ON AMENABLE SEMIGROUPS WITH A FINITE-DIMENSIONAL SET OF INVARIANT MEANS II[1]

BY

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1. Introduction

This paper is a sequel to [4], and we shall freely use the notation of [4].

Let $G = \{e_1, \cdots, e_n\}$ be a finite semigroup in which the relations $e_i e_j = e_j$ hold for $1 \leqq i, j \leqq n$. Let $A_i = \{e_i\}$, $i = 1, \cdots, n$; then $A_i \subset G$ are evidently finite groups containing identity element only. G possesses obviously the following properties:

(1) G is a semigroup in which left cancellation holds.

(2) G is the union of n disjoint finite groups $A_1, \cdots, A_n$ each isomorphic to the other, and each of which is a left ideal in G. (Since left cancellation holds in G, every A_i is a (l.i.l.c.); see [4, Section 2].)

Now if a semigroup possesses the properties (1) and (2), it does not follow necessarily that its finite subgroups $A_1, \cdots, A_n$ are all isomorphic to the trivial group containing the identity element only. In fact for any finite group A and integer $n > 0$ a semigroup possessing the properties (1) and (2) is constructed in [6, p. 1081], such that $A_1, \cdots, A_n$ are all isomorphic to A. Applying Theorem 3.1 of [4] we get that dim $Ml(G) = n$ for semigroups which have properties (1) and (2) (dim $Ml(G) = n$ means "the linear manifold spanned by the left invariant means is n-dimensional").

It is the main purpose of this paper to prove

THEOREM E. *If G is a semigroup with left cancellation, then* dim $Ml(G) = n$, $0 < n < \infty$, *if and only if G is finite and is the union of n finite disjoint groups $A_1, \cdots, A_n$, each isomorphic to the other, and each of which is a minimal left ideal in G.*

Taking $n = 1$ one gets: If G is a semigroup with left cancellation, then dim $Ml(G) = 1$ (in other words, G has a unique left invariant mean) if and only if G is a finite group.

It is interesting to note that left cancellation and dim $Ml(G) = n$, $0 < n < \infty$, which are not, seemingly, strict conditions imposed on G, determine to such a great extent the algebraic structure of G and mostly, that they imply that G has to be finite.

Theorem E is a generalization of Theorem A of the author (see [4, Section 1])

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[1] This paper contains partial results from the Ph.D. thesis of the author under the advisorship of Professor A. Dvoretzky and Dr. H. Kesten. I wish to thank both and especially Dr. H. Kesten for the real interest he took in my work and for his helpful advice.

to semigroups in which left cancellation holds. For groups, bearing in mind that they do not contain nontrivial ideals, one gets that dim $Ml(G) = 1$ or ∞ (see Theorem B in [4, Section 1]).

The author was not able till now to drop entirely the countability condition imposed on G in Theorem A [4, Section 1].

In order to prove Theorem E we prove some theorems which are interesting for their own sake:

THEOREM E_1. *If G is a left amenable semigroup, then for every countable subsemigroup $G_0 \subset G$ there exists a countable subsemigroup G_0', $G_0 \subset G_0' \subset G$, which is left amenable.*

As is well known (see [1, p. 516]) not every subsemigroup of a left amenable semigroup is left amenable.

THEOREM E_2. *If G is a left amenable semigroup with left cancellation and $G_0 \subset G$ is a left amenable subsemigroup, then there exists a linear isometry from the subspace of left invariant elements of $m(G_0)^*$ into the subspace of left invariant elements of $m(G)^*$ which maps left invariant means into left invariant means.*

For groups this theorem is a result of Day [1, p. 533]. Since the decomposition of a semigroup into left cosets with respect to a subsemigroup is much more complicated than the same decomposition for a group with respect to a subgroup, the above generalization is not trivial.

2. The main theorem

In order to prove the main theorem of this paper we need the following results:

THEOREM E_1. *Let G be a left amenable semigroup, and $G_0 \subset G$ a countable subsemigroup. Then there exists a countable left amenable[2] subsemigroup G_0' such that $G_0 \subset G_0' \subset G$.*

Proof. By [1, p. 524] there is a net of finite means in $m(G)^*$, $\{\varphi_\alpha\}$, such that $\lim_\alpha \| L_g \varphi_\alpha - \varphi_\alpha \| = 0$ for each g in G. Thus for $\varepsilon > 0$ and for every finite set $\{a_1, \cdots, a_k\} \subset G$ there is an α_0, which depends on this finite set, such that $\| L_{a_i} \varphi_\alpha - \varphi_\alpha \| < \varepsilon$ for $1 \leqq i \leqq k$, if $\alpha \geqq \alpha_0$. Let now $G_0 = \{g_1, g_2, g_3, \cdots\}$, and for a finite mean $\varphi_\alpha \in m(G)^*$ put

$$\tau(\varphi_\alpha) = \{g; g \in G, \varphi_\alpha(1_g) > 0\}$$

($1_g \in m(G)$; see [4, Section 2]). Since the φ_α are finite means, $\tau(\varphi_\alpha) \subset G$ are finite sets for each α. Let α_1 be such that $\| L_{g_1} \varphi_{\alpha_1} - \varphi_{\alpha_1} \| < 1$, and let α_2 be such that $\| L_a \varphi_{\alpha_2} - \varphi_{\alpha_2} \| < \frac{1}{2}$ for $a \in \{g_1, g_2\} \cup \tau(\varphi_{\alpha_1})$. If $\varphi_{\alpha_{k-1}}$

[2] The theorem remains true if "left" is replaced by "right" and also if "left amenable" is replaced by "amenable".

was chosen so that

$$\| L_a \varphi_{\alpha_{k-1}} - \varphi_{\alpha_{k-1}} \| < 1/(k-1) \quad \text{for} \quad a \in \{g_1, \cdots, g_{k-1}\} \cup [\bigcup_{i=1}^{k-2} \tau(\varphi_{\alpha_i})],$$

then let α_k be such that

$$\| L_a \varphi_{\alpha_k} - \varphi_{\alpha_k} \| < 1/k \quad \text{for} \quad a \in \{g_1, \cdots, g_k\} \cup [\bigcup_{i=1}^{k-1} \tau(\varphi_{\alpha_i})].$$

We choose in this way a sequence of finite means $\{\varphi_{\alpha_k}\}$.

Let $A = G_0 \cup [\bigcup_{i=1}^{\infty} \tau(\varphi_{\alpha_i})]$, and let G_0' be the subsemigroup of G generated by A. Our sequence φ_{α_k} satisfies $\lim_{k\to\infty} \| L_a \varphi_{\alpha_k} - \varphi_{\alpha_k} \| = 0$ for each a in A, since if $a \in A$, then either $a = g_k$ for some k, and then

$$\| L_a \varphi_{\alpha_n} - \varphi_{\alpha_n} \| < 1/n \quad \text{for} \quad n \geqq k,$$

or $a \in \tau(\varphi_{\alpha_k})$ for some k, and then

$$\| L_a \varphi_{\alpha_n} - \varphi_{\alpha_n} \| < 1/n \quad \text{for} \quad n \geqq k+1.$$

If φ is an arbitrary element of the sequence $\{\varphi_{\alpha_n}\}$, then $\tau(\varphi) \subset G_0'$, and therefore $\varphi = \sum_{i=1}^{n} \beta_i Q1_{g_i}$ (where[3] g_i are in G_0' and $(Q1_{g_i})f = f(g_i)$ for f in $m(G)$). But

$$L_a(Q1_{g_i})f = (Q1_{g_i})(l_a f) = (l_a f)(g_i) = f(ag_i) = Q(1_{ag_i})(f),$$

which implies

$$L_a \varphi = \sum_{i=1}^{n} \beta_i L_a(Q1_{g_i}) = \sum_{i=1}^{n} \beta_i(Q1_{ag_i}).$$

Let now[4] $\varphi' = \sum_{i=1}^{n} \beta_i Q'1_{g_i}$, where Q' is the natural embedding of $l_1(G_0')$ into $m(G_0')^*$ (now 1_{g_i} is in $l_1(G_0')$), and for a in G_0' let l_a' be the left translation operator of $m(G_0')$, by the element a. If $L_a' = (l_a')^*$, then as above

$$L_a' \varphi' = \sum_{i=1}^{n} \beta_i Q'1_{ag_i};$$

and therefore we can write for a in G_0'

$$\begin{aligned} \| L_a \varphi - \varphi \| &= \sup_{f \in m(G), \|f\| \leqq 1} \left| \sum \beta_i Q(1_{ag_i} - 1_{g_i})f \right| \\ &= \sup_{f \in m(G), \|f\| \leqq 1} \left| \sum \beta_i(f(ag_i) - f(g_i)) \right| \\ &= \sup_{f \in m(G_0'), \|f\| \leqq 1} \left| \sum \beta_i(f(ag_i) - f(g_i)) \right| \\ &= \sup_{f \in m(G_0'), \|f\| \leqq 1} \left| \sum \beta_i Q'(1_{ag_i} - 1_{g_i})f \right| \\ &= \| L_a' \varphi' - \varphi' \|. \end{aligned}$$

The third equality is true since a and g_i for $i = 1, \cdots, n$ are in G_0'. The last norm is of $m(G_0')^*$. We can thus write for each a in A

$$\| L_a' \varphi_{\alpha_n}' - \varphi_{\alpha_n}' \| = \| L_a \varphi_{\alpha_n} - \varphi_{\alpha_n} \|,$$

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[3] From now on the g_i's stand for elements of G_0' and not necessarily of G_0.

[4] For the above chosen $\varphi \in \{\varphi_{\alpha_n}\} \subset Ql_1(G)$ we define now a $\varphi' \in Ql_1(G_0')$. Thus to each $\varphi_{\alpha_n} \in Ql_1(G)$ we define a $\varphi_{\alpha_n}' \in Ql_1(G_0')$.

and therefore

$$\lim_{n\to\infty} \| L'_a \varphi'_{\alpha_n} - \varphi'_{\alpha_n} \| = 0 \qquad \text{for each } a \text{ in } A.$$

Let now φ'_0 be a w^*-cluster point of φ'_{α_n} (in $m(G'_0)^*$).

Then φ'_0 is a mean, as a w^*-cluster point of means, (see [1, p. 513, (C)]), and moreover $L'_a \varphi'_0 = \varphi'_0$ for each a in A. (See the inequalities of Remark 5.1 of [4].) The end of the same Remark 5.1 of [4] implies that $L'_a \varphi'_0 = \varphi'_0$ for each a in G'_0, which finishes the proof of our theorem.

Remark. It is known (see [1, p. 516, (F)]) that if every finitely generated subsemigroup of a semigroup G is left amenable, then so is G. The converse is not true (otherwise left amenability of a semigroup would imply left amenability of each subsemigroup, which is not true; see [1, p. 516, (D) and (F)]). Nevertheless, Theorem E_1 shows that every finitely generated subsemigroup of a left amenable semigroup is included in a countable left amenable semigroup.

In order to prove Theorem E_2 we need the following remarks:

Remark 2.1. Let G be a semigroup, and $G_0 \subset G$ a subsemigroup, and suppose that

$$\pi : m(G) \to m(G_0) \quad \text{is such that} \quad (\pi x)(g) = x(g) \quad \text{for } g \text{ in } G_0 \text{ and } x \text{ in } m(G),$$

$$l_a : m(G) \to m(G) \quad \text{is such that} \quad (l_a x)(g) = x(ag) \quad \text{for } g \text{ in } G,$$

$$l_a^0 : m(G_0) \to m(G_0) \quad \text{is such that} \quad (l_a^0 x)(g) = x(ag) \quad \text{for } g \text{ in } G_0 \text{ and } a \text{ in } G_0.$$

Then

$$\pi l_a = l_a^0 \pi \qquad \text{for each } a \text{ in } G_0.$$

Let x be in $m(G)$; then $(\pi l_a)(x) = \pi(l_a x)$ is in $m(G_0)$, and $(l_a^0 \pi)x = l_a^0(\pi x)$ is in $m(G_0)$. Thus we have to prove that for each g of G_0,

$$[\pi(l_a x)](g) = [l_a^0(\pi x)](g).$$

But for g in G_0 one has

$$[\pi(l_a x)](g) = (l_a x)(g) = x(ag) \quad \text{and}$$

$$[l_a^0(\pi x)](g) = (\pi x)(ag) = x(ag) \qquad \text{since } a \text{ is in } G_0.$$

Remark 2.2. If $\nu_0 \in m(G_0)^*$ is a mean, then so is $\pi^* \nu_0$ since if $x \in m(G)$ with $x(g) \geqq 0$ for each g in G, then $x(g) = (\pi x)(g) \geqq 0$ for g in G_0, and $(\pi^* \nu_0)x = \nu_0(\pi x) \geqq 0$ since ν_0 is a mean. By $\pi(1_G) = 1_{G_0}$ we get that $(\pi^* \nu_0)(1_G) = \nu_0(1_{G_0}) = 1.$

Remark 2.3. Let G be a left amenable semigroup, and $G_0 \subset G$ a left amenable subsemigroup. Let ν_0 be a left invariant element of $m(G_0)^*$ (i.e.,

$\nu_0(l_a^0 f) = \nu_0(f)$ for f in $m(G_0)$ and a in G_0) and $x \in m(G)$. Then the function

$$y(a) = \nu_0(\pi l_a x)$$

is constant on left cosets of G with respect to G_0. In other words, if $a \sim b$ (see for instance the definition after Lemma 5.1 in [4]), then $y(a) = y(b)$.

If $a \approx b$, then by definition there are g_1, g_2 in G_0 such that $ag_1 = bg_2$, and then

$$l_{g_1}^0(\pi l_a x) = (l_{g_1}^0 \pi)(l_a x) = (\pi l_{g_1})(l_a x) = \pi(l_{g_1} l_a)(x) = \pi(l_{ag_1} x).$$

For the second equality, see Remark 2.1; the last is true since $l_c l_d = l_{dc}$. In the same way $l_{g_2}^0(\pi l_b x) = \pi(l_{bg_2} x)$ which implies

$$l_{g_1}^0(\pi l_a x) = l_{g_2}^0(\pi l_b x).$$

However,

$$y(a) = \nu_0(\pi l_a x) = \nu_0[l_{g_1}^0(\pi l_a x)] = \nu_0[l_{g_2}^0(\pi l_b x)] = \nu_0(\pi l_b x) = y(b).$$

The second and fourth equalities hold since by our assumption ν_0 is a left invariant element in $m(G_0)^*$.

If now $a \sim b$, then by definition there are $a_1, \cdots, a_k$ in G such that

$$a \approx a_1 \approx a_2 \approx \cdots \approx a_n \approx b.$$

By the above, $y(a) = y(a_1) = \cdots = y(a_n) = y(b)$ which proves the remark.

We are now ready to prove

THEOREM E_2. *If G is a left amenable semigroup with left cancellation, and $G_0 \subset G$ is a left amenable subsemigroup, then there exists a linear isometry, from the subspace of left invariant elements of $m(G_0)^*$ into the subspace of left invariant elements of $m(G)^*$, which maps left invariant means into left invariant means.*

Proof. If φ, ψ are in $m(G)^*$, then let $\varphi \odot \psi \in m(G)^*$ be defined by

$$(\varphi \odot \psi)x = \varphi_h[\psi_g(x(hg))] \qquad \text{for } x \text{ in } m(G).$$

(ψ_g is the functional ψ with respect to the variable g for fixed h in G.) This multiplication renders $m(G)^*$ a Banach algebra (see [1, p. 527]).

Let $\mu \in Ml(G)$ be fixed, and π, l_a, l_a^0 as in Remark 2.1.

We define as in Day [1, p. 533] for every left invariant element ν_0 of $m(G_0)^*$ (i.e., for $\nu_0 \in m(G_0)^*$ such that $(l_a^0)^*\nu_0 = \nu_0$ for each a in G_0):

$$T\nu_0 = \mu \odot \pi^*\nu_0 .$$

$T\nu_0$ is a left invariant element of $m(G)^*$ (see proof of Corollary 2 in [1, p. 529]), and if ν_0 is a mean, so is $\pi^*\nu_0$ (Remark 2.2), and therefore so is $\mu \odot \pi^*\nu_0$. (If φ, ψ are means, so is $\varphi \odot \psi$, as is easily proved.) Since $\odot$ is distributive and π^* is linear, T is linear. Moreover, by [1, p. 527, Lemma 1] and bearing in mind that π^* is isometric [1, p. 512, (2)] we have for every left invariant element ν_0 of $m(G_0)^*$ that

$$\| T\nu_0 \| = \| \mu \odot \pi^*\nu_0 \| \leqq \| \mu \| \, \| \pi^*\nu_0 \| = \| \pi^*\nu_0 \| = \| \nu_0 \|.$$

In order to finish the proof of the theorem we have still to prove that $\| T\nu_0 \| = \| \nu_0 \|$.

In what follows we construct (see Day [1, p. 533]) a function x of $m(G)$ which satisfies $\| x \| \leqq 1$ and $| (T\nu_0)x | \geqq \| \nu_0 \| - \varepsilon$.

Let $x_0 \in m(G_0)$, $\| x_0 \| \leqq 1$ be such that

$$\nu_0(x_0) \geqq \| \nu_0 \| - \varepsilon,$$

and let $\{H_\alpha\}$ be a decomposition of G into left cosets with respect to G_0. Then G_0 is included in exactly one left coset, for G_0 is by assumption left amenable which implies that every two right ideals have nonvoid intersection (this is easily proved; see end of proof of Corollary 5.5 in [4]), and therefore for each a, b in G_0, $aG_0 \cap bG_0 \neq \emptyset$ which yields the existence of g_1, g_2 in G_0 such that $ag_1 = bg_2$. We get thus that $a \sim b$, which implies that G_0 is included in exactly one left coset of the H_α's (let it be H_0). We choose now some $g_0 \in G_0 \subset H_0$, and some h_α from each other H_α, such that $\{g_0\} \cup \bigcup_\alpha \{h_\alpha\}$ form a set of representatives of the left cosets of G with respect to G_0. Now for each α, $h_\alpha G_0 \subset H_\alpha$, since for g in G_0, $(h_\alpha g)g = h_\alpha g^2$, and thus $h_\alpha \sim h_\alpha g$.

We define the function x of $m(G)$ as follows:

$$x(g) = x_0(g) \quad \text{for } g \text{ in } G_0.$$

If $g \in h_\alpha G_0$, then $g = h_\alpha g_1$ for exactly one g_1 of G_0 (otherwise $g = h_\alpha g_1 = h_\alpha g_2$ which contradicts the left cancellation which holds in G). (Please note that this is the only place where left cancellation is used.)

We define for this $g = h_\alpha g_1$

$$x(g) = x(h_\alpha g_1) = x_0(g_1).$$

$x(g)$ is defined in this way on $G_0 \cup [\bigcup_\alpha h_\alpha G_0]$ (the union is over $\{\alpha; h_\alpha \neq g_0\}$). For any other g in G we define $x(g) = 0$. Obviously $\| x \| \leqq 1$ and $\pi x = x_0$. Moreover,

$$(T\nu_0)x = (\mu \odot \pi^*\nu_0)x = \mu_\sigma[(\pi^*\nu_0)_g\, x(\sigma g)] = \mu_\sigma[(\pi^*\nu_0)(l_\sigma x)] = \mu_\sigma[\nu_0(\pi l_\sigma x)].$$

If we let $y(\sigma) = \nu_0(\pi l_\sigma x)$, then by Remark 2.3, $y(\sigma)$ is constant on each H_α and on H_0. In order to know the value of $y(\sigma)$ on H_α or H_0 it is sufficient to know $y(h_\alpha)$ or $y(g_0)$. Now

$$y(g_0) = \nu_0[(\pi l_{g_0})(x)] = \nu_0[(l^0_{g_0}\pi)x] = \nu_0(l^0_g(\pi x)) = \nu_0(\pi x) = \nu_0(x_0).$$

For the second equality, see Remark 2.1. And if $g \in G_0$, then

$$[\pi(l_{h_\alpha} x)](g) = (l_{h_\alpha} x)g = x(h_\alpha g) = x_0(g);$$

thus $\pi(l_{h_\alpha} x) = x_0$. We get

$$y(h_\alpha) = \nu_0(\pi l_{h_\alpha} x) = \nu_0(x_0) \quad \text{and} \quad y(g_0) = \nu_0[\pi(l_{g_0} x)] = \nu_0(x_0).$$

Thus for each g in H_α or in H_0, $y(g) = \nu_0(x_0)$; in other words, $y(g) = \nu_0(x_0) \cdot 1_G(g)$. But

$$(T\nu_0)x = \mu_\sigma[\nu_0(\pi l_\sigma x)] = \mu(y) = \mu(\nu_0(x_0) \cdot 1_G) = \nu_0(x_0) \geqq \| \nu_0 \| - \varepsilon,$$

which implies that

$$\| T\nu_0 \| = \sup_{\| x \| \leqq 1} | (T\nu_0)x | \geqq \| \nu_0 \| - \varepsilon \quad \text{for each } \varepsilon > 0.$$

We have proved that $\| T\nu_0 \| \geqq \| \nu_0 \|$, and since $\| T\nu_0 \| \leqq \| \nu_0 \|$, we get that for each left invariant element of $m(G_0)^*$, $\| T\nu_0 \| = \| \nu_0 \|$, which finishes the proof of Theorem E_2.

Remark 2.4. That this theorem is not true for general left amenable semigroups is proved by the following example: Let N_0 be the multiplicative semigroup of nonnegative integers $N_0 = \{0, 1, 2, \cdots\}$, and let $N_1 = \{1, 2, 3, \cdots\}$ be the multiplicative subsemigroup of natural numbers. Then N_0 and N_1 are obviously a left amenable semigroup and left amenable subsemigroup. (Every commutative semigroup is amenable; see [1, p. 516].) $\{0\} \subset N_0$ is obviously a finite group and left ideal in N_0. It is obviously a (l.i.l.c.) since $m\{0\} = \{0\}$ for m in N_0. But $\{0\}$ is the only finite left ideal in N_0, and therefore dim $Ml(N_0) = 1$ (see Theorem 3.1 in [4]). But N_1 has no finite ideals, and Corollary 5.2 of [4] yields that dim $Ml(N_1) = \infty$.

Thus there cannot exist a linear isometry from the space of left invariant elements of $m(N_1)^*$ into the space of left invariant elements of $m(N_0)^*$.

Remark 2.5. T maps the linear manifold spanned by the left invariant means in $m(G_0)^*$ isometrically into the linear manifold spanned by the left invariant means in $m(G)^*$. Thus if dim $Ml(G) = n < \infty$, then

$$\dim Ml(G_0) \leqq \dim Ml(G) = n$$

(where G_0, G, and T are defined in Theorem E_2).

In fact for left amenable semigroups in which left cancellation holds, every left invariant element μ of $m(G)^*$ can be decomposed into $\mu = \alpha\mu^+ - \beta\mu^-$, where μ^+, μ^- are left invariant means and $\alpha, \beta \geqq 0$. (See [2, p. 281] for semigroups in which two-sided cancellation holds.) Therefore dim $Ml(G)$, if finite, equals the dimension of the space of all left invariant elements of $m(G)^*$, if left cancellation holds.[5]

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[5] The left cancellation is not needed in order that this should be true, as can be easily seen. In fact, if φ is a left invariant element of $m(G)^*$, then it can be represented as $\varphi = \varphi_1 - \varphi_2$, where $\varphi_1, \varphi_2 \in m(G)^*$ are nonnegative (for instance by Jordan's decomposition theorem). Let $\alpha_i = \| \varphi_i \|$, and $\psi_i = \varphi_i / \| \varphi_i \|$ if $\varphi_i \neq 0$, and $\psi_i = 0$ if $\varphi_i = 0$ $(i = 1, 2)$. Then if $\mu \in Ml(G)$, we have by Day [1, p. 530], $\varphi = \mu \odot \varphi = \alpha_1 \mu \odot \psi_1 - \alpha_2 \mu \odot \psi_2$ and $\mu \odot \psi_i$ is either 0, if ψ_i is 0, or belongs to $Ml(G)$ if ψ_i is a mean (see [1, p. 529]) which proves: For any left amenable semigroup G, $\{\varphi \in m(G)^*; L_g \varphi = \varphi, g \in G\}$ coincides with the linear manifold spanned by $Ml(G)$.

Remark 2.6. In the proof of Theorem E_2 we assumed only that $hg_1 = hg_2$, for h in G and g_1, g_2 in G_0 implies $g_1 = g_2$, which is a weaker condition than left cancellation in G.

We are now ready to prove

THEOREM *E*. *If G is a left amenable semigroup with left cancellation, then* $\dim Ml(G) = n < \infty$ *if and only if G is finite and the union of n disjoint finite groups, each isomorphic to the other, and each of which is a left ideal in G.*[6]

Proof. If G is the union of n finite groups $A_1, \cdots, A_n$ each of which is a left ideal, then G has exactly n finite groups and (l.i.l.c.) (since left cancellation holds in G), and therefore by Theorem 3.1 of [4], $\dim Ml(G) = n$.

Let now $\dim Ml(G) = n$, $0 < n < \infty$. We claim that G has to be finite. Otherwise we could choose an infinite sequence $\{g_n\}$ of different elements of G. If G_0 is the countable subsemigroup generated by $\{g_n\}$, then by Theorem E_1 there is a left amenable countable subsemigroup G_0' such that $G_0 \subset G_0' \subset G$. But by Theorem E_2 and Remark 2.5, $\dim Ml(G_0') = m \leqq n < \infty$. Since G_0' is countable, it contains, by Corollary 5.2 of [4], exactly m finite groups $A_1, \cdots, A_m$ which are (l.i.l.c.). By Lemma 3.1 in [4], $A = \bigcup_{i=1}^{m} A_i$ is a right ideal in G_0'. Moreover, $G_0' = A$. Otherwise let g_0 be an element of G_0' which is not in A. If e_j is the identity element of A_j, then $e_1 g_0$ is in A and therefore in A_i for some $1 \leqq i \leqq m$. Thus $e_1 g_0 = e_1 g_0 e_i$, and by the left cancellation $g_0 = g_0 e_i$. But $g_0 e_i$ is in A_i; thus g_0 is in A, which contradicts the assumption that $G_0' \neq A$. Thus $G_0' = A$ and G_0' is finite. However $G_0' \supset \{g_n\}$, which implies that G has to be a finite semigroup.[7] Since G is finite, $l_1(G)$ is finite-dimensional, and

$$n = \dim Ml(G) = \dim [Ml(G) \cap Ql_1(G)] = \dim I_1(G).$$

($l_1(G)$ is reflexive in this case; see Remark 4.2 in [4].) We apply now Corollary 4.2 of [4] and get that G contains exactly n finite groups $A_1, \cdots, A_n$ which are (l.i.l.c.). As groups and left ideals, the A_i's are minimal left ideals and therefore disjoint. But as above, $G = \bigcup_{i=1}^{n} A_i$, since otherwise there would exist a g_0 in G and not in $A = \bigcup_{i=1}^{n} A_i$. If e_j is the identity element of A_j, then $e_1 g_0$ is in A_i for some $1 \leqq i \leqq n$, and therefore $e_1 g_0 e_i = e_1 g_0$. By the left cancellation $g_0 e_i = g_0$, which proves that g_0 is in A, and thus that $G = \bigcup_{i=1}^{n} A_i$.

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[6] Let $Z_n = \{e_1, \cdots, e_n\}$ be the semigroup in which the relations $e_i e_j = e_j$ for $1 \leqq i, j \leqq n$, hold. Then Theorem E can be paraphrased as follows:

If G is a semigroup with left cancellation, then $\dim Ml(G) = n$, $0 < n < \infty$, if and only if $G = A \times Z_n$, where A is a finite group. ($A \times Z_n$ is the direct product of A and Z_n.)

[7] The main part of the theorem is that G has to be finite. For compact semigroups (which applies to our case, since we know already from above that G has to be finite) it is proved by Rosen in [6, pp. 1079–1080] that they have to contain finite groups isomorphic to one another which are also left ideals. We prove more in what follows, i.e., we connect their number with $\dim Ml(G)$.

We define now $\varphi_{ij} : A_i \to A_j$, $\varphi_{ij}(g) = ge_j$ for each g in A_i.

$$\varphi_{ij}(ab) = abe_j = a(e_j(be_j)) = ae_j be_j = \varphi_{ij}(a)\varphi_{ij}(b).$$

(be_j is in A_j, and e_j is its two-sided identity element.) φ_{ij} is thus a homomorphism of A_i into A_j. Let now $a \in A_i$ with $\varphi_{ij}(a) = e_j$. Thus

$$e_j = \varphi_{ij}(a) = ae_j$$

which yields $ae_j e_i = e_j e_i$. But $a \in A_i$, $e_j e_i \in A_i$, and A_i is a group with e_i as identity element. Therefore $a = e_i$ and φ_{ij} is one-to-one, from A_i into A_j. But in the same way $\varphi_{ji} : A_j \to A_i$ is also one-to-one, and since the A_k's are finite we get that φ_{ij} are isomorphisms from A_i onto A_j. (The isomorphisms φ_{ij} satisfy obviously $\varphi_{jk}\varphi_{ij} = \varphi_{ik}$, and φ_{ii} is the identity mapping from A_i into A_i. Thus $\varphi_{ji} = (\varphi_{ij})^{-1}$.) We remark that the above lines yield a proof of the footnote to Theorem 4.1 of [4].

Remark 2.7. For commutative semigroups in which cancellation holds we get that dim $Ml(G) = 1$ or ∞ (see Corollary 5.4 in [4]). In this connection see Luthar's nice work in [5] where he obtains, among other results, that a commutative semigroup (not necessarily countable!) has a unique invariant mean if and only if it contains a finite ideal in it.

If we take $n = 1$ in Theorem E, we get

COROLLARY 2.1. *A left amenable semigroup with left cancellation has a unique left invariant mean* (*i.e.* dim $Ml(G) = 1$) *if and only if it is a finite group.*

Remark 2.8. *Applications.* Let us take $G = Z_1$, the additive semigroup of natural numbers. Then $Ml(Z_1)$ are all the Banach limits, i.e., all the means of $m(Z_1)^*$ which satisfy for every $k > 0$ (for $k = 1$ is sufficient)

$$\varphi\{a_1, a_2, \cdots, a_n, \cdots\} = \varphi(l_k\{a_1, a_2, \cdots, a_n, \cdots\})$$

where l_k is defined by $l_k\{a_1, a_2, \cdots, a_n, \cdots\} = \{a_{k+1}, a_{k+2}, \cdots, a_{n+k}, \cdots\}$. If $L_k = l_k^*$, then $Ml(Z_1)$ are all the means of $m(Z_1)^*$ which satisfy $L_k\varphi = \varphi$ for every $k > 0$. Since Z_1 has no finite ideals, dim $Ml(Z_1) = \infty$.

Let now Z_2 be the multiplicative semigroup of natural numbers. Then $Ml(Z_2)$ are all the means of $m(Z_2)^*$ which satisfy for each positive integer k

$$\varphi\{a_1, a_2, \cdots, a_n, \cdots\} = \varphi\{a_k, a_{2k}, \cdots, a_{nk}, \cdots\}.$$

Since Z_2 does not contain finite ideals and has cancellation, we get that dim $Ml(Z_2) = \infty$.

Another example is the multiplicative semigroup $G = [a, \infty) = \{x; x \geqq a\}$, where $a \geqq 0$ is a real number.

We have to handle two cases:

(1) $a = 0$. Then G has exactly one finite group which is also (l.i.l.c.) which is $\{0\}$. By Theorem 3.1 in [4], dim $Ml(G) = 1$, and if φ_0 is the unique invariant mean of G, then $\varphi_0(f) = f(0)$ for each f in $m(G)$.

(2) $a \geqq 1$. In this case[8] G is an infinite amenable semigroup in which cancellation holds, and therefore dim $Ml(G) = \infty$.

Added September 12, 1962. We give now an example of an infinite semigroup with dim $Ml(G) = n$, $0 < n < \infty$, which will show that the left cancellation of the semigroup G in Theorem E cannot be entirely dropped.

Let $Z_n = \{e_1, \cdots, e_n\}$ with the relations $e_i e_j = e_j$ for $1 \leqq i, j \leqq n$, and let A be some finite group. Then the direct product of A by Z_n is defined, as is well known, by $A \times Z_n = \{(a, e_i); a \in A, e_i \in Z_n\}$ with the multiplication $(a, e_i)(b, e_j) = (ab, e_i e_j) = (ab, e_j)$. As is easily seen, the set

$$A_i = \{(a, e_i); a \in A\}$$

is a finite group isomorphic to A (which is included in $A \times Z_n$). (See [6].) Let now G_0 be an infinite group. We define in the set $G = G_0 \cup A \times Z_n$ a multiplication $\circ$, which as is easily seen renders it a semigroup, as follows: If g', g'' are both in G_0 (or both in $A \times Z_n$), then $g' \circ g''$ means multiplication in G_0 (or in $A \times Z_n$). For g in G_0 and h in $A \times Z_n$, we define $g \circ h = h \circ g = h$. We claim that $A_1, \cdots, A_n$ are the only finite groups and (l.i.l.c.) in G. First of all, A_i is a (l.i.l.c.) since for g in G_0, $g(a, e_i) = (a, e_i)$, which implies that $gA_i = A_i$. And if $g = (b, e_j) \in A \times Z_n$, then $(b, e_j)A_i = (bA, e_i) = A_i$. Let now B be a finite group and (l.i.l.c.) in G. And let us assume the existence of a g_0 in $B \cap G_0$. Then $G_0 = G_0 g_0 \subset B$, which cannot be since B is assumed to be finite. This proves that $B \subset A \times Z_n$. Therefore $B \cap A_j$ is nonvoid for some $1 \leqq j \leqq n$. Since both of B and A_j are minimal left ideals, one gets that $B = A_j$. Theorem 3.1 and Remark 3.2 of [4] yield that dim $Ml(G) = n$ which is the required result.

Another theorem which one gets easily from Theorem E, and whose proof is identical with that on the last page of [4] is the following:

THEOREM. *The radical of the second conjugate algebra $m(G)^*$ for any infinite, left amenable, left cancellation semigroup is infinite-dimensional.*

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[8] For $0 < a < 1$, G is not a semigroup!