ON AMENABLE SEMIGROUPS WITH A FINITE-DIMENSIONAL SET OF INVARIANT MEANS I[1]

BY

E. GRANIRER

1. Introduction

A semigroup G is left [right] amenable if there exists a positive linear functional of norm one on the Banach space of bounded real functions on G with the sup. norm which is invariant under the left [right] translation operator (see Section 2).

It is proved by I. S. Luthar in [12] that a commutative semigroup G has a unique invariant mean if and only if G contains a finite ideal. It is proved by M. M. Day in [2] that infinite solvable groups or infinite amenable nontorsion groups or infinite locally finite groups (see [2, p. 535]) have more than one left invariant mean. It is also proved in [2] that if a left amenable group G has a subgroup or a factor group with more than one left invariant mean, then G has more than one left invariant mean.

It is the purpose of this paper to prove the following theorems:

THEOREM A. *If G is a left amenable countable semigroup, then the linear manifold spanned by the set of left invariant means has dimension $n < \infty$ if and only if G contains exactly n finite disjoint groups $A_1, \cdots, A_n$ which are left ideals with left cancellation (abbreviated (l.i.l.c.); i.e., $ga = gb$, $a, b \in A_i$, $g \in G$ implies $a = b$).*[2]

In the "if" part, countability of G may be dropped. From the "only if" part it follows that all the left invariant means have to be finite means.

The author was not able till now to drop entirely the countability condition imposed on G (though one expects this theorem to be true without imposing on G any countability condition) but only to replace it by some weaker one (see Section 5).

THEOREM B. *If G is a left amenable group (not necessarily countable), then the dimension of the linear manifold spanned by the left invariant means is either one or not finite. It is one if and only if G is finite.*[3]

Received June 22, 1961.

[1] This paper contains partial results from the Ph.D. thesis of the author under the advisorship of Professor A. Dvoretzky and Dr. H. Kesten. I wish to express my thanks to both and especially to Dr. H. Kesten for the real interest he took in my work and for his helpful advice.

[2] The finite groups A_i are even isomorphic to one another.

[3] This author proved meanwhile: Let G be a semigroup with left cancellation. Then the dimension of the linear manifold spanned by the left invariant means is n, $0 < n < \infty$, if and only if G is finite and is the union of n finite disjoint groups each of which is a left ideal in G (and each isomorphic to the other). (See the next paper in this journal.)

THEOREM C. *If G is a left amenable semigroup, then G admits left invariant countable means (i.e., means which belong to $Q(l_1(G))$) if and only if G contains finite groups which are (l.i.l.c.).*

If for each such finite group and (l.i.l.c.) A we define the left invariant mean

$$\varphi_A(f) = (1/N(A)) \sum_{g_i \epsilon A} f(g_i),$$

where $N(A)$ is the number of elements in A, and f is a bounded real function on G, then it follows that the set $\{\varphi_{A_\alpha}\}$ is a basis (topological) for the norm-closed linear space of left invariant elements in $Q(l_1(G))$.

THEOREM D. *If G is a left amenable semigroup, then G admits only countable left invariant means if and only if G contains a finite number of finite groups which are (l.i.l.c.).*

There are in this paper some lemmas and remarks which are interesting for their own sake.

2. Definitions and notations

Let G be a semigroup. $l_1(G)$ will be the space of all real-valued functions φ on G such that $\| \varphi \| = \sum_{g \epsilon G} | \varphi(g) |$ is finite.

$m(G)$ will denote the space of real-valued bounded functions on G with norm $\| f \| = \sup_{g \epsilon G} | f(g) |$. By [3, p. 29], $l_1(G)^*$ (the star means conjugate space) is linearly isometric with $m(G)$. We shall identify $l_1(G)^*$ with $m(G)$.

$Q : l_1(G) \rightarrow m(G)^*$ will be the natural embedding $(Q\varphi)(f) = \sum_{g \epsilon G} \varphi(g) f(g)$ for $f \epsilon m(G)$.

Let l_a [r_a] be the left [right] translation operator in $m(G)$

$$(l_a f)(g) = f(ag) \qquad [(r_a f)(g) = f(ga)],$$

and let $L_a = l_a^*$, $R_a = r_a^*$, where $(l_a^* \varphi) f = \varphi(l_a f)$ and $(r_a^* \varphi) f = \varphi(r_a f)$, $\varphi \epsilon m(G)^*$, $f \epsilon m(G)$.

An element $\varphi \epsilon m(G)^*$ is called a mean if $\| \varphi \| = 1$ and $\varphi(f) \geqq 0$ for $f \epsilon m(G)$ such that $f(g) \geqq 0$ for every $g \epsilon G$. $M(G) \subset m(G)^*$ will denote the set of means. $\varphi \epsilon m(G)^*$ is a left [right] invariant mean if it is a mean and $L_g \varphi = \varphi$ [$R_g \varphi = \varphi$] for every $g \epsilon G$. $Ml(G)$ [$Mr(G)$] will denote the set of left [right] invariant means. $M(G)$, $Ml(G)$, $Mr(G)$ are w^*-compact convex sets in $m(G)^*$ (if nonvoid!). See M. M. Day [2].

A semigroup G is left [right] amenable if $Ml(G) \neq \emptyset$ [$Mr(G) \neq \emptyset$]. G is amenable if $Ml(G) \neq \emptyset$ and $Mr(G) \neq \emptyset$ (and it is shown in [2] that this implies $Ml(G) \cap Mr(G) \neq \emptyset$). $\varphi \epsilon Ml(G) \cap Mr(G)$ will be called an invariant mean.

An element $\varphi \epsilon l_1(G)$ is called a finite mean if $Q\varphi$ is a mean and $\{g; | \varphi(g) | > 0\}$ is finite. $\varphi \epsilon l_1(G)$ is a countable mean if $Q\varphi$ is a mean (or equivalently, if $\varphi(g) \geqq 0$ for $g \epsilon G$ and $\sum_{g \epsilon G} \varphi(g) = 1$). The set of finite means when embedded in $m(G)^*$ is w^*-dense in $M(G)$ (see [2, p. 513]).

A net of means (for convergence of nets see [10]) φ_λ converges w^* (strongly) to left [right] invariance if $L_g \varphi_\lambda - \varphi_\lambda$ $[R_g \varphi_\lambda - \varphi_\lambda]$ converges w^* (in norm) to zero for each $g \in G$. When no ambiguity arises, we shall drop Q and say that the net of finite means φ_λ converges w^* (strongly) to left invariance if $L_g \varphi_\lambda - \varphi_\lambda$ converges w^* (in norm) to zero for every $g \in G$.

If A is a subset of G, then 1_A will denote

$$1_A(g) = 1 \quad \text{if} \quad g \in A,$$
$$= 0 \quad \text{if} \quad g \notin A.$$

If $a \in G$, then 1_a will be

$$1_a(g) = 1 \quad \text{if} \quad g = a,$$
$$= 0 \quad \text{if} \quad g \neq a.$$

A set $A \subset G$ is a left [right] ideal if $GA \subset A$ $[AG \subset A]$. $A \subset G$ is a left ideal with left cancellation (l.i.l.c.) [right ideal with right cancellation (r.i.r.c.)] if it is a left [right] ideal and $ga = gb$ for $g \in G$, a, $b \in A$ implies $a = b$ $[ag = bg$ for $g \in G$, $a, b \in A$ implies $a = b]$. If $A \subset G$ is a finite (l.i.l.c.), then φ_A will denote the left invariant mean whose value on $f \in m(G)$ is

$$\varphi_A(f) = (1/N(A)) \textstyle\sum_{a_i \in A} f(a_i).$$

Here $N(A)$ denotes the number of elements in A, and this convention will be used throughout this paper. (In other words, $\varphi_A = Q[(1/N(A)) \cdot 1_A]$.) φ_A is obviously a mean, and since for finite ideals to be (l.i.l.c.) is equivalent to the condition $gA = A$ for every $g \in G$, φ_A is also left invariant.

(If A is a finite (r.i.r.c.), then φ_A will be a right invariant mean.)

If $K \subset m(G)^*$ (or $K \subset l_1(G)$, or $K \subset m(G)$) is a nonvoid set, then the meaning of $\dim K = n$, $n < \infty$, will be throughout this paper that the linear manifold (in the algebraic sense) spanned by K is finite-dimensional and its dimension is n. $\dim K = \infty$ will mean that the linear manifold spanned by K (in the algebraic sense) is not finite-dimensional.

3. Semigroups with n $(0 < n < \infty)$ finite groups which are left ideals with left cancellation

It is the purpose of this section to prove

THEOREM 3.1. *If G is a semigroup with exactly n finite groups* $(0 < n < \infty)$ *which are (l.i.l.c.), $A_1, \cdots, A_n$, then G is left amenable, and*

$$(3.1) \qquad Ml(G) = \left\{\varphi; \varphi = \textstyle\sum_{i=1}^{n} \alpha_i \varphi_{A_i},\ \alpha_i \geqq 0,\ \sum_1^n \alpha_i = 1\right\}.$$

We need first a lemma which is proved partly in [1] by Clifford.

LEMMA 3.1. *If G is a semigroup with exactly n finite groups $A_1, \cdots, A_n$ which are (l.i.l.c.), then $A = \bigcup_{i=1}^{n} A_i$ is a finite minimal right ideal.*

Proof. Let $g \in G$ be arbitrary. Clearly $A_i g$ is a finite left ideal. It is even a minimal left ideal. For let $I \subset A_i g$ be a left ideal, and let $ag \in I$,

$a \in A_i$. Then $A_i ag \subset I$. But $A_i a = A_i$ since A_i is a group, and thus $A_i g = A_i ag \subset I$.

Consequently the left ideal $A_i gag \subset A_i g$ must equal $A_i g$ which shows that $c, d \in A_i g$, $c \neq d$ implies

$$cag \neq dag. \tag{3.2}$$

Moreover if h is any element of G, then $hA_i = A_i$ since A_i is a finite (l.i.l.c.). In other words, $hA_i g$ contains as many elements as $A_i g$, and thus if $c \neq d$, $c, d \in A_i g$, one must have $hc \neq hd$. Thus $A_i g$ is a finite (l.i.l.c.). From (3.2) it follows that $A_i g$ is also a group (since it is a finite semigroup in which right and left cancellation hold) and by the conditions of the lemma

$$A_i g = A_j \subset \bigcup_{k=1}^{n} A_k$$

for some j. So far we have that $\bigcup_1^n A_k$ is a finite right ideal. It is a minimal right ideal. To see this, let $I \subset \bigcup_1^n A_k$ be a right ideal, and let $a_1 \in I$. Without loss of generality we may assume $a_1 \in A_1$. Let $a \in A_i$, and let e be the unit element of the group A_i. Then $a_1 e \in A_i$, and if $(a_1 e)^{-1}$ is its inverse in A_i, then

$$a = ea = (a_1 e)(a_1 e)^{-1} a = a_1[e(a_1 e)^{-1} a] \in a_1 G \subset IG \subset I.$$

Hence $A_i \subset I$ for all i which shows that $\bigcup_1^n A_i \subset I$.

Remark 3.1. In fact it is proved above that if $\{A_\alpha\}$ is the set of all finite groups in G which are (l.i.l.c.), then $A = \bigcup_\alpha A_\alpha$ is a right minimal ideal.

Proof of Theorem 3.1. Every φ_{A_i} is a left invariant mean; thus G is left amenable. $\varphi_{A_i} \in Ml(G)$, and since $Ml(G)$ is convex, it follows that

$$\{\varphi; \varphi = \textstyle\sum_1^n \alpha_i \varphi_{A_i}, \alpha_i \geqq 0, \sum_1^n \alpha_i = 1\} \subset Ml(G).$$

Let now $\varphi \in Ml(G)$ and $a \in A = \bigcup_{i=1}^n A_i$. Then $\varphi(f) = \varphi(l_a f)$ for $f \in m(G)$. Let $h(g) = (l_a f)(g) = f(ag)$ and $A = \{g_1, \cdots, g_N\}$. For the above fixed $a \in A$ let

$$B_i = \{g \in G; ag = g_i\}, \qquad i = 1, \cdots, N.$$

Then $B_i \cap B_j = \emptyset$, $i \neq j$, and $\bigcup_1^N B_i = G$ because A is a right ideal by Lemma 3.1. But $h(g) = f(ag) = f(g_i)$ for $g \in B_i$. It follows that

$$h(g) = \textstyle\sum_1^N f(g_i) 1_{B_i}(g),$$

and thus

$$\varphi(f) = \varphi(h) = \varphi(\textstyle\sum_1^N f(g_i) 1_{B_i}) = \sum_1^N f(g_i)\varphi(1_{B_i}).$$

Now $1_{B_i}(g) = 1_{g_i}(ag)$, because if $g \in B_i$, then $ag = g_i$ and $1_{g_i}(ag) = 1$. And if $g \notin B_i$, then $ag \neq g_i$ and $1_{g_i}(ag) = 0$. Therefore

$$\varphi(1_{B_i}) = \varphi(l_a 1_{g_i}) = \varphi(1_{g_i}) = \varphi(l_{g_i} 1_{g_i}). \tag{3.3}$$

Let i be fixed, and let $g_i \in A_k$, and denote by e_k the identity of A_k. Then $1_{g_i}(g_i g) \geqq 1_{e_k}(g)$ for every $g \in G$. (If $g = e_k$, the sides are equal, and if

$g \neq e_k$, then the right side is zero while the left side is nonnegative). We get

$$\varphi(l_{g_i} 1_{g_i}) \geqq \varphi(1_{e_k}) = \varphi(l_{g_i^{-1}} 1_{e_k}), \tag{3.4}$$

where $g_i^{-1} \in A_k$, $g_i^{-1} g_i = e_k = g_i g_i^{-1}$. However $(l_{g_i^{-1}} 1_{e_k})(g) = 1_{e_k}(g_i^{-1} g) \geqq 1_{g_i}(g)$ for every $g \in G$ (because if $g = g_i$ the sides are equal, and if $g \neq g_i$ the right side is zero while the left is nonnegative). Thus

$$\varphi(1_{e_k}) = \varphi(l_{g_i^{-1}} 1_{e_k}) \geqq \varphi(1_{g_i}), \tag{3.5}$$

and (3.3), (3.4), (3.5) imply

$$\varphi(1_{B_i}) = \varphi(1_{g_i}) = \varphi(1_{e_k}) \qquad \text{for every } g_i \in A_k .$$

We can now write

$$\varphi(f) = \sum f(g_i)\varphi(1_{B_i}) = \sum_{k=1}^{n} \varphi(1_{e_k})[\textstyle\sum_{g_i \in A_k} f(g_i)]$$
$$= \sum_{k=1}^{n} \varphi(1_{e_k}) N(A_k)[(1/N(A_k)) \textstyle\sum_{g_i \in A_k} f(g_i)] = \sum_{k=1}^{n} \varphi(1_{e_k}) N(A_k) \varphi_{A_k}(f).$$

Since B_i are N disjoint sets such that $\bigcup_1^N B_i = G$, we get that

$$1 = \sum_1^N \varphi(1_{B_i}) = \sum_{k=1}^{n} \sum_{g_i \in A_k} \varphi(1_{B_i}) = \sum_{k=1}^{n} \varphi(1_{e_k}) N(A_k),$$

and $\varphi(1_{e_k}) N(A_k) \geqq 0$. If we denote $\alpha_k = \varphi(1_{e_k}) N(A_k)$, then we have proved that every $\varphi \in Ml(G)$ is of the form $\sum_1^n \alpha_k \varphi_{A_k}$, and thus that

$$Ml(G) = \{\varphi; \varphi = \textstyle\sum_1^n \alpha_i \varphi_{A_i}, \alpha_i \geqq 0, \sum_1^n \alpha_i = 1\}.$$

Remark 3.2. A_i as minimal left ideals (left ideals and groups!) are disjoint, and therefore φ_{A_i}, $i = 1, \cdots, n$, are linearly independent. (If $\sum \beta_i \varphi_{A_i} = 0$, then $0 = (\sum_1^n \beta_i \varphi_{A_i})(1_{A_k}) = \beta_k$.) Therefore we get that φ_{A_i}, $i = 1, \cdots, n$, is the set of vertices of the convex set $Ml(G)$.

Remark 3.3. Theorem 3.1 obviously remains true if we replace "left" by "right" and (l.i.l.c.) by (r.i.r.c.).

4. Amenable semigroups with countable left invariant means

THEOREM 4.1. *Let G be a left amenable semigroup which admits countable left invariant means. Then for each such mean φ_0' there is in G a sequence $\{A_i\}$ (not necessarily infinite but not empty) of finite groups which are (l.i.l.c.) and such that*[4] $\varphi_0' = \sum_{i=1}^{\infty} \alpha_i \varphi_{A_i}$ *and* $\alpha_i \geqq 0$, $\sum_1^{\infty} \alpha_i = 1$.

Proof. Let $\varphi_0 \in l_1(G)$ be such that $Q\varphi_0 = \varphi_0'$. Then $\varphi_0(g) = \varphi_0'(1_g) \geqq 0$ and $\| \varphi_0 \| = \sum_{g \in G} \varphi_0(g) = \| \varphi_0' \| = 1$. Thus

$$I = \{g; \varphi_0(g) > 0\}$$

is countable and nonvoid. If $a \in I$ and $b = g_0 a \in Ga$, then

[4] If $\{A_\alpha\}$ is the set of all finite groups and (l.i.l.c.) in G, such that e_α is the unit of A_α, then for $g \in A_\alpha$, $\varphi_{\alpha\beta}(g) = g e_\beta$ is an isomorphism of A_α onto A_β, such that $\varphi_{\alpha\alpha}$ is the identity mapping and $\varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} = \varphi_{\alpha\gamma}$. This was communicated to the author by M. Perles and can be directly proved.

$$\varphi_0(b) = \varphi_0'(1_b) = \varphi_0'(l_{g_0} 1_b) = \varphi_0'(l_{g_0} 1_{g_0 a}).$$

Since for every $g \in G$, $l_{g_0} 1_{g_0 a}(g) = 1_{g_0 a}(g_0 g) \geqq 1_a(g)$, we get

$$\varphi_0(b) = \varphi_0'(l_{g_0} 1_b) \geqq \varphi_0'(1_a) = \varphi_0(a) > 0.$$

Thus for every $b \in Ga$, $a \in I$,

$$\varphi_0(b) \geqq \varphi_0(a) > 0, \tag{4.1}$$

and therefore $Ga \subset I$. We proved actually more. Since $\varphi_0(g) \geqq 0$ and $\sum_{g \in G} \varphi_0(g) = 1$, we have $Ga \subset \{g; \varphi_0(g) \geqq \varphi_0(a)\}$, which is a finite set if $a \in I$. Thus for every $a \in I$, Ga is a finite left ideal.

Let now $H \subset Ga$ be a minimal left ideal, $H = (b_1, \cdots, b_n)$ say. $Hb_i \subset H$ is a left ideal, and since H is minimal, $Hb_i = H$ for each $b_i \in H$. Thus for each b_i, $b_k \in H$ there is a $b_j \in H$ such that $b_j b_i = b_k$. It follows that $\varphi_0(b_k) \geqq \varphi_0(b_i)$ because $b_k \in Gb_i$, $b_i \in I$, and (4.1) holds. But interchanging i and k we get that $\varphi_0(b_i) = \varphi_0(b_k)$ for every b_i, $b_k \in H$, in other words, that φ_0 is constant on H. However H is a left ideal with left cancellation. Were this not so, then there would exist $g_0 \in G$, b_i, $b_j \in H$ such that

$$g_0 b_i = g_0 b_j = b_k, \qquad b_i \neq b_j. \tag{4.2}$$

But $\varphi_0(b_k) = \varphi_0'(1_{b_k}) = \varphi_0'(l_{g_0} 1_{b_k})$, and by (4.2)

$$(l_{g_0} 1_{b_k})(g) = 1_{b_k}(g_0 g) \geqq 1_{b_i}(g) + 1_{b_j}(g) \quad \text{for every } g \in G.$$

Thus $\varphi_0(b_k) = \varphi_0'(1_{b_k}) = \varphi_0'(l_{g_0} 1_{b_k}) \geqq \varphi_0'(1_{b_i} + 1_{b_j}) = \varphi_0(b_i) + \varphi_0(b_j)$. It follows that

$$\varphi_0(b_k) \geqq \varphi_0(b_i) + \varphi_0(b_j) = 2\varphi_0(b_k) > 0$$

because φ_0 is constant on H and $b_k \in I$. This is a contradiction, and consequently H is a (l.i.l.c.). But then for $b \in H$, $bH = H$, and from the above also $Hb = H$. Thus H is a finite group which is a (l.i.l.c.).

We prove now that $H = Ga$. It is sufficient to prove that $a \in H$ because then $Ga \subset H$, but H was a minimal left ideal in Ga; thus $H = Ga$. If $a \notin H$, then let $b_1 = g_1 a \in H$. Since H is a group,

$$b_1 = (g_1 b_1)(g_1 b_1)^{-1} b_1 = g_1[b_1(g_1 b_1)^{-1} b_1] = g_1 b_j, \tag{4.3}$$

where $b_j = b_1(g_1 b_1)^{-1} b_1 \in H$. Thus $b_1 = g_1 b_j = g_1 a$ and $a \notin H$. But $\varphi_0(b_1) = \varphi_0'(1_{b_1}) = \varphi_0'(l_{g_1} 1_{b_1})$, and by (4.3), $(l_{g_1} 1_{b_1})(g) = 1_{b_1}(g_1 g) \geqq 1_a(g) + 1_{b_j}(g)$ for every $g \in G$, so that

$$\varphi_0(b_1) = \varphi_0'(l_{g_1} 1_{b_1}) \geqq \varphi_0'(1_a + 1_{b_j}) = \varphi_0(a) + \varphi_0(b_j) = \varphi_0(a) + \varphi_0(b_1),$$

and this contradicts the assumption that $a \in I$ (or $\varphi_0(a) > 0$).

We conclude that for each $a \in I$, $Ga \subset I$ is a finite group and (l.i.l.c.) on which φ_0 is constant (Ga is also a minimal left ideal). Thus if $a, b \in I$, then either $Ga = Gb$, or $Ga \cap Gb = \emptyset$. Let

$$I = \cup_n Ga_n \tag{4.4}$$

be a decomposition of I into disjoint finite groups which are (l.i.l.c.). I is countable and $I = \bigcup_{b \in I} Gb$; therefore there are $\{a_n\} \subset I$ which satisfy (4.4). Let $Ga_n = A_n$. We proved above that $\varphi_0(g)$ is constant on A_k and $a_k \in A_k$. Thus

$$\varphi_0(g) = \varphi_0(a_k) \quad \text{if} \quad g \in A_k . \tag{4.5}$$

But

$$1 = \| \varphi_0 \| = \textstyle\sum_{g \in I} \varphi_0(g) = \sum_k \sum_{g \in A_k} \varphi_0(g) = \sum_k N(A_k)\varphi_0(a_k).$$

Let $\alpha_k = \varphi_0(a_k)N(A_k)$; then $\alpha_k \geqq 0$, $\sum \alpha_k = 1$. Since (4.5) can be written $\varphi_0(g) = \sum_k \varphi_0(a_k)1_{A_k}(g)$ ($1_{A_k}(g) \in l_1(G)$ because A_k is finite, and since the A_i's are disjoint for each $g \in G$, only one element of this sum is not zero), we get

$$\begin{aligned}\varphi_0 &= \textstyle\sum \varphi_0(a_k)1_{A_k} \\ &= \textstyle\sum \varphi_0(a_k)N(A_k)[(1/N(A_k))1_{A_k}] = \sum \alpha_k[(1/N(A_k))1_{A_k}].\end{aligned}$$

Thus $\varphi_0' = Q\varphi_0 = Q\sum \alpha_k[(1/N(A_k))1_{A_k}] = \sum \alpha_k Q[(1/N(A_k))1_{A_k}] = \sum \alpha_k \varphi_{A_k}$, because the convergence of $\sum \varphi_0(a_k)1_{A_k}$ is in $l_1(G)$ norm and Q is isometric. And this finishes the proof of Theorem 4.1.

Remark 4.1. The above theorem can be paraphrased in the following way: If G is a left amenable semigroup, and $Ml(G) \cap Q(l_1(G)) \neq \emptyset$, then G has finite groups which are (l.i.l.c.). The converse holds also because if G contains at least one finite group and (l.i.l.c.) A, then $\varphi_A \in Ml(G) \cap Q(l_1(G))$.

Thus we can state

THEOREM 4.2. *G is a semigroup such that $Ml(G) \cap Ql_1(G) \neq \emptyset$ if and only if G contains at least one finite group which is a (l.i.l.c.).*

COROLLARY 4.1. *If G is an infinite left amenable group, then*

$$Ml(G) \cap Ql_1(G) = \emptyset.$$

Otherwise G would have to contain finite left ideals. This corollary can be proved much more easily directly.

It is proved in [7, p. 9], and formerly asserted in [13], that if G is an amenable group and $\varphi \in m(G)^*$ is a left invariant element ($L_g \varphi = \varphi$ for every $g \in G$), then there exist $\varphi_1, \varphi_2 \in Ml(G)$ and $\alpha \geqq 0, \beta \geqq 0$ such that $\varphi = \alpha\varphi_1 - \beta\varphi_2$. In fact $\alpha\varphi_1 = \varphi^+$ and $\beta\varphi_2 = \varphi^-$ of Jordan's decomposition theorem (see [5, p. 98]) for bounded additive real set functions, and it is proved in [7] that φ^+, φ^- are also left invariant if φ is left invariant. In this direction we prove the following:

LEMMA 4.1. *If G is a left amenable semigroup and $\varphi' \in Ql_1(G)$ is a left invariant element ($L_g \varphi' = \varphi'$ for every $g \in G$), then there exist $\varphi_1', \varphi_2' \in Ml(G) \cap Q(l_1(G))$ and $\alpha \geqq 0, \beta \geqq 0$ such that $\varphi' = \alpha\varphi_1' - \beta\varphi_2'$ and $\{g; \varphi_1'(1_g) > 0\} \cap \{g; \varphi_2'(1_g) > 0\} = \emptyset$.*

Proof. Let $\varphi \in l_1(G)$, $Q\varphi = \varphi'$, and

$$A = \{g;\ \varphi(g) > 0\}, \qquad B = \{g;\ \varphi(g) < 0\}.$$

Let $\varphi_1(g) = \varphi(g)1_A(g)$, $\varphi_2(g) = -\varphi(g)1_B(g)$. Then $\varphi_1, \varphi_2 \in l_1(G)$ and $\varphi(g) = \varphi_1(g) - \varphi_2(g)$ and $\| \varphi_1 \| = \sum_{g \in A} \varphi(g)$, $\| \varphi_2 \| = -\sum_{g \in B} \varphi(g)$. If $f \in m(G)$ and $g_0 \in G$, then

$$(4.6) \qquad \varphi'(f) = \sum_{g \in G} \varphi(g)f(g) = \varphi'(l_{g_0} f) = \sum_{g \in G} \varphi(g)f(g_0\, g).$$

If $f = 1_A$, then $\varphi'(1_A) = \sum_{g \in G} \varphi(g)1_A(g) = \sum_{g \in A} \varphi(g) = \sum_{g \in G} \varphi(g)1_A(g_0\, g)$. Let now $A_1 = \{g;\ g_0\, g \in A\}$; then $1_A(g_0\, g) = 1_{A_1}(g)$. We get

$$\sum_{g \in G} \varphi(g)1_A(g_0\, g) = \sum_{g \in G} \varphi(g)1_{A_1}(g) = \sum_{g \in A_1} \varphi(g)$$
$$= \sum_{g \in A \cap A_1} \varphi(g) + \sum_{g \in B \cap A_1} \varphi(g)$$

because $\varphi(g) = 0$ for $g \notin A \cup B$. But $\varphi(g) < 0$ for $g \in B$, and $\varphi(g) > 0$ for $g \in A$. Thus

$$\sum_{g \in A \cap A_1} \varphi(g) + \sum_{g \in B \cap A_1} \varphi(g) \leqq \sum_{g \in A \cap A_1} \varphi(g) \leqq \sum_{g \in A} \varphi(g)$$
$$= \sum_{g \in A \cap A_1} \varphi(g) + \sum_{g \in B \cap A_1} \varphi(g) \leqq \sum_{g \in A \cap A_1} \varphi(g).$$

We get that $\sum_{g \in A} \varphi(g) = \sum_{g \in A \cap A_1} \varphi(g)$ and $\sum_{g \in B \cap A_1} \varphi(g) = 0$ which implies

$$(4.7) \qquad A = A \cap A_1 \quad \text{and} \quad B \cap A_1 = \emptyset$$

(thus $A \subset A_1 = \{g;\ g_0\, g \in A\}$ which implies $g_0 A \subset A$ and A is a left ideal). Let now $f \in m(G)$.

$$\sum_{g \in G} \varphi_1(g)f(g) = \sum_{g \in G} \varphi(g)1_A(g)f(g) = \sum_{g \in A} \varphi(g)f(g).$$

But

$$\sum_{g \in G} \varphi(g)1_A(g)f(g) = \sum_{g \in G} \varphi(g)1_A(g_0\, g)f(g_0\, g)$$

because of (4.6) when we look at $1_A(g)f(g)$ as belonging to $m(G)$. And

$$\sum_{g \in G} \varphi(g)1_A(g_0\, g)f(g_0\, g) = \sum_{g \in A_1} \varphi(g)f(g_0\, g)$$
$$= \sum_{g \in A \cap A_1} \varphi(g)f(g_0\, g) + \sum_{g \in B \cap A_1} \varphi(g)f(g_0\, g)$$

because $1_A(g_0\, g) = 1_{A_1}(g)$ and $\varphi(g) = 0$, $g \notin A \cup B$, $A \cap B = \emptyset$. But by (4.7) we get that

$$\sum_{g \in B \cap A_1} \varphi(g)f(g_0\, g) = 0 \quad \text{and} \quad \sum_{g \in A \cap A_1} \varphi(g)f(g_0\, g) = \sum_{g \in A} \varphi(g)f(g_0\, g).$$

Thus

$$\sum_{g \in G} \varphi_1(g)f(g) = \sum_{g \in A} \varphi(g)f(g_0\, g) = \sum_{g \in G} \varphi(g)1_A(g)f(g_0\, g)$$
$$= \sum_{g \in G} \varphi_1(g)f(g_0\, g).$$

We have proved that, for every $g \in G$, $\varphi_1(g)$ satisfies $L_g(Q\varphi_1) = Q\varphi_1$. Since $\varphi_2(g) = -\varphi(g)1_B(g)$ and $B = \{g;\ -\varphi(g) > 0\}$, the same proof holds for the left invariance of $\varphi_2(g)$. By their definition $\varphi_i(g) \geqq 0$ for every $g \in G$. Let now

$$\psi_i(g) = \varphi_i(g)/\| \varphi_i \| \quad \text{if} \quad \varphi_i \neq 0,$$
$$= 0 \qquad \text{if} \quad \varphi_i = 0, \qquad i = 1, 2.$$

Then $\varphi_i' = Q\psi_i$ (if not zero) are countable left invariant means, and

$$\varphi' = Q[\varphi_1 - \varphi_2] = \| \varphi_1 \| \varphi_1' - \| \varphi_2 \| \varphi_2' .$$

Remark 4.2. Let $I(G) = \{\varphi; \varphi \in m(G)^*, L_g \varphi = \varphi \text{ for every } g \in G\}$. Since $L_g = l_g^*$, $I(G)$ is a w^*-closed subspace of $m(G)^*$ and a fortiori norm-closed. If we denote $I_1(G) = I(G) \cap Q(l_1(G))$, then $I_1(G)$ is a norm-closed (since Q is isometric) linear space, and Lemma 4.1 implies that $I_1(G)$ is the linear manifold spanned by $Ml(G) \cap Q(l_1(G))$. If $\{A_\alpha\}$ is the set of finite groups which are (l.i.l.c.), then each $\varphi \in Ml(G) \cap Q(l_1(G))$ is of the form $\varphi = \sum \alpha_i \varphi_{A_i}$ for some sequence $\{A_i\} \subset \{A_\alpha\}$. Thus we get that the norm-closed linear space spanned by the $\{\varphi_{A_\alpha}\}$ equals $I_1(G)$. Moreover, the $\{\varphi_{A_\alpha}\}$ form a generalized basis for $I_1(G)$ because every $\varphi \in I_1(G)$ can be represented as $\varphi = \sum_1^\infty \alpha_i \varphi_{A_i}$ for some sequence $\{A_i\} \subset \{A_\alpha\}$, and the assumption

$$\lim_{N\to\infty} \textstyle\sum_1^N \alpha_n \varphi_{A_n} = 0$$

in $m(G)^*$ norm implies

$$0 = \lim_{N\to\infty} \left(\textstyle\sum_1^N \alpha_n \varphi_{A_n}\right)(1_{A_k}) = \lim_{N\to\infty} \left[\textstyle\sum_1^N \alpha_n \varphi_{A_n}(1_{A_k})\right] = \alpha_k$$

(since $\varphi_{A_i}(1_{A_k}) = \delta_{ik}$). And now it can be easily seen that the $\{\varphi_{A_\alpha}\}$ are the vertices of the norm-closed convex set $Ml(G) \cap Ql_1(G)$, and $Ml(G) \cap Ql_1(G)$ is the norm closure of the convex set spanned by φ_{A_α}. We have proved

LEMMA 4.2. *$I_1(G)$ is the norm-closed linear space spanned by $\{\varphi_{A_\alpha}\}$, and the set $\{\varphi_{A_\alpha}\}$ is a generalized basis in $I_1(G)$.*

COROLLARY 4.2. *For any semigroup G,* dim $I_1(G) = n$, $0 < n < \infty$, *if and only if G contains exactly n finite groups $A_1, \cdots, A_n$ which are (l.i.l.c.).*

If dim $I_1(G) = n > 0$, then Lemma 4.1 implies the existence of countable left invariant means, and by Theorem 4.1 we get that G contains finite groups which are (l.i.l.c.) (let all of them be $\{A_\alpha\}$). By Lemma 4.2 the $\{\varphi_{A_\alpha}\}$ are a basis for $I_1(G)$, so that G has exactly n finite groups which are (l.i.l.c.). Conversely, let $A_1, \cdots, A_n$ be the finite groups and (l.i.l.c.) in G; then $\varphi_{A_i} \in I_1(G)$, $i = 1, \cdots, n$, and by Lemma 4.2 the φ_{A_i} are a basis for $I_1(G)$, and therefore dim $I_1(G) = n$. (This result will be needed in the following section.)

Remark 4.3. If G contains an infinite number of finite groups $\{A_\alpha\}$ which are (l.i.l.c.), then $\varphi_{A_\alpha} \in Ml(G) \cap Q(l_1(G))$, and so $Ml(G) \cap Q(l_1(G)) \neq \emptyset$. Moreover $Ml(G) \cap Q(l_1(G)) \subset Ml(G)$, and the sides are not equal (in other words, G admits also left invariant means which are not in $Q(l_1(G))$. For let $\{A_n\}_1^\infty$ be an infinite sequence of finite groups in G which are (l.i.l.c.).

For every $f \in m(G)$ let $Tf \in m(Z)$ (Z are the natural numbers with addition) be defined as follows:

$$(4.8) \qquad (Tf)(i) = (1/N(A_i)) \sum_{g_j \in A_i} f(g_j).$$

It follows easily that $(T1_G)(i) = 1_Z(i)$, and if $f \in m(G)$ and $f(g) \geqq 0$ for every $g \in G$, then $(Tf)(i) \geqq 0$ for every $i \in Z$. Moreover

$$T(l_a f)(i) = (1/N(A_i)) \sum_{g_j \in A_i} (l_a f)(g_j) = (1/N(A_i)) \sum_{g_j \in A_i} f(ag_j)$$
$$= (1/N(A_i)) \sum_{g_j \in A_i} f(g_j) = (Tf)(i)$$

since A_i is a (l.i.l.c.). Thus $T(f) = T(l_a f)$ for every $a \in G$. Let now φ be a mean in $m(Z)^*$ which has the property that $\varphi(1_B) = 0$ for every finite set $B \subset Z$ (for the existence of such means see [9, p. 80]). Let now φ_0 be defined for $f \in m(G)$:

$$\varphi_0(f) = \varphi(Tf).$$

It follows from above that $\varphi_0(1_G) = \varphi(T1_G) = \varphi(1_Z) = 1$ since φ is a mean in $m(Z)^*$. Now if $f \in m(G)$ is such that $f(g) \geqq 0$ for every $g \in G$, then $(Tf)(i) \geqq 0$ for every $i \in Z$, and therefore $\varphi_0(f) = \varphi(Tf) \geqq 0$. This implies also that $\varphi_0 \in m(G)^*$, as is easily seen. But $T(l_a f) = T(f)$ for $a \in G$; thus

$$\varphi_0(l_a f) = \varphi(Tl_a f) = \varphi(Tf) = \varphi_0(f),$$

which implies that $\varphi_0 \in Ml(G)$. If $g_0 \in G$, then since

$$(T1_{g_0})(i) = (1/N(A_i)) \sum_{g_j \in A_i} 1_{g_0}(g_j),$$

we get that $(T1_{g_0})(i) = 1/N(A_i) \neq 0$ if $g_0 \in A_i$, and $(T1_{g_0})(i) = 0$ if $g_0 \notin A_i$. Anyway $T1_{g_0}$ is either the function $(1/N(A_i))1_i$ for some $i \in Z$ or identically zero. Since the choice of φ implies $\varphi(1_i) = 0$ for every $i \in Z$, we get that $\varphi_0(1_{g_0}) = \varphi(T1_{g_0}) = 0$ for every $g_0 \in G$. Thus $\varphi_0 \notin Q(l_1(G))$, but $\varphi_0 \in Ml(G)$.

We can now state

COROLLARY 4.3. *If G is a left amenable semigroup, then $Ml(G) = Ml(G) \cap Q(l_1(G))$ if and only if G contains a finite number (at least one!) of finite groups which are (l.i.l.c.).*

The "if" part is Theorem 3.1. For the "only if" part, since G admits left invariant countable means, then by Theorem 4.1, G contains finite groups which are (l.i.l.c.). If their number were infinite, then by Remark 4.3, G would admit left invariant means which are not countable means, which contradicts $Ml(G) = Ml(G) \cap Q(l_1(G))$.

5. Left amenable semigroups with a finite-dimensional set of left invariant means

In this section we prove the main result of this paper, i.e.,

THEOREM 5.1. *If G is a left amenable denumerable semigroup such that* *dim $Ml(G) = n < \infty$, then $Ml(G) \subset Q(l_1(G))$, and G contains exactly n finite groups which are (l.i.l.c.).*

In fact some stronger result will be proved in which the denumerability condition on G will be replaced by: G has a denumerable subsemigroup $G_0 \subset G$ such that the set of left cosets of G with respect to G_0 is denumerable.

From Theorem 5.1 we get, using a result of Day in [2], that for left amenable infinite groups (not necessarily denumerable) dim $Ml(G) = \infty$. After the proof of the main theorem, we give some corollaries which may be interesting for their own sake.

For the proofs we need the following lemmas and remarks:

LEMMA 5.1. *Let G be a left amenable semigroup, and $\{\varphi_\alpha\}$ a net of finite means converging strongly to left invariance. If $A \subset G$ is a denumerable set, then there exists a sequence $\{\varphi_{\alpha_n}\} \subset \{\varphi_\alpha\}$ such that*

$$\lim_{n\to\infty} \| L_a \varphi_{\alpha_n} - \varphi_{\alpha_n} \| = 0 \qquad \textit{for every } a \in A.$$

Proof. Let $A = \{a_i\}_1^\infty$, and let D be the directed index set of the net $\{\varphi_\alpha\}$. By assumption $\lim_\alpha \| L_g \varphi_\alpha - \varphi_\alpha \| = 0$. There is then $\alpha_1 \in D$ such that $\| L_{a_1} \varphi_{\alpha_1} - \varphi_{\alpha_1} \| < 1$. There also exist α_2^1, α_2^2 in D such that

$$\| L_{a_1} \varphi_\alpha - \varphi_\alpha \| < \tfrac{1}{2} \quad \text{if} \quad \alpha \geqq \alpha_2^1$$

and

$$\| L_{a_2} \varphi_\alpha - \varphi_\alpha \| < \tfrac{1}{2} \quad \text{if} \quad \alpha \geqq \alpha_2^2 .$$

Let $\alpha_2 \in D$ be such that $\alpha_2 \geqq \alpha_2^1$ and $\alpha_2 \geqq \alpha_2^2$. Then $\| L_{a_i} \varphi_{\alpha_2} - \varphi_{\alpha_2} \| < \frac{1}{2}$, $i = 1, 2$. If $\alpha_1, \cdots, \alpha_{k-1}$ have been chosen so that

$$\| L_{a_i} \varphi_{\alpha_j} - \varphi_{\alpha_j} \| < 1/j, \qquad i = 1, 2, \cdots, j, \quad j = 1, 2, \cdots, k-1,$$

then α_k will be chosen in the following way: There exist $\alpha_k^1, \cdots, \alpha_k^k$ such that $\alpha \geqq \alpha_k^i$ implies $\| L_{a_i} \varphi_\alpha - \varphi_\alpha \| < 1/k$. Let $\alpha_k \geqq \alpha_k^i$ for $1 \leqq i \leqq k$. Then $\| L_{a_i} \varphi_{\alpha_k} - \varphi_{\alpha_k} \| < 1/k$, $1 \leqq i \leqq k$. The sequence $\{\alpha_k\}$, $k = 1, 2, \cdots$, satisfies the requirements because if $a \in A$, then $a = a_{i_0}$ for some i_0, and hence

$$\| L_{a_{i_0}} \varphi_{\alpha_j} - \varphi_{\alpha_j} \| < 1/j \qquad \text{for } j \geqq i_0,$$

so that $\lim_{n\to\infty} \| L_a \varphi_{\alpha_j} - \varphi_{\alpha_j} \| = 0$, which proves the lemma.

DEFINITION (see [4, p. 215]). Let G be a semigroup, and $G_0 \subset G$ a subsemigroup; then for $a, b \in G$ we write $a \approx b$ if there exist $g', g'' \in G_0$ such that $ag' = bg''$. For $c, d \in G$ we write $c \sim d$ if there exist a finite set $a_1, \cdots, a_k$ of elements in G such that $c \approx a_1 \approx a_2 \approx \cdots \approx a_n \approx d$. The relation $\sim$ is an equivalence relation, and the decomposition of G into the disjoint equivalence classes with respect to this relation are the left cosets of G with respect to G_0.

LEMMA 5.2. *Let G be a semigroup, and $G_0 \subset G$ a subsemigroup. Let*

$\varphi \in m(G)^*$ satisfy $L_g \varphi = \varphi$ if $g \in G_0$. If for some $a \in G$, $L_a \varphi = \varphi$, then $L_g \varphi = \varphi$ for every g in the left coset containing a.

Proof. Let $b \approx c$; then $bg_1 = cg_2$ for some $g_1, g_2 \in G_0$. Since $L_{g_i} \varphi = \varphi$, $i = 1, 2$, and $L_s L_t = L_{st}$ if $s, t \in G$, we get

$$L_b \varphi = L_b(L_{g_1} \varphi) = L_{bg_1} \varphi = L_{cg_2} \varphi = L_c(L_{g_2} \varphi) = L_c \varphi.$$

In other words, $b \approx c$ implies $L_b \varphi = L_c \varphi$. Let now b be in the left coset containing a. Then there exist $a_1, \cdots, a_k \in G$ such that

$$a \approx a_1 \approx \cdots \approx a_k \approx b,$$

and by assumption $L_a \varphi = \varphi$. Consequently

$$\varphi = L_a \varphi = L_{a_1} \varphi = \cdots = L_{a_k} \varphi = L_b \varphi.$$

COROLLARY 5.1. *If $G_0 \subset G$ is as in Lemma 5.2, and if $\{g_\alpha\}$ is a set of representatives of the left cosets of G with respect to G_0, then left invariance of $\varphi \in m(G)^*$ by the elements of the set $G_0 \cup \{g_\alpha\}$ implies left invariance by every $g \in G$. (In other words, $L_g \varphi = \varphi$ for $g \in G_0 \cup \{g_\alpha\}$ implies $L_g \varphi = \varphi$ for every $g \in G$.)*

We shall further use the following known facts:

$(5.1)^*$ If φ is a w^*-cluster point of the sequence $\{\varphi_{\alpha_n}\}$ of Lemma 5.1 ($\varphi \in \bigcap_{n=1}^{\infty} \overline{\{\varphi_{\alpha_n}, \varphi_{\alpha_{n+1}}, \cdots\}}$, and the bar means w^*-closure), then $L_a \varphi = \varphi$ for each $a \in A$. (This is a trivial generalization of [2, p. 520, (B)].) Let $\varphi_{\alpha_n} = \varphi_n$, $f \in m(G)$; then

$$\begin{aligned} |(L_a \varphi - \varphi)f| &\leqq |L_a(\varphi - \varphi_n)f| + |(L_a \varphi_n - \varphi_n)f| + |(\varphi_n - \varphi)f| \\ &\leqq \| L_a \varphi_n - \varphi_n \| \, \| f \| + |(\varphi - \varphi_n) l_a f| + |(\varphi - \varphi_n)f|. \end{aligned}$$

But there is an n_1 such that $\| L_a \varphi_n - \varphi_n \| \, \| f \| < \varepsilon/3$ whenever $n \geqq n_1$, and an $n_2 \geqq n_1$ such that $\varphi_{n_2} \in \{\psi; |(\psi - \varphi) l_a f| < \varepsilon/3; |(\psi - \varphi)f| < \varepsilon/3\}$. Thus $L_a \varphi = \varphi$ for each $a \in A$. Since $L_a L_b = L_{ab}$, one even has $L_c \varphi = \varphi$ for every c which is in the semigroup generated by A.

$(5.2)^*$ The set of finite means is w^*-dense in the w^*-compact convex set of means on $m(G)$ (see M. M. Day [2, p. 513, (C)]).

$(5.3)^*$ It is well known that if φ_n is a sequence of means on $m(G)$ with exactly one w^*-cluster point φ_0, then $\lim_{n\to\infty} \varphi_n(f) = \varphi_0(f)$ for every $f \in m(G)$. Otherwise there would exist $f_0 \in m(G)$ and a sequence n_i such that $|(\varphi_{n_i} - \varphi_0)f_0| \geqq \varepsilon$ for some $\varepsilon > 0$. But by $(5.2)^*$, φ_{n_i} has a w^*-cluster point φ_0' which has to belong to the w^*-closed set $\{\psi; |(\psi - \varphi_0)f_0| \geqq \varepsilon\}$. Thus $\varphi_0' \neq \varphi_0$. Since φ_0' is also a w^*-cluster point of φ_n, we get a contradiction to the uniqueness of φ_0.

$(5.4)^*$ If Y is a linear topological space (l.t.s.) and $X \subset Y$ is a finite-dimensional linear manifold, then X is closed (see [3, p. 14, Corollary 4 (a)].

(5.5)* If Y is a finite-dimensional (l.t.s.) with two topologies τ_1, τ_2 (both make Y a (l.t.s.)), then τ_1 and τ_2 are equivalent (see [3, p. 14, Corollary 4 (a)]).

(5.6)* If Y is a (l.t.s.) and $X \subset Y$ is a linear manifold, then the induced topology from Y in X makes X a (l.t.s.) (and if Y is convex, so is X).

(5.7)* If X is a separable Banach space with a Schauder basis x_i, then there exist $\delta_i > 0$ such that each sequence $y_i \in X$ for which $\| x_i - y_i \| < \delta_i$ is also a Schauder basis for X. (See [11]. We use only the case where X has a finite Schauder basis which is entirely trivial.)

(5.8)* If G is a left amenable semigroup and $\varphi_0 \in Ml(G)$, and if $\{\varphi_\alpha\}$ is a net of finite means converging w^* to φ_0, then $\{\varphi_\alpha\}$ is converging w^* to left invariance (see [2, p. 520, (A)]). But by [2, p. 524, Theorem 1], there is a net of finite averages far out in φ_α (for definition see [2, p. 523, Definition 5])—let it be ψ_β—which converges in norm to left invariance, and by [2, p. 523, Lemma 3], the net $\{\psi_\beta\}$ also converges w^* to φ_0.

(5.9)* $l_1(G)$ is w^*-sequentially complete (see [5, p. 374]).

We are now ready to prove Theorem 5.1.[5]

THEOREM 5.1. *Let G be a left amenable semigroup with $G_0 \subset G$ a denumerable subsemigroup such that the set of left cosets (of G with respect to G_0) is denumerable.*

If $\dim Ml(G) = n < \infty$, *then* $Ml(G) \subset Q(l_1(G))$, *and G has exactly n finite disjoint groups which are (l.i.l.c.).*

(*Remark.* If G has a denumerable left ideal, then the condition of the theorem holds.)

Proof. Let E be the linear manifold spanned by $Ml(G)$. Let $\varphi_1, \cdots, \varphi_n \in Ml(G)$ be a basis for E. If $\delta_1, \cdots, \delta_n$ are chosen as in (5.7)*, then $\delta = \min_{1 \leq i \leq n} \delta_i$ satisfies: Each set $\psi_i \in E, i = 1, \cdots, n, \| \psi_i - \varphi_i \| < \delta$, is also a basis for E. We shall prove that there are such ψ_i's which are in $Q(l_1(G))$. It will then follow that $Ml(G) \subset Q(l_1(G))$, i.e., E is the linear manifold spanned by $Ml(G) \cap Q(l_1(G))$; in other words, E is exactly $I_1(G)$ of Remark 4.2. Hence by the assumption of the theorem, $\dim I_1(G) = n$, and Corollary 4.2 implies then the existence of exactly n finite groups which are (l.i.l.c.).

Let now φ_0 be one of $(\varphi_1, \cdots, \varphi_n)$, and $S(x_0, \varepsilon) = \{x; \| x - x_0 \| < \varepsilon\}$. E is finite-dimensional and by (5.4)* is closed in both the w^* and norm topology of $m(G)^*$. By (5.6)* both topologies induce in E topologies which make E a linear topological space, and by (5.5)* they are equivalent. Therefore there exists a w^*-neighborhood N_0 of φ_0 such that

$$\varphi_0 \in N_0 \cap E \subset S(\varphi_0, \delta) \cap E,$$

[5] This author is very grateful to Professor Day for kindly pointing out an error in the original proof of this theorem.

and we can assume that N_0 is w^*-closed and convex. (If $N_1 = N(\varphi_0, f_1, \cdots, f_n, \varepsilon) = \{\varphi; |(\varphi - \varphi_0)f_i| < \varepsilon, 1 \leqq i \leqq n, f_i \in m(G)\}$ satisfies $\varphi_0 \in N_1 \cap E \subset S(\varphi_0, \delta) \cap E$, then $N_0 = \{\varphi; |(\varphi - \varphi_0)f_i| \leqq \varepsilon/2, 1 \leqq i \leqq n\}$ is a w^*-closed and convex neighborhood of φ_0 and satisfies

$$\varphi_0 \in N_0 \cap E \subset S(\varphi_0, \delta) \cup E.)$$

By $(5.2)^*$ we can choose for each w^*-neighborhood W of φ_0 a finite mean $\varphi_W \in N_0 \cap W$. $\{\varphi_W\}$ is a net of finite means defined on the directed set of w^*-neighborhoods of φ_0 (directed by inclusion) which converges w^* to φ_0. By $(5.8)^*$ there is a net $\{\psi_\beta\}$ of finite averages far out in $\{\varphi_W\}$ which converges in norm to left invariance, and $\{\psi_\beta\}$ converges w^* to φ_0. But $\varphi_W \in N_0$ and N_0 is convex; thus ψ_β as a finite average of the finite means φ_W is also a finite mean, and $\psi_\beta \in N_0$. By the assumption of the theorem there exists a countable set $\{g_n\}$ of representatives of the left cosets of G with respect to G, and also G_0 is countable.

By Lemma 5.1 there exists a sequence $\{\psi_{\beta_n}\} \subset \{\psi_\beta\}$ such that

$$\lim_{n\to\infty} \| L_a \psi_{\beta_n} - \psi_{\beta_n} \| = 0 \qquad \text{for every } a \in G_0 \cup \{g_n\}.$$

But by $(5.2)^*$ the sequence ψ_{β_n} has some w^*-cluster point $\psi_0 \in N_0$ (because $\psi_{\beta_n} \in N_0$ and N_0 is w^*-closed). By $(5.1)^*$, ψ_0 satisfies $L_a \psi_0 = \psi_0$ for every $a \in G_0 \cup \{g_n\}$; hence by Corollary 5.1, $L_g \psi_0 = \psi_0$ for every $g \in G$. But ψ_0 as a w^*-cluster point of finite means is by $(5.2)^*$ a mean. Thus we have constructed a ψ_0 which is a left invariant mean, and $\psi_0 \in N_0 \cap E \subset S(\varphi_0, \delta) \cap E$. We shall show that $\psi_0 \in Q(l_1(G))$ and thus finish the proof of the theorem. We shall construct a sequence ψ_k' of finite means such that $\lim_{k\to\infty} \psi_k'(f) = \psi_0(f)$ for $f \in m(G)$, and it will follow by $(5.9)^*$ that $\psi_0 \in Q(l_1(G))$.

There are w^*-neighborhoods V_n of ψ_0 such that $\psi_0 \in V_n \cap E \subset S(\psi_0, 1/n) \cap E$, and we can assume that V_n are w^*-closed (as we did for N_0). Let $W_n = V_1 \cap \cdots \cap V_n$. W_n is a w^*-closed neighborhood of ψ_0, such that $\psi_0 \in W_n \cap E \subset V_n \cap E \subset S(\psi_0, 1/n) \cap E$ and $W_n \supset W_{n+1}$, $n = 1, 2, \cdots$. From now on, let $\psi_{\beta n} = \psi_n$. Since ψ_0 is a w^*-cluster point of ψ , we can choose a sequence $n_1 < n_2 < \cdots < n_k < \cdots$ such that $\psi_{n_k} \in W_k$. $\{\psi_{n_k}\}$ as a subsequence of ψ_n satisfies also $\lim_{k\to\infty} \| L_a \psi_{n_k} - \psi_{n_k} \| = 0$ for every $a \in G_0 \cup \{g_n\}$. But ψ_{n_k} as a sequence of finite means has by $(5.2)^*$ some w^*-cluster point ψ_0' (which by $(5.2)^*$ is a mean). By $(5.1)^*$, ψ_0' also satisfies $L_a \psi_0' = \psi_0'$ for every $a \in G_0 \cup \{g_n\}$, and by Corollary 5.1, $L_g \psi_0' = \psi_0'$ for every $g \in G$, i.e., ψ_0' is a left invariant mean and $\psi_0' \in E$. But $\psi_{n_k} \in W_k \subset W_{k_0}$ for $k \geqq k_0$, and W_{k_0} is w^*-closed; therefore $\psi_0' \in W_{k_0}$ for each k_0, and therefore $\psi_0' \in W_k \cap E$ for $k = 1, 2, \cdots$. We can now write $\psi_0' \in W_k \cap E \subset S(\psi_0, 1/k) \cap E$; in other words, $\| \psi_0' - \psi_0 \| < 1/k$ for every k, and thus $\psi_0' = \psi_0$. We have proved that ψ_{n_k} has exactly one w^*-cluster point which is ψ_0. It follows by $(5.3)^*$ that $\lim_{k\to\infty} \psi_{n_k}(f) = \psi_0(f)$ for every $f \in m(G)$. But the ψ_{n_k} are finite means. Let $\psi_k' \in l_1(G)$ be such that $Q\psi_k' = \psi_{n_k}$. Then ψ_k' is a weak Cauchy sequence in $l_1(G)$, because if $f \in m(G)$, $\lim_{k\to\infty} f(\psi_k') = \lim_{k\to\infty} \psi_{n_k}(f) = \psi_0(f)$, and therefore $f(\psi_k')$ is a Cauchy sequence of reals. But by $(5.9)^*$, $l_1(G)$ is weakly sequentially complete, and therefore there exists a $\psi_0' \in l_1(G)$ such that $\lim_{k\to\infty} f(\psi_k') = f(\psi_0')$ for $f \in m(G)$. However $\psi_0(f) = \lim_{k\to\infty} \psi_{n_k}(f) = \lim_{k\to\infty} f(\psi_k') = f(\psi_0')$, so that $Q\psi_0' = \psi_0$. Thus $\psi_0 \in Q(l_1(G))$, which completes the proof.

COROLLARY 5.2. *If G is a denumerable semigroup with* $\dim Ml(G) = n$, $0 < n < \infty$, *then $Ml(G) \subset Q(l_1(G))$, and G contains exactly n finite groups which are (l.i.l.c.).*

Remark 5.1. We remark here that $\dim Ml(G) = 1$ is equivalent to G having unique left invariant mean, as is easily seen.

COROLLARY 5.3. *If G is an infinite left amenable group (not necessarily denumerable), then* $\dim Ml(G) = \infty$.

Proof. If G_0 is a denumerable left amenable group, and if $\dim Ml(G_0) < \infty$, then Corollary 5.2 implies the existence of finite left ideals in G_0, which cannot be.

If G is the group of our theorem, since $0 < n$, G is left amenable. If $G_0 \subset G$ is a subgroup, then G_0 is left amenable (see [2, p. 513]. There exists by [2, p. 534] an isometric linear operator[6] from the space $\{\varphi; \varphi \in m(G_0)^*, L_g' \varphi = \varphi$ for $g \in G_0\}$ into $\{\psi; \psi \in m(G)^*, L_g \psi = \psi$ for $g \in G\}$.

But $\{\psi; \psi \in m(G)^*, L_g \psi = \psi$ for $g \in G\}$ is by [13, pp. 280–281] (see also [7, p. 9]) the linear manifold spanned by $Ml(G)$. Thus if $\dim Ml(G) = n < \infty$, then $\dim Ml(G_0) \leqq n$ for each subgroup $G_0 \subset G$. Now since G is infinite, we can find a sequence $\{g_i\}$ of different elements in G. Let G_0 be the countable group generated by $\{g_i\}$; then $\dim Ml(G_0) \leqq n$, which is a contradiction (since G_0 is infinite and denumerable, and therefore $\dim Ml(G_0) = \infty$).

Remark 5.2. If G is a finite group, then it is known that it has a unique left [right] invariant mean. We can now rephrase Corollary 5.3 as follows: G is a left amenable group with unique left invariant mean if and only if G is finite (see Remark 5.1). We have proved thus

THEOREM 5.2. *If G is a left amenable group, then* $\dim Ml(G)$ *is either one or not finite. It is one if and only if G is finite.*

COROLLARY 5.4. *If G is a commutative countable semigroup, then*

$$\dim Ml(G) = 1 \quad or \quad \dim Ml(G) = \infty.$$

A commutative semigroup has at most one minimal ideal. We get thus a partial result of a more general result of I. S. Luthar in [12].

COROLLARY 5.5. *Let G be a countable semigroup which is left amenable and right amenable. If* $\dim Ml(G) = n < \infty$, *then $n = 1$ and $Ml(G) = Mr(G)$.*

---

[6] $L_a' = (l_a')^*$ and $(l_a' f)(g) = f(ag)$ for $f \in m(G_0)$ and $a, g \in G_0$.

(In other words, G has a unique right invariant mean which coincides with the left invariant one.)[7]

Proof. By Corollary 5.2, G has n finite groups $A_1, \cdots, A_n$ which are (l.i.l.c.), and by Lemma 3.1, $A = \bigcup A_i$ is a right minimal ideal. But if $n > 1$, then A_1, A_2 are different left minimal ideals and therefore disjoint. Let now $\varphi_0 \in Mr(G)$ ($Mr(G) \neq \emptyset$ by assumption). For any $g_0 \in G$, $(r_{g_0} 1_{Gg_0})(g) = 1_{Gg_0}(gg_0) = 1_G(g)$, and thus $1 = \varphi_0(1_G) = \varphi_0(1_{Gg_0})$. Let now $g_i \in A_i$, $i = 1, 2$. Then $A_i = Gg_i$, and thus

$$(*) \qquad 2 = \varphi_0(1_{A_1}) + \varphi_0(1_{A_2}) = \varphi_0(1_{A_1} + 1_{A_2}).$$

But A_1, A_2 are disjoint, and thus

$$1_{A_1}(g) + 1_{A_2}(g) \leqq 1_G(g),$$

so that $\varphi_0(1_{A_1} + 1_{A_2}) \leqq \varphi_0(1_G) = 1$, which contradicts $(*)$, and therefore $n = 1$. In other words $A = A_1$, and A is a finite group and (l.i.l.c.) and also a right minimal ideal. But A is also a (r.i.r.c.) because for $g \in G$, $Ag \subset A$ is a left ideal. But A is a minimal left ideal so that $A = Ag$ for $g \in G$, which implies that A is also a (r.i.r.c.). Moreover, A is the only group which is (r i.r.c.), for if B were another one, then A, B would be different right minimal ideals and therefore disjoint. If $a \in A$, $b \in B$, then

$$1 = \varphi_1(1_G) = \varphi_1(1_{bG}) = \varphi_1(1_B) \qquad \text{where } \varphi_1 \in Ml(G).$$

and

$$1 = \varphi_1(1_G) = \varphi_1(1_{aG}) = \varphi_1(1_A).$$

As before however $1_A(g) + 1_B(g) \leqq 1_G(g)$, so that

$$1 = \varphi_1(1_G) \geqq \varphi_1(1_A) + \varphi_1(1_B) = 2,$$

which is again a contradiction. Therefore G has exactly one finite group and (l.i.l.c.) which coincides with the only finite group which is (r.i.r.c.). Remark 3.3 implies now that G has a unique right invariant mean and a unique left invariant mean both of which coincide with φ_A.[8]

Remark 5.3. The above corollaries are true also for left amenable semigroups G which have a countable subsemigroup $G_0 \subset G$ such that the set of left cosets of G with respect to G_0 is countable (for instance, if G has some countable left ideal).

Remark 5.4. All the results in this paper are true if left is replaced by right and (l.i.l.c.) by (r.i.r.c.).

---

[7] And G contains a unique finite group and (l.i.l.c.) which coincides with the unique finite group and (r.i.r.c.) of G.

[8] Please note also the following result which in this author's opinion is of considerable interest: If G is a countable semigroup with right cancellation, then dim $Ml(G) = n$, $0 < n < \infty$, implies that G is a finite group and that $n = 1$. (If A_1 is some finite group and (l.i.l.c.) of G (see Corollary 5.2) with identity e_1 and $g \in G$, then $ge_1^2 = ge_1$. But the right cancellation implies that $ge_1 = g$, and since A_1 is a left ideal, we get that $G = A_1$.)

Added September 12, 1962. In a recent paper, *The second conjugate space of a Banach algebra as an algebra* (Pacific J. Math., vol. 11 (1961), pp. 847–870), P. Civin and B. Yood conjecture (p. 853) that for any infinite *commutative* group G, the radical of the second conjugate algebra $m(G)^*$ is infinite-dimensional. This is proved there (p. 853) only for the additive group of integers.

In our paper here we prove much more than this conjecture, namely: The radical of the second conjugate algebra $m(G)^*$, for any infinite *amenable* group G, is infinite-dimensional.

In order to see this we have only to remark the following: By the paper cited above (pp. 849–850),

$$J_1 = \{\varphi \in m(G)^*; \varphi(1_G) = 0 \text{ and } L_g \varphi = \varphi \text{ for each } g \in G\}$$

satisfies $J_1^2 = \{0\}$ and hence is included in the radical of $m(G)^*$. We choose now a fixed $\varphi_0 \in Ml(G)$. For each $\varphi \in Ml(G)$ we have $\varphi = (\varphi - \varphi_0) + \varphi_0$ and $\varphi - \varphi_0 \in J_1$, which implies that $Ml(G) \subset J_1 + \varphi_0$. The assumption that $\dim J_1 < \infty$ would imply that $\dim Ml(G) < \infty$ which contradicts Corollary 5.3 of our paper. (In this connection see also the last page of the next paper in this journal.)

References

1. A. H. Clifford, *Semigroups containing minimal ideals*, Amer. J. Math., vol. 70 (1948), pp. 521–526.
2. M. M. Day, *Amenable semigroups*, Illinois J. Math., vol. 1 (1957), pp. 509–544.
3. ———, *Normed linear spaces*, Berlin, Springer, 1958.
4. J. Dixmier, *Les moyennes invariantes dans les semi-groupes et leurs applications*, Acta Sci. Math. Szeged, vol. 12 (1950), pp. 213–227.
5. N. Dunford and J. T. Schwartz, *Linear operators*, New York, Interscience Publishers, 1958.
6. E. Følner, *On groups with full Banach mean values*, Math. Scand., vol. 3 (1955), pp. 243–254.
7. ———, *Note on groups with and without full Banach mean value*, Math. Scand., vol. 5 (1957), pp. 5–11.
8. ———, *Generalization of a theorem of Bogolioùboff to topological abelian groups, with an appendix on Banach mean values in non-abelian groups*, Math. Scand., vol. 2 (1954), pp. 1–18.
9. M. Jerison, *The set of all generalized limits of bounded sequences*, Canadian J. Math., vol. 9 (1957), pp. 79–89.
10. J. L. Kelley, *General topology*, New York, Van Nostrand, 1955.
11. M. Krein, D. Milman, and M. Rutman, *A note on basis in Banach space*, Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4), vol. 16 (1940), pp. 106–110 (in Russian).
12. I. S. Luthar, *Uniqueness of the invariant mean on an abelian semigroup*, Illinois J. Math., vol. 3 (1959), pp. 28–44.
13. M. M. Day, *Means for the bounded functions and ergodicity of the bounded representations of semigroups*, Trans. Amer. Math. Soc., vol. 69 (1950), pp. 276–291.

The Hebrew University of Jerusalem
Jerusalem, Israel