K of C, consisting of all Lipschitz functions, let K_1 be the completion of K with respect to d, show that K_1 contains all equivalence classes of summable functions with finite Φ area, extend $A = E$ to a functional Λ on K_1 by the Fréchet process, and then prove

$$\Lambda(f) = \Phi(f) \tag{2}$$

for every $f \in K_1$. We prove the formulas (1) and (2) for functions on the unit cube in n-dimensional space.

The proof of the above formulas makes important use of some of the results of Fleming [2].

2. Preliminaries

We will denote by $\mathcal{B}$ the class of all real-valued functions f on R^n such that

(i) f vanishes outside a compact subset of R^n;

(ii) f is summable on R^n; and

(iii) for each i, the i^{th} partial derivative of f, considered in the sense of the Schwartz distribution theory, is a bounded measure μ_i.

$\mathfrak{D}$ denotes the set of all infinitely differentiable functions on R^n with compact support.

Let $\mathfrak{I}^k$ be the set of all transformations $\psi = (\psi_1, \cdots, \psi_k)$ of R^n into R^k such that $\psi_i \in \mathfrak{D}$ for $i = 1, \cdots, k$. Define

$$\| \psi \| = \sup_{x \in R^n} \| \psi(x) \|.$$

Following Fleming [2] we let, for each Borel subset E of R^n, $I(f, E)$ be the total variation on E of the vector-valued measure $(\mu_1, \cdots, \mu_n)$; i.e.,

$$I(f, E) = \inf [\sup \textstyle\sum_{i=1}^n \mu_i(\psi_i)],$$

where the infimum is taken over all open sets U containing E, and the supremum is taken over all $\psi \in \mathfrak{I}^n$ such that $\| \psi \| \leqq 1$, spt $\psi \subseteq U$.

For a fixed f, $I(f, E)$ is thus a completely additive, nonnegative bounded Borel measure.

When $E = R^n$, we write $I(f)$ for $I(f, R^n)$.

We also associate with $f \in \mathcal{B}$, the vector-valued measure

$$\nu = (m, \mu_1, \mu_2, \cdots, \mu_n),$$

where m denotes the Radon measure corresponding to ordinary n-dimensional Lebesgue measure. For each Borel set E of R^n, we let $L(f, E)$ be the total variation of ν on E; i.e.,

$$L(f, E) = \inf_U [\sup \{m(\psi) + \textstyle\sum_{i=2}^{n+1} \mu_{i-1}(\psi_i)\}],$$

where the infimum is taken over all open sets containing E, and the supremum is taken over all $\psi \in \mathfrak{I}^{n+1}$ such that $\| \psi \| \leqq 1$ and spt $\psi \subseteq U$.

For a fixed f, $L(f, E)$ is a completely additive, nonnegative, Borel measure that is finite on every bounded E.

$\mathfrak{L}_1$ denotes the set of all real-valued functions that are summable on R^n.

For each $f \in \mathfrak{L}_1$ and each positive integer r, we shall use the symbol $\mathcal{J}_r(f)$ to denote the well-known integral mean

$$\{\mathcal{J}_r(f)\}(x) = r^n \int_{x_1}^{x_1+1/r} \int_{x_2}^{x_2+1/r} \cdots \int_{x_n}^{x_n+1/r} f(\xi)\, d\xi.$$

$\mathfrak{B}$, I, L, and $\mathcal{J}$ have the following properties:

2.1. *$\mathfrak{B}$ is a vector space over the real numbers.*

2.2.
$$I(f + g, B) \leqq I(f, B) + I(g, B), \quad \textit{and}$$
$$I(\alpha f, B) = |\alpha| \, I(f, B),$$

where α is a real number, and B a Borel set.

2.3. *If U is a fixed open set of R^n, then $I(f, U)$ and $L(f, U)$ are lower semicontinuous on $\mathfrak{B}$ with respect to the $\mathfrak{L}_1$ topology.*

2.4. *$I(f, B) \leqq L(f, B)$ for every $f \in \mathfrak{B}$ and every Borel set B.*

2.5. *If $f \in \mathfrak{L}_1$, then $\{\mathcal{J}_r(f)\}(x)$ is continuous with respect to x.*

2.6. *If f is continuous, then $\mathcal{J}_r(f)$ has continuous first-order partial derivatives.*

2.7. *If $f \in \mathfrak{L}_1$ and is bounded, then $\mathcal{J}_r(f)$ is Lipschitz.*

2.8. *If f is Lipschitz, then*

$$\frac{\partial}{\partial x_i} [\mathcal{J}_r(f)] = \mathcal{J}_r \left(\frac{\partial f}{\partial x_i} \right)$$

almost everywhere.

2.9. *If $f \in \mathfrak{L}_1$, then $\mathcal{J}_r(f)$ approaches f almost everywhere.*

2.10. *If f is Lipschitz with compact support, then*

$$I[f - \mathcal{J}_r(f)] \to 0 \qquad \textit{as} \quad r \to \infty.$$

2.11. *If f is continuous with compact support, then $\mathcal{J}_r(f)$ approaches f uniformly on R^n.*

2.12. *If f, $g \in \mathfrak{B}$ and B is a Borel set, then*

$$L(f + g, B) \leqq L(f, B) + I(g, B).$$

2.13. *If f is Lipschitz with compact support and E is a Borel set, then*

(i) $$I(f, E) = \int_E \left[\sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}\right)^2\right]^{1/2} dx, \quad and$$

(ii) $$L(f, E) = \int_E \left[1 + \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}\right)^2\right]^{1/2} dx.$$

2.14. *If $f \in \mathcal{B}$, then $\mathcal{J}_r(f) \in \mathcal{B}$, and for each compact set B of R^n*

$$\limsup_{r\to\infty} I\{\mathcal{J}_r(f), B\} \leqq I(f, B), \quad and$$
$$\limsup_{r\to\infty} L\{\mathcal{J}_r(f), B\} \leqq L(f, B).$$

2.15. *If $f \in \mathcal{B}$, $N \geqq 0$ and we define,*

$$\begin{aligned} f_N(x) &= f(x) \quad \text{if} \quad -N \leqq f(x) \leqq N, \\ &= -N \quad \text{if} \quad f(x) < -N, \\ &= N \quad \text{if} \quad f(x) > N, \end{aligned}$$

then $f_N \in \mathcal{B}$, and

$$I(f_N, B) \leqq I(f, B), \qquad L(f_N, B) \leqq L(f, B)$$

for each Borel set B of R^n.

2.16 THEOREM. *If h_1, $h_2 \in \mathcal{B}$ and are bounded, h_2 is Lipschitz, B is a bounded Borel set with finite perimeter, $|h_1(x) - h_2(x)| \leqq K$ for all $x \in R^n$, and χ_B is the characteristic function of B, then $(1 - \chi_B)h_1 + \chi_B h_2 \in \mathcal{B}$, and*

$$L\{(1 - \chi_B)h_1 + \chi_B h_2, E\} \leqq L(h_1, E) + I(h_2, B \cap E) + K \cdot P(B)$$

for every Borel set E (where $P(B)$ denotes the perimeter of B).

Proof. (i) Suppose to begin with that h_1 is also Lipschitz and E is open. Using integral means we can construct a sequence $\{\chi^{(r)}\}$ of functions such that

(A) each $\chi^{(r)}$ has continuous first-order partial derivatives and $0 \leqq \chi^{(r)}(x) \leqq 1$ for all $x \in R^n$;

(B) there is a compact set containing the support of every $\chi^{(r)}$;

(C) $\chi^{(r)} \to \chi_B$ almost everywhere;

(D) $$\limsup_{r\to\infty} \int \left[\sum_{i=1}^n \left(\frac{\partial \chi^{(r)}}{\partial x_i}\right)^2\right]^{1/2} dx \leqq P(B).$$

It follows from (A), (B), and (C) that $\chi^{(r)} \to \chi_B$ in the $\mathfrak{L}_1$ topology, and hence since h_1 and h_2 are bounded,

$$(1 - \chi^{(r)})h_1 + \chi^{(r)}h_2 \to (1 - \chi_B)h_1 + \chi_B h_2$$

in the $\mathfrak{L}_1$ topology; hence by 2.3,

$$\begin{aligned} &L\{(1 - \chi_B)h_1 + \chi_B h_2, E\} \\ &\qquad \leqq \liminf_{r\to\infty} L\{(1 - \chi^{(r)})h_1 + \chi^{(r)}h_2, E\}, \end{aligned}$$

which by 2.13 (ii),

$$= \liminf_{r\to\infty} \int_E \left[1 + \sum_{i=1}^n \left\{(1 - \chi^{(r)}) \frac{\partial h_1}{\partial x_i} + \chi^{(r)} \frac{\partial h_2}{\partial x_i} + \frac{\partial \chi^{(r)}}{\partial x_i} (h_2 - h_1)\right\}^2\right]^{1/2} dx$$

$$\leqq \liminf_{r\to\infty} \left[\int_E \left[1 + \sum_{i=1}^n (1 - \chi^{(r)})^2 \left(\frac{\partial h_1}{\partial x_i}\right)^2\right]^{1/2} dx\right.$$

$$\left. + \int_E \chi^{(r)} \left[\sum_{i=1}^n \left(\frac{\partial h_2}{\partial x_i}\right)^2\right]^{1/2} dx + \int_E | h_2 - h_1 | \left[\sum_{i=1}^n \left(\frac{\partial \chi^{(r)}}{\partial x_i}\right)^2\right]^{1/2} dx\right]$$

$$\leqq \liminf_{r\to\infty} \left[L(h_1, E) + \int_E \chi^{(r)} \left[\sum_{i=1}^n \left(\frac{\partial h_2}{\partial x_i}\right)^2\right]^{1/2} dx\right.$$

$$\left. + K \int_{R^n} \left[\sum_{i=1}^n \left(\frac{\partial \chi^{(r)}}{\partial x_i}\right)^2\right]^{1/2} dx\right],$$

and by (D)

$$\leqq \liminf_{r\to\infty} \left[L(h_1, E) + \int_E \chi^{(r)} \left[\sum_{i=1}^n \left(\frac{\partial h_2}{\partial x_i}\right)^2\right]^{1/2} dx + KP(B)\right].$$

But since $\chi^{(r)} \to \chi_B$ in the $\mathfrak{L}_1$ topology, we have

$$\lim_{r\to\infty} \int_E \chi^{(r)} \left[\sum_{i=1}^n \left(\frac{\partial h_2}{\partial x_i}\right)^2\right]^{1/2} dx = \int_E \chi_B \left[\sum_{i=1}^n \left(\frac{\partial h_2}{\partial x_i}\right)^2\right]^{1/2} dx$$

$$= \int_{B\cap E} \left[\sum_{i=1}^n \left(\frac{\partial h_2}{\partial x_i}\right)^2\right]^{1/2} dx = I(h_2, B \cap E),$$

so that the proof is complete in this case.

(ii) h_1 is arbitrary, B is arbitrary, and E is open. Take $\varepsilon > 0$ or $N > 0$ according as $L\{(1 - \chi_B)h_1 + \chi_B h_2, E\}$ is finite or infinite. Let E_1 be a compact subset of E such that

$$L\{(1 - \chi_B)h_1 + \chi_B h_2, \operatorname{Int}(E_1)\} > L\{(1 - \chi_B)h_1 + \chi_B h_2, E\} - \varepsilon \quad \text{or} \quad > N.$$

Put $h_1^{(r)} = I_r(h_1)$. Then

(E) each $h_1^{(r)}$ is Lipschitz and $| h_1^{(r)}(x) - h_2(x) | \leqq K + \delta_r$ for all $x \in R^n$, where $\delta_r \to 0$ as $r \to \infty$;

(F) there is a compact set containing the support of every $h_1^{(r)}$ and the $h_1^{(r)}$'s are uniformly bounded;

(G) $h_1^{(r)} \to h_1$ almost everywhere.

By (E), (F), and (G)

$$(1 - \chi_B)h_1^{(r)} + \chi_B h_2 \to (1 - \chi_B)h_1 + \chi_B h_2$$

in the $\mathfrak{L}_1$ topology, so that by 2.3,

$$L\{(1 - \chi_B)h_1 + \chi_B h_2, \operatorname{Int}(E_1)\}$$

$$\leqq \liminf_{r\to\infty} L\{(1 - \chi_B)h_1^{(r)} + \chi_B h_2, \operatorname{Int}(E_1)\},$$

which by (i)

$$\leqq \liminf_{r\to\infty} [L\{h_1^{(r)}, \operatorname{Int}(E_1)\} + I\{h_2, B \cap \operatorname{Int}(E)\}] + K\cdot P(B)$$

$$\leqq L(h_1, E_1) + I(h_2, B \cap E_1) + K\cdot P(B) \qquad \text{by } 2.14.$$

Thus

$$L\{(1 - \chi_B)h_1 + \chi_B h_2, E\} \leqq L(h_1, E) + I(h_2, B \cap E) + K\cdot P(B).$$

(iii) h_1, h_2, B, and E are arbitrary. Let $\{U_s\}$, $\{V_s\}$ be decreasing sequences of open sets such that $B \cap E \subseteq U_s$, $E \sim B \subseteq V_s$ for all s, and

$$L(h_1, B \cap E) = \lim_{s\to\infty} L(h_1, U_s),$$

$$L(h_1, E \sim B) = \lim_{s\to\infty} L(h_1, V_s),$$

$$I(h_2, B \cap E) = \lim_{s\to\infty} I(h_2, U_s),$$

$$I(h_2, E \sim B) = \lim_{s\to\infty} I(h_2, V_s).$$

Then

$$L\{(1 - \chi_B)h_1 + \chi_B h_2, E\}$$

$$\leqq \lim_{s\to\infty} L\{(1 - \chi_B)h_1 + \chi_B h_2, U_s \cup V_s\},$$

which by (ii)

$$\leqq \lim_{s\to\infty} [L(h_1, U_s \cup V_s) + I(h_2, B \cap (U_s \cup V_s))] + K\cdot P(B)$$

$$\leqq L(h_1, E) + I(h_2, B \cap E) + \lim_{s\to\infty} I(h_2, B \cap V_s) + K\cdot P(B).$$

Since $B \cap V_s \subseteq V_s \sim (E \sim B)$, it follows that $\lim_{s\to\infty} I(h_2, B \cap V_s) = 0$, and the proof is complete.

2.17 Theorem. *If $f \in \mathfrak{B}$ and is bounded and B is a Borel set with bounded frontier and finite perimeter, then $\chi_B \cdot f \in \mathfrak{B}$.*

Proof. Let J be a closed interval such that f vanishes outside J and $\operatorname{Fr}(B) \subseteq \operatorname{Int}(J)$. Put $B_1 = J \cap ({\sim}B)$. Then B_1 is bounded, and $P(B_1) < \infty$. Put $h_1 = f$, $h_2 = 0$, whence by 2.16,

$$L\{(1 - \chi_{B_1})\cdot f, E\} \leqq L(f, E) + K\cdot P(B)$$

for every Borel set E. Since $(1 - \chi_{B_1})\cdot f = \chi_B \cdot f$, then $\chi_B \cdot f \in B$.

3. The main approximation theorems

3.1 Theorem. *If $f \in \mathfrak{B}$ and $\varepsilon > 0$, then there exists a Lipschitz function g on R^n with compact support and agreeing with f except on a set with measure less than ε.*

This theorem can be obtained from Goffman [3, Theorem 6], Saks [4, Theorem (12.2), p. 300], and Federer [1, 5.2]. However, it can also be proved directly, in the following way. The basic idea for the first part of this proof has been obtained from Federer [1, 5.2].

Proof of 3.1. It follows from 2.15 that we may assume f to be bounded. Let Z and l satisfy conditions (i), (ii), and (iii) of Lemma 2 (appearing below) with $\varepsilon = 2^{-(n+2)}$. Let $L > 0$ be such that if

$$G = \{x; l(x) \leqq L\},$$

then the set $R^n \sim G$ is bounded and has measure less than ε. Then

$$|f(x) - f(y)| \leqq 2n^{1/2}L \, \| x - y \| \tag{1}$$

for all $x, y \in G$, because, if we let J_x be the largest open cube with centre at x but not containing y, let J_y be the largest open cube with centre at y but not containing x, and let ρ be the edge-length of J_x, J_y, then $J_x \cap J_y$ contains an open cube with edge-length $\frac{1}{2}\rho$; hence

$$m(J_x \cap J_y)/m(J_x) \geqq 1/2^n. \tag{2}$$

By Lemma 2 (iii), the set

$$A\{x, J_x, l(x)\} \cup A\{y, J_y, l(y)\}$$

has measure $\leqq 2^{-(n+1)}m(J_x)$; hence by (2), the sets

$$J_x \sim A\{x, J_x, l(x)\}, \qquad J_y \sim A\{y, J_y, l(y)\}$$

have a common point z. Thus

$$|f(x) - f(z)| = l(x) \, \| x - z \| \leqq L \, \| x - z \|, \quad \text{and}$$

$$|f(y) - f(z)| \leqq L \, \| y - z \|,$$

so that

$$|f(x) - f(y)| \leqq L\{\| x - z \| + \| y - z \|\}$$

$$\leqq 2n^{1/2}L \, \| x - y \|.$$

Since, by (1), f is Lipschitz on G, there exists a Lipschitz function g on R^n with compact support and agreeing with f on G.

LEMMA 1. *Let $f \in \mathcal{B}$ and be bounded. There exists a subset W of R^n such that*

(i) *$R^n \sim W$ has zero measure, and*

(ii) *for every $\xi \in W$, every $L > 0$, and every open cube J with centre at ξ, the set*

$$A(\xi, J, L) = \{y; y \in J \quad \text{and} \quad |f(y) - f(\xi)| > L \, \| y - \xi \|\}$$

has measure $\leqq (1/L)I(f, J)$.

Proof. (A) Suppose first of all that f is Lipschitz. In this case we can prove the lemma with $W = R^n$.

In the integral

$$I(f, J) = \int_J \left[\sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \right)^2 \right]^{1/2} dx,$$

make the substitution

$$\begin{aligned} x_1 &= \xi_1 + r\cos\theta_1 , \\ x_2 &= \xi_2 + r\sin\theta_1\cos\theta_2 , \\ x_3 &= \xi_3 + r\sin\theta_1\sin\theta_2\cos\theta_3 , \\ &\vdots \\ x_{n-1} &= \xi_{n-1} + r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\cos\theta_{n-1} , \\ x_n &= \xi_n + r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\theta_{n-1} \end{aligned}$$

when $n > 1$, and $x_1 = \xi_1 + r$ when $n = 1$. Then

$$(1)\quad I(f, J) = \int_0^\pi \cdots \int_0^\pi \left[\int_{-\psi(\theta_1,\cdots,\theta_{n-1})}^{\psi(\theta_1,\cdots,\theta_{n-1})} \left[\sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}\right)^2\right]^{1/2} r^{n-1}\,dr\right] \cdot\phi(\theta_1, \cdots, \theta_{n-1})\,d\theta_1\cdots d\theta_{n-1}.$$

But since

$$\frac{\partial f}{\partial r}\,dr = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\,dx_i ,$$

we have

$$\left|\frac{\partial f}{\partial r}\right| |dr| \leqq \left[\sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}\right)^2\right]^{1/2} \left[\sum_{i=1}^n (dx_i)^2\right]^{1/2},$$

$$\left|\frac{\partial f}{\partial r}\right| \leqq \left[\sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}\right)^2\right]^{1/2},$$

so that by (1)

$$(2)\quad I(f, J) \geqq \int_0^\pi \cdots \int_0^\pi \left[\int_{-\psi(\theta_1,\cdots,\theta_{n-1})}^{\psi(\theta_1,\cdots,\theta_{n-1})} \left|\frac{\partial f}{\partial r}\right| r^{n-1}\,dr\right] \cdot\phi(\theta_1,\cdots, \theta_{n-1})\,d\theta_1\cdots d\theta_{n-1}.$$

Now let $p(\theta_1, \cdots, \theta_{n-1})$ denote the one-dimensional measure of the set
$B(\theta_1, \cdots, \theta_{n-1})$

$$= \{r;\ -\psi \leqq r \leqq \psi \ \text{ and } \ |f(r, \theta_1, \cdots, \theta_{n-1}) - f(\xi)| > L\cdot r\},$$

and let $\alpha(\theta_1, \cdots, \theta_{n-1})$, $\beta(\theta_1, \cdots, \theta_{n-1})$ denote the infimum and supremum of all $r \in B(\theta_1, \cdots, \theta_{n-1})$. Then $\beta - \alpha \geqq p$ and

$$(\beta - \alpha)\cdot L < \int_{-\psi}^{\psi} \left|\frac{\partial f}{\partial r}\right| dr;$$

hence by (2)

$$\begin{aligned} I(f, J) &\geqq \int_0^\pi \cdots \int_0^\pi \int_{-\psi}^{\psi} L\cdot pr^{n-1}\phi\,dr\,d\theta\cdots d\theta_{n-1} \\ &= L\cdot(\text{measure of } A(\xi, J, L)). \end{aligned}$$

(B) When f is arbitrary, put

$$f^{(r)} = \mathscr{I}_r(f).$$

By 2.7, each $f^{(r)}$ is Lipschitz. By 2.9, $f^{(r)} \to f$ almost everywhere, and hence $f^{(r)} \to f$ in the $\mathfrak{L}_1$ topology. By 2.3 and 2.14

$$\lim_{r\to\infty} I(f^{(r)}, J) = I(f, J) \tag{3}$$

for all open cubes J, not in a countable collection $\mathcal{J}$.

Let W be the set where $f^{(r)} \to f$. Let $\xi \in W$ and let J be an open cube with centre at ξ but not in $\mathcal{J}$. Put

$$A_r = \{y, y \in J \quad \text{and} \quad | f^{(r)}(y) - f^{(r)}(\xi) | > L \| y - \xi \|\}.$$

Then

$$A \cap W \subseteq \liminf_{r\to\infty} A_r ;$$

hence

$$\begin{aligned} m(A) &\leqq \liminf_{r\to\infty} m(A_r) \\ &\leqq \lim_{r\to\infty} (1/L) I(f^{(r)}, J) && \text{by (A)} \\ &= (1/L) I(f, J) && \text{by (3).} \end{aligned}$$

When $J \in \mathcal{J}$, we can show that $m(A) \leqq (1/L)I(f, J)$, by approximating J with an open cube $J_1 \subseteq J$ and $\notin \mathcal{J}$.

LEMMA 2. *If $f \in \mathcal{B}$ and is bounded and $\varepsilon > 0$, then there exist a subset Z of R^n and a positive, finite-valued, measurable function l on Z such that*

(i) *$l(x) \leqq 1$ outside a bounded set;*

(ii) *$R^n \sim Z$ has zero measure; and*

(iii) *for every $x \in Z$ and every open cube J with centre at x, the set*

$$A\{x, J, l(x)\}$$

has measure $\leqq \varepsilon \cdot m(J)$. ($A$ is defined in Lemma 1.)

Proof. Since

$$\mu(B) = I(f, B)$$

is a bounded, nonnegative, completely additive, Borel measure, there exists a subset Y of R^n such that $m(R^n \sim Y) = 0$ and, for all $x \in Y$,

$$\lim_{\eta\to 0+} I\{f, J(x, \eta)\}/m\{J(x, \eta)\} \tag{4}$$

exists, where $J(x, \eta)$ is the open cube with centre at x and edge-length η. Let W be a subset of R^n with the properties (i) and (ii) of Lemma 1. Put

$$Z = Y \cap W.$$

Then $R^n \sim Z$ has zero measure. Define

$$\begin{aligned} l(x) &= 1 + (1/\varepsilon) \sup_{\eta>0} I\{f, J(x, \eta)\}/m\{J(x, \eta)\} \\ &= 1 + (1/\varepsilon) \sup_{\eta} I\{f, J(x, \eta)\}/m\{J(x, \eta)\}, \end{aligned} \tag{5}$$

where in the second case the supremum is taken over a countable dense subset D of the positive reals, which does not intersect the countable set E consisting of all $\eta > 0$ for which there is an open cube J with edge-length 2η and $I\{f, \mathrm{Fr}\,(J)\} > 0$. Since there exists a sequence $f^{(r)}$ of Lipschitz functions such that

$$\lim_{r\to\infty} I\{f^{(r)}, J(x, \eta)\} = I\{f, J(x, \eta)\} \qquad \text{for all } \eta \text{ in } D,$$

and since $I\{f^{(r)}, J(x, \eta)\}$ is continuous with respect to x, it follows that $I\{f, J(x, \eta)\}$ is measurable with respect to x, and hence $l(x)$ is measurable.

By Lemma 1, we have for all $x \in Z$,

$$\begin{aligned} m[A\{x, J(x, \eta), l(x)\}] &\leqq (1/l(x))I\{f, J(x, \eta)\} \\ &\leqq (1/l(x))\varepsilon\{l(x) - 1\}m\{J(x, \eta)\} \qquad \text{by (5)} \\ &\leqq \varepsilon \cdot m\{J(x, \eta)\}. \end{aligned}$$

3.2 THEOREM. *Let $f \in \mathcal{B}$ and be bounded and Borel measurable. Let $\varepsilon > 0$. There exists a bounded Borel set B such that*

(i) $P(B) < \infty$;
(ii) *B is contained in the set $F = \{x; f(x) \neq 0\}$ and $m(F \sim B) < \varepsilon$;*
(iii) $I\{(1 - \chi_B)\cdot f\} < \varepsilon$; *and*
(iv) *for every Borel set G with $P(G) < 1 + P(B)$, one has*

$$I(\chi_{G\sim B}\cdot f) < \varepsilon.$$

Proof. For each positive integer r, let

$$F_r = \{x; |f(x)| > 1/r\}.$$

Then

$$\lim_{r\to\infty} F_r = F,$$

and hence there exists a positive integer r_1 such that

$$m(F \sim F_{r_1}) < \varepsilon. \tag{1}$$

Let

$$E_z^+ = \{x; f(x) > z\},$$
$$E_z^- = \{x; -f(x) > z\} = \{x; f(x) < -z\}.$$

By the co-area formula [2, 3.3]

$$I(f) = \int_{-\infty}^{\infty} P(E_z^+)\, dz, \quad \text{and} \quad I(-f) = \int_{-\infty}^{\infty} P(E_z^-)\, dz;$$

hence

$$\int_0^{\infty} [P(E_z^+) + P(E_z^-)]\, dz < \infty.$$

We can therefore choose a δ such that $0 < \delta < \min\,(1/r_1, \varepsilon/6)$,

$$\int_0^{\delta} [P(E_z^+) + P(E_z^-)]\, dz < \varepsilon/12, \quad \text{and}$$

$$\int_{-\delta}^{\delta} P(E_z^+)\, dz < \varepsilon/3. \tag{2}$$

There now exists a z_1 such that $0 < z_1 \leqq \delta$ and

$$\delta[P(E^+_{z_1}) + P(E^-_{z_1})] < \varepsilon/12;$$

hence

$$\delta P(E^+_{z_1} \cup E^-_{z_1}) < \varepsilon/12. \tag{3}$$

Put

$$B = E^+_{z_1} \cup E^-_{z_1} .$$

Then B is a bounded Borel set with finite perimeter. Also $B \subseteq F$ and $F_{r_1} \subseteq B$; hence by (1), $m(F \sim B) < \varepsilon$.

It remains to show that (iii) and (iv) are true. To verify (iv), let G be a Borel set such that $P(G) < 1 + P(B)$. Put

$$R_z = \{x; \chi_{G \sim B} \cdot f > z\}.$$

By 2.17, $\chi_{G\sim B} \cdot f \in \mathcal{B}$, and hence by the co-area formula

$$I(\chi_{G\sim B} \cdot f) = \int_{-z_1}^{z_1} P(R_z)\, dz.$$

But

$$R_z = \{x; f(x) > z\} \cap (G \sim B) \quad \text{or} \quad \{x; f(x) > z\} \cup \{\sim(G \sim B)\}$$

according as $z \geqq 0$ or < 0, and hence

$$P(R_z) \leqq P(E^+_z) + P(B) + P(G)$$
$$< P(E^+_z) + 2P(B) + 1,$$

so that

$$I\{\chi_{G\sim B} \cdot f\} < \int_{-\delta}^{\delta} P(E^+_z)\, dz + 4\delta P(B) + 2\delta$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \qquad \text{by (2) and (3).}$$

One obtains (iii) from (iv) by putting $G = R^n$.

3.3 Theorem. *Let $f \in \mathcal{B}$ and be bounded, and let g be a Lipschitz function on R^n with compact support. Let $\varepsilon > 0$. There exists a bounded open set G such that*

(i) Fr (G) *has zero measure*;

(ii) *the set $G \triangle \{x; f(x) \neq g(x)\}$ has measure less than ε (where $\triangle$ denotes symmetric difference)*; *and*

(iii) *$L\{g + \chi_G \cdot (f - g), E\} < L(f, E) + \varepsilon$ for every Borel set E.*

Proof. Since f is equal almost everywhere to a Borel measurable function, we can assume that f itself is Borel measurable. By 3.2, there exists a bounded Borel set B such that

(A) $P(B) < \infty$;

(B) B is contained in the set $F = \{x; f(x) - g(x) \neq 0\}$ and $m(F \sim B) < \varepsilon/2$,;

(C) $I\{(1 - \chi_B)\cdot(f - g)\} < \varepsilon/4$; and
(D) for every Borel set H with $P(H) < 1 + P(B)$, one has

$$I(\chi_{H\sim B}\cdot(f - g)) < \varepsilon/8.$$

Since g is Lipschitz, there exists a constant $M > 0$ and such that for every Borel set E of R^n

$$I(g, E) \leqq M\cdot m(E). \tag{1}$$

Let $K > 0$ be a constant such that

$$|f(x) - g(x)| \leqq K \tag{2}$$

for all $x \in R^n$.

By [2, 11.3], there exists a bounded open set G such that
(E) Fr (G) has zero measure;
(F) $m(G \triangle B) < \min(\varepsilon/4M, \varepsilon/2)$; and
(G) $P(G \triangle B) < \min(\varepsilon/4K, 1)$.

It follows from 2.16 that for every Borel set E,

$$L\{(1 - \chi_{G\triangle B})\cdot f + \chi_{G\triangle B}\cdot g, E\} \leqq L(f, E) + I(g, (G \triangle B) \cap E) + K\cdot P(G \triangle B),$$

so that by (1), (F), and (G)

$$L\{(1 - \chi_{G\triangle B})\cdot f + \chi_{G\triangle B}\cdot g, E\} \leqq L(f, E) + \varepsilon/2. \tag{3}$$

But

$$\begin{aligned} g + \chi_G(f - g) - [(1 - \chi_{G\triangle B})\cdot f + \chi_{G\triangle B}\cdot g] &= (-1 + \chi_G + \chi_{G\triangle B})\cdot(f - g) \\ &= (-1 + \chi_B + 2\chi_{G\sim B})\cdot(f - g). \end{aligned} \tag{4}$$

However since $B \cup G = B \cup (G \triangle B)$, it follows that

$$P(B \cup G) \leqq P(B) + P(G \triangle B),$$

so that by (G), $P(B \cup G) < 1 + P(B)$; hence by (D)

$$I\{\chi_{G\sim B}\cdot(f - g)\} < \varepsilon/8. \tag{5}$$

By (4), (5), and (C)

$$I[g + \chi_G\cdot(f - g) - \{(1 - \chi_{G\triangle B})\cdot f + \chi_{G\triangle B}\cdot g\}] < \varepsilon/2. \tag{6}$$

It now follows from (3), (6), and 2.12 that

$$L\{g + \chi_G\cdot(f - g), E\} < L(f, E) + \varepsilon.$$

3.4 THEOREM. *If $f \in \mathcal{B}$, B is a compact set, and $\varepsilon > 0$, then there exists a Lipschitz function g on R^n with compact support and such that*

(i) $m\{x; f(x) \neq g(x)\} < \varepsilon$; *and*
(ii) $L(g, B) \leqq L(f, B) + \varepsilon$.

Proof. (a) Assume to begin with that f is bounded. By 3.1, there exists a Lipschitz function g_1 on R^n with compact support and such that

$$m\{x; f(x) \neq g_1(x)\} < \varepsilon/4.$$

By 3.3, there exists a bounded open set G such that
(A) Fr (G) has zero measure;
(B) the set $G \triangle \{x; f(x) \neq g_1(x)\}$ has measure less than $\varepsilon/4$; and
(C) $L\{g_1 + \chi_G(f - g_1), B\} \leqq L(f, B) + \varepsilon/2$.

Then $m(G) < \frac{1}{2}\varepsilon$. Choose a positive integer r, such that for all $r \geqq r_1$, the set where $\mathcal{J}_r\{\chi_G \cdot (f - g_1)\}$ is nonzero has measure less than $\frac{3}{4}\varepsilon$.

Now by 2.14

$$L\{g_1 + \chi_G(f - g_1), B\}$$
$$\geqq \limsup_{r\to\infty} L[\mathcal{J}_r(g_1) + \mathcal{J}_r\{\chi_G \cdot (f - g_1)\}, B],$$

and by 2.12

$$\geqq \limsup_{r\to\infty} [L[g_1 + \mathcal{J}_r\{\chi_G \cdot (f - g_1)\}, B] - I\{g_1 - \mathcal{J}_r(g_1)\}],$$

which by 2.10

$$\geqq \limsup_{r\to\infty} L[g_1 + \mathcal{J}_r\{\chi_G \cdot (f - g_1)\}, B].$$

Hence we can choose a positive integer $r_2 \geqq r_1$ and such that if we put

$$g = g_1 + \mathcal{J}_{r_2}\{\chi_G \cdot (f - g_1)\},$$

we have

$$L(g, B) \leqq L\{g_1 + \chi_G \cdot (f - g_1), B\} + \varepsilon/2,$$

so that by (C), $L(g, B) \leqq L(f, B) + \varepsilon$. Also

$$m\{x; f(x) \neq g(x)\} < \varepsilon,$$

and it follows from 2.7 that g is Lipschitz.

(b) Suppose now that f is arbitrary. Choose a positive number N such that if we define

$$\begin{aligned} f_N(x) &= f(x) \quad \text{if} \quad |f(x)| \leqq N, \\ &= N \quad \text{if} \quad f(x) > N, \\ &= -N \quad \text{if} \quad f(x) < -N, \end{aligned}$$

we have

$$(1) \qquad m\{x; f(x) \neq f_N(x)\} < \varepsilon/2.$$

By (a) there exists a Lipschitz function g on R^n with compact support and such that
(D) $m\{x; f_N(x) \neq g(x)\} < \varepsilon/2$; and
(E) $L(g, B) \leqq L(f_N, B) + \varepsilon$.

By (E) and 2.15,

$$L(g, B) \leqq L(f, B) + \varepsilon,$$

and by (1) and (D),

$$m\{x; f(x) \neq g(x)\} < \varepsilon.$$

4. The equality of the three areas

Let Q denote the unit cube in R^n. Associated with each summable function f on Q is the Goffman area $\Phi(f)$, which we discussed in the introduction and which is given by

$$(1) \qquad \Phi(f) = \inf [\liminf_{r\to\infty} E(f^{(r)})],$$

where the infimum is taken over all sequences $\{f^{(r)}\}$ of polyhedral functions on Q that converge to f in the $\mathfrak{L}_1$ topology, and where E denotes elementary area.

Also, there is associated with each measurable f, the Goffman area

$$(2) \qquad \Psi(f) = \inf [\liminf_{r\to\infty} A(g^{(r)})],$$

where this time the infimum is taken over all sequences $\{g^{(r)}\}$ of continuous functions on Q, converging to f with respect to the metric

$$d(g^{(r)}, f) = m\{x; x \in Q \quad \text{and} \quad g^{(r)}(x) \neq f(x)\}.$$

Now let K_1 denote the class of all equivalence classes of those functions f on Q such that for every $\varepsilon > 0$ there exists a Lipschitz function g on Q with $d(g, f) < \varepsilon$. For each $f \in K_1$ we can define an area

$$(3) \qquad \Lambda(f) = \inf [\liminf_{r\to\infty} A(g^{(r)})] = \inf [\liminf_{r\to\infty} E(g^{(r)})],$$

where the infimum is taken over all sequences $\{g^{(r)}\}$ of Lipschitz functions such that $d(g^{(r)}, f) \to 0$ as $r \to \infty$.

We are going to prove

(i) for every summable f on Q

$$\Phi(f) = \Psi(f),$$

(ii) if f is summable and $\Phi(f) < \infty$, then $f \in K_1$, and

(iii) for every summable $f \in K_1$

$$\Lambda(f) = \Phi(f).$$

Because we are going to prove (i), (ii), and (iii), it is not necessary for us to verify that Λ is an extension of the Lebesgue area functional.

Evidently

4.1. $\Phi(f) \leqq \Lambda(f)$ *for every* $f \in K_1$.

4.2 THEOREM. $\Phi(f) \leqq \Psi(f)$ *for every summable* f.

Proof. We can assume $\Psi(f)$ to be finite. Take a sequence $\{f^{(r)}\}$ of continuous functions such that

$$(1) \qquad A(f^{(r)}) \to \Psi(f)$$

as $r \to \infty$ and the set

$$\{x; f^{(r)}(x) \neq f(x)\}$$

has measure less than 2^{-r}. For each r we can now choose a polyhedral function $g^{(r)}$ on Q such that

$$\sup_{x \in Q} | f^{(r)}(x) - g^{(r)}(x) | < 1/r, \quad \text{and} \tag{2}$$

$$| A(f^{(r)}) - E(g^{(r)}) | < 1/r. \tag{3}$$

For each positive integer s define

$$\begin{aligned} g_s^{(r)}(x) &= g^{(r)}(x) && \text{if} \quad | g^{(r)}(x) | \leqq s, \\ &= s && \text{if} \quad g^{(r)}(x) > s, \\ &= -s && \text{if} \quad g^{(r)}(x) < -s, \end{aligned}$$

and similarly define $f_s^{(r)}, f_s$. Now $f_s^{(r)} \to f_s$ almost everywhere as $r \to \infty$, because for each r_1 the set

$$Q \sim \{x; f_s^{(r)}(x) = f_s(x) \quad \text{for all} \quad r \geqq r_1\}$$

has measure less than $\sum_{r=r_1}^{\infty} 2^{-r} = 2^{-r_1+1}$. By bounded convergence $f_s^{(r)} \to f_s$ in the $\mathfrak{L}_1$ topology. Also $f_s \to f$ in the $\mathfrak{L}_1$ topology. Hence there exists a sequence of positive integers $\{r_s\} \to \infty$ and such that $f^{(r_s)} \to f$ in the $\mathfrak{L}_1$ topology. Then

$$g^{(r_s)} \to f$$

in the $\mathfrak{L}_1$ topology. But $g_s^{(r_s)}$ is polyhedral, and

$$E(g_s^{(r_s)}) \leqq E(g^{(r_s)});$$

hence

$$\begin{aligned} \Phi(f) &\leqq \liminf_{s \to \infty} E(g_s^{(r_s)}) \\ &\leqq \liminf_{s \to \infty} E(g^{(r_s)}) \\ &\leqq \Psi(f) && \text{by (1) and (3).} \end{aligned}$$

4.3 Theorem. *If f is summable and $\Phi(f)$ is finite, then the function f defined by*

$$\begin{aligned} f_1(x) &= f(x) && \textit{if} \quad x \in Q, \\ &= 0 && \textit{if} \quad x \notin Q, \end{aligned}$$

belongs to the class $\mathfrak{B}$.

Proof. (i) Suppose first of all that f is Lipschitz on Q. Take a $\phi \in \mathfrak{Z}^n$ such that $\| \phi \| \leqq 1$. By Fubini's theorem,

$$- \sum_{i=1}^{n} \int_{R^n} f_1(x) \frac{\partial \phi_i}{\partial x_i} dx$$

$$= - \sum_{i=1}^{n} \int_{Q'} \left[\int_0^1 f(x) \frac{\partial \phi_i}{\partial x_i} dx_i \right] dx_1 \cdots dx_{i-1}\, dx_{i+1} \cdots dx_n$$

(where Q' denotes the unit cube in R^{n-1}) and, integrating by parts,

$$= -\sum_{i=1}^{n} \int_{Q'} \left\{ \left[f(x)\phi_i(x) \right]_{x_i=0}^{x_i=1} - \int_0^1 \frac{\partial f}{\partial x_i} \phi_i \, dx_i \right\} dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n$$

$$(1) \qquad \leqq \sum_{i=1}^{n} \int_{Q'} \left[(-1)^{x_i+1} | f(x) | \right]_{x_i=0}^{x_i=1} dx_1 \cdots dx_{i-1} \, dx_{i+1} \cdots dx_n + \int_Q \left[\sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \right)^2 \right]^{1/2} dx.$$

Put

$$\eta(t) = f(x_1, \cdots, x_{i-1}, t, x_{i+1}, \cdots, x_n).$$

Then there exists a point $u \in [0, 1]$ such that

$$\int_0^1 \eta(t) \, dt = \eta(u),$$

and since

$$| \eta(1) - \eta(u) | = \left| \int_u^1 \eta'(t) \, dt \right| \leqq \int_u^1 | \eta'(t) | \, dt,$$

and similarly

$$| \eta(u) - \eta(0) | \leqq \int_0^u | \eta'(t) | \, dt,$$

it follows that

$$| \eta(1) | + | \eta(0) | \leqq 2 \int_0^1 | \eta(t) | \, dt + \int_0^1 | \eta'(t) | \, dt.$$

Therefore by (1) and Fubini's theorem,

$$-\sum_{i=1}^{n} \int_{R^n} f_1(x) \frac{\partial \phi_i}{\partial x_i} dx \leqq 2n \int_Q | f(x) | \, dx + \sum_{i=1}^{n} \int_Q \left| \frac{\partial f}{\partial x_i} \right| dx + \int_Q \left[\sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \right)^2 \right]^{1/2} dx.$$

Hence

$$(2) \qquad I(f_1) \leqq 2n \int_Q |f(x)| \, dx + (n+1) \int_Q \left[\sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \right)^2 \right]^{1/2} dx,$$

so that $f_1 \in \mathfrak{B}$.

(ii) f is arbitrary. Let $\{f^{(r)}\}$ be a sequence of polyhedral functions converging $\mathfrak{L}_1$ to f and such that

$$\lim_{r \to \infty} \int_Q \left[1 + \sum_{i=1}^{n} \left(\frac{\partial f^{(r)}}{\partial x_i} \right)^2 \right]^{1/2} dx = \Phi(f).$$

Let

$$\begin{aligned} f_1^{(r)}(x) &= f^{(r)}(x) \quad \text{if} \quad x \in Q, \\ &= 0 \qquad\quad\ \text{if} \quad x \notin Q. \end{aligned}$$

Then $f_1^{(r)} \to f_1$ in the $\mathcal{L}_1$ topology, and hence by 2.3,

$$\begin{aligned} I(f_1) &\leqq \liminf_{r\to\infty} I(f_1^{(r)}) \\ &\leqq 2n \int_Q |f(x)|\,dx + (n+1)\Phi(f) \qquad \text{by (2).} \end{aligned}$$

Thus $I(f_1) < \infty$, so that $f_1 \in \mathcal{B}$.

4.4 THEOREM. *If f is summable and $\Phi(f) < \infty$, then $f \in K_1$ and*

$$\Lambda(f) \leqq \Phi(f).$$

Proof. Define

$$\begin{aligned} f_1(x) &= f(x) \quad \text{if} \quad x \in Q, \\ &= 0 \quad \text{if} \quad x \notin Q. \end{aligned}$$

By 4.3, $f_1 \in \mathcal{B}$.

Let $\{f^{(r)}\}$ be a sequence of polyhedral functions on Q that converge $\mathcal{L}_1$ to f and are such that

$$\Phi(f) = \lim_{r\to\infty} E(f^{(r)}). \tag{1}$$

Define

$$\begin{aligned} f_1^{(r)}(x) &= f^{(r)}(x) \quad \text{if} \quad x \in Q, \\ &= 0 \quad \text{if} \quad x \notin Q. \end{aligned}$$

It follows from 2.9 and 2.14 that we can choose for each positive integer r a positive integer s_r such that

$$\int_{R^n} | \mathcal{J}_{s_r}(f_1^{(r)}) - f_1^{(r)} |\,dx < 1/r, \quad \text{and} \tag{2}$$

$$L\{\mathcal{J}_{s_r}(f_1^{(r)}), \text{Int}(Q)\} < L(f_1^{(r)}, Q) + 1/r;$$

hence

$$L\{\mathcal{J}_{s_r}(f_1^{(r)}), \text{Int}(Q)\} < E(f^{(r)}) + 1/r. \tag{3}$$

By (2), $\mathcal{J}_{s_r}(f_1^{(r)}) \to f_1$ in the $\mathcal{L}_1$ topology, so that by 2.3,

$$\begin{aligned} L\{f_1, \text{Int}(Q)\} &\leqq \liminf_{r\to\infty} L\{\mathcal{J}_{s_r}(f_1^{(r)}), \text{Int}(Q)\} \\ &\leqq \liminf_{r\to\infty} E(f^{(r)}) \qquad \text{by (3);} \end{aligned}$$

hence by (1)

$$L\{f_1, \text{Int}(Q)\} \leqq \Phi(f).$$

The theorem now follows immediately from 4.5.

4.5 LEMMA. *Let $f \in \mathcal{B}$ and $\varepsilon > 0$. There exists a Lipschitz function g on Q such that the set $\{x;\ x \in Q$ and $f(x) \neq g(x)\}$ has measure less than ε and $E(g) < L\{f, \text{Int}(Q)\} + \varepsilon$.*

Proof. (i) Suppose first that $|f(x)| \leqq K$ for all $x \in R^n$. Let

$$a = (\tfrac{1}{2}, \tfrac{1}{2}, \cdots, \tfrac{1}{2}),$$

and for each $t \in [0, \frac{1}{2}]$, put

$$Q_t = \{2t(x - a) + a; x \in Q\}.$$

Let D be the set of all $t \in [0, \frac{1}{2}]$ for which $L\{f, \mathrm{Fr}\,(Q_t)\} = 0$. The complement of D in $[0, \frac{1}{2}]$ will be countable. Let $t_0 \in D$ be such that $t_0 > 0$ and

(1) $$m(Q \sim Q_{t_0}) < \tfrac{1}{2}\varepsilon, \quad \text{and}$$

(2) $$L\{f, \mathrm{Int}\,(Q) \sim Q_{t_0}\} < n^{-1}\left(\frac{\sqrt{n}}{4t_0} + 1\right)^{-1}\left(\frac{\sqrt{n}}{2t_0} + 3\right)^{-n}\frac{\varepsilon}{4}.$$

Let $t_1 \in D$ be such that $t_0 < t_1 < \frac{1}{2}$ and $\frac{1}{2} - t_1 > t_1 - t_0$.

By 3.4, there exists for each r a Lipschitz function $g^{(r)}$ on R^n with compact support and such that

$$m\{x; f(x) \neq g^{(r)}(x)\} < 1/r, \quad \text{and}$$

(3) $$L(g^{(r)}, Q_{t_1}) < L(f, Q_{t_1}) + 1/r.$$

We can assume that $| g^{(r)}(x) | \leqq K$ for all x and the $g^{(r)}$'s have uniformly bounded supports. Then $g^{(r)} \to f$ in the $\mathfrak{L}_1$ topology, so that by 2.3,

$$\liminf_{r\to\infty} L(g^{(r)}, Q_{t_0}) \geqq L(f, Q_{t_0}), \quad \text{and}$$

$$\liminf_{r\to\infty} L(g^{(r)}, Q_{t_1}) \geqq L(f, Q_{t_1}).$$

But by (3)

$$\limsup_{r\to\infty} L(g^{(r)}, Q_{t_1}) \leqq L(f, Q_{t_1}),$$

so that

$$\limsup L(g^{(r)}, Q_{t_1} \sim Q_{t_0}) \leqq L(f, Q_{t_1} \sim Q_{t_0}).$$

Hence one can choose a large r, put $h = g^{(r)}$, and obtain

(4) $$m\{x; f(x) \neq h(x)\} < \tfrac{1}{2}\varepsilon,$$

(5) $$L(h, Q_{t_1}) < L(f, Q_{t_1}) + \tfrac{1}{2}\varepsilon, \quad \text{and}$$

(6) $$L(h, Q_{t_1} \sim Q_{t_0}) < n^{-1}\left(\frac{\sqrt{n}}{4t_0} + 1\right)^{-1}\left(\frac{\sqrt{n}}{2t_0} + 3\right)^{-n}\frac{\varepsilon}{2}.$$

For each $x \in Q \sim Q_{t_0}$ define $\theta(x)$ by

$$x \in \mathrm{Fr}\,\{Q_{\theta(x)}\}.$$

Then

$$| \theta(x) - \theta(x') | \leqq \| x - x' \|$$

for all $x, x' \in Q \sim Q_{t_0}$. For $x \in Q \sim Q_{t_0}$, define

$$p(x) = \left[\frac{t_0(\frac{1}{2} - t_1)}{\theta(x)(\frac{1}{2} - t_0)} + \frac{t_1 - t_0}{\frac{1}{2} - t_0}\right](x - a) + a.$$

Then p maps $Q \sim Q_{t_0}$ onto $Q_{t_1} \sim Q_{t_0}$. Also

$$p^{-1}(y) = \left[\frac{t_0(t_1 - \frac{1}{2})}{\theta(y)(t_1 - t_0)} + \frac{\frac{1}{2} - t_0}{t_1 - t_0}\right](y - a) + a.$$

Then

$$\| p(x) - p(x') \| \leqq \left(\frac{\sqrt{n}}{4t_0} + 1\right) \| x - x' \| \tag{7}$$

for all $x, x' \in Q \sim Q_{t_0}$, and

$$\| p^{-1}(y) - p^{-1}(y') \| \leqq \left(\frac{\sqrt{n}}{2t_0} + 3\right) \| y - y' \| \tag{8}$$

for all $y, y' \in Q_{t_1} \sim Q_{t_0}$. Define

$$\begin{aligned} g(x) &= h(x) \qquad \text{if} \quad x \in Q_{t_0}, \\ &= h\{p(x)\} \quad \text{if} \quad x \in Q \sim Q_{t_0}. \end{aligned}$$

Then g is Lipschitz on Q, and by (1) and (4), the set

$$\{x; x \in Q \quad \text{and} \quad f(x) \neq g(x)\}$$

has measure less than ε. For almost all $x \in Q \sim Q_{t_0}$, one has

$$\begin{aligned} \sum_{i=1}^{n} \left(\frac{\partial g}{\partial x_i}\right)^2 &= \sum_{i=1}^{n} \left[\sum_{j=1}^{n} \left(\frac{\partial h}{\partial y_j}\right)_{y=p(x)} \frac{\partial p_i}{\partial x_i}\right]^2 \\ &\leqq \sum_{i=1}^{n} \left[\sum_{j=1}^{n} \left\{\left(\frac{\partial h}{\partial y_j}\right)_{y=p(x)}\right\}^2\right] \left[\sum_{j=1}^{n} \left(\frac{\partial p_j}{\partial x_i}\right)^2\right] \\ &\leqq n^2 \left(\frac{\sqrt{n}}{4t_0} + 1\right)^2 \sum_{j=1}^{n} \left\{\left(\frac{\partial h}{\partial y_j}\right)_{y=p(x)}\right\}^2, \end{aligned}$$

so that by (7),

$$\begin{aligned} &\int_{Q \sim Q_{t_0}} \left[1 + \sum_{i=1}^{n} \left(\frac{\partial g}{\partial x_i}\right)^2\right]^{1/2} dx \\ &\qquad \leqq n \left(\frac{\sqrt{n}}{4t_0} + 1\right) \int_{Q \sim Q_{t_0}} \left[1 + \sum_{j=1}^{n} \left\{\left(\frac{\partial h}{\partial y_j}\right)_{y=p(x)}\right\}^2\right]^{1/2} dx \end{aligned}$$

and by (8),

$$\begin{aligned} &\leqq n \left(\frac{\sqrt{n}}{4t_0} + 1\right) \left(\frac{\sqrt{n}}{2t_0} + 3\right)^n \int_{Q \sim Q_{t_0}} \left[1 + \sum_{j=1}^{n} \left\{\left(\frac{\partial h}{\partial y_j}\right)_{y=p(x)}\right\}^2\right]^{1/2} \frac{\partial(p)}{\partial(x)} dx \\ &= n \left(\frac{\sqrt{n}}{4t_0} + 1\right) \left(\frac{\sqrt{n}}{2t_0} + 3\right)^n \int_{Q_{t_1} \sim Q_{t_0}} \left[1 + \sum_{j=1}^{n} \left(\frac{\partial h}{\partial y_j}\right)^2\right]^{1/2} dy \end{aligned}$$

which by (6), $< \varepsilon/2$. Then

$$E(g) < L(h, Q_{t_0}) + \varepsilon/2 \leqq L(h, Q_{t_1}) + \varepsilon/2,$$

and by (5), $E(g) < L\{f, \text{Int}\,(Q)\} + \varepsilon$.

(ii) When f is unbounded, the lemma follows from (i) and 2.15.

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University of Adelaide
Adelaide, South Australia