DECOMPOSITION OF SUPERMARTINGALES: THE UNIQUENESS THEOREM

BY

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In our preceding paper [1], *A decomposition theorem for supermartingales* (to which we shall refer from now on as the "existence paper"), we have proved that Doob's decomposition problem was solvable for a supermartingale belonging to the class (D). We shall be concerned here with the uniqueness of such a decomposition. It will be proved more precisely that, among all the possible decompositions for a potential, there is one and only one "natural" decomposition. Just as in the existence paper, we are adapting to the supermartingale case some proofs (from the paper [2]) which had been originally designed for the case of excessive functions and Markov processes.

The reader is referred to the existence paper for hypotheses, terminology, and notations. The numbering of theorems and definitions also follows that of the existence paper.

We begin with a proof of the general uniqueness theorem, which leads to a somewhat awkward characterization of the natural decomposition. A simpler one is given in the next section, with the help of a notion of "accessibility" of stopping times. We conclude with some applications to the theory of square integrable martingales.

1. DEFINITION 4. Let $Y = \{Y_t\}$ be a stochastic process, well adapted to the $\mathfrak{F}_t$ family, whose sample functions a.s. are right continuous and have limits from the left (including for $t = \infty$). Let ε and τ be two positive numbers ($\varepsilon > 0$, $0 < \tau \leqq \infty$). A sequence (T_n) of stopping times will be called a *(Y, ε)-chain over* $[0, \tau]$ if the following conditions are fulfilled:

(i) $T_1 = 0$; $T_n(\omega) \leqq T_{n+1}(\omega)$ a.s.

(ii) The equality $T_n(\omega) = \tau$ holds for n large enough, except for an ω set of measure 0.

(iii) The function $s \to Y_s(\omega)$ has a.s. an oscillation smaller than ε on each interval $[T_n(\omega), T_{n+1}(\omega)[$.

Some remarks may be made about this definition. We note first that it is not empty. If we take indeed $T_1 = 0$ and, inductively,

$$T_{n+1}(\omega) = \inf\{r \leqq \tau : r \geqq T_n(\omega), |Y_r(\omega) - Y_{T_n}(\omega)| > \varepsilon/2\},$$

we get a (Y, ε)-chain over $[0, \tau]$.

Let (S_n) and (T_n) be two chains of stopping times (i.e., sequences of stopping times which satisfy conditions (i) and (ii)). We construct a new chain (R_n), which will be called the *refinement* of (S_n) and (T_n), by ordering

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the values of $S_n(\omega)$ and $T_n(\omega)$ in their natural order, and taking for $R_p(\omega)$ the p^{th} such value. If (S_n) or (T_n) is a (Y, ε)-chain, the same is true for (R_n).

The hypotheses about $\{Y_t\}$ being the same as above, let $A = \{A_t\}$ be a right continuous increasing process. We may define, for each ε, the sum

$$\sum\nolimits_{t \leqq \tau, |Y_t(\omega) - Y_t^-(\omega)| \geqq \varepsilon} (A_t(\omega) - A_t^-(\omega))(Y_t(\omega) - Y_t^-(\omega)),$$

as it involves only a finite number of terms. If $\{A_t\}$ is an integrable process and $\{Y_t\}$ is bounded by some constant K, then this sum has a limit in the L^1 sense as ε tends to 0; we designate this limit by the notation

$$\sum\nolimits_{t \leqq \tau} (A_t - A_t^-)(Y_t - Y_t^-).$$

The following lemma is the key to the proof of the uniqueness theorem.

LEMMA 7. *Let $\{A_t\}$ be an integrable right continuous increasing process, and let $Y = \{Y_t\}$ be a right continuous martingale, whose absolute value is bounded by a constant K. Let (T_m) be a (Y, ε)-chain over $[0, \tau]$. The difference*

$$(*) \qquad \mathbf{E}[Y_\tau A_\tau - \sum\nolimits_m Y_{T_m}(A_{T_{m+1}} - A_{T_m}) \mid \mathfrak{F}_0]$$

tends in the L^1 norm, as ε tends to 0, to the random variable

$$(**) \qquad \mathbf{E}[\sum\nolimits_{t \leqq \tau} (A_t - A_t^-)(Y_t - Y_t^-) \mid \mathfrak{F}_0].$$

Proof. We establish first that, if we can prove the lemma using one particular sequence of (Y, ε)-chains $(\varepsilon \to 0)$, then it is true for all chains. Let indeed (S_n) and (T_n) be two (Y, ε)-chains, and let (R_n) be their refinement. We have, from the definition of (Y, ε)-chains over $[0, \tau]$,

$$\mathbf{E}[|\sum\nolimits_m Y_{T_m}(A_{T_{m+1}} - A_{T_m}) - \sum\nolimits_n Y_{R_n}(A_{R_{n+1}} - A_{R_n})|] \leqq \varepsilon \mathbf{E}[A_\tau]$$

and the same relation with (T_m) replaced by (S_m); the difference of the two corresponding sums $(*)$ is thus smaller than $2\varepsilon\mathbf{E}[A_\tau]$ in norm. According to this result, we may compute the limit of $(*)$ with the particular sequence of chains we are going to construct.

Let D_n be the set of all $\omega \in \Omega$ such that the function $t \to Y_t(\omega)$ possesses more than n jumps whose absolute value exceeds ε in the interval $[0, \tau]$. D_n decreases as n increases, and the lack of oscillatory discontinuities for the sample functions of $\{Y_t\}$ implies that $\bigcap_n D_n = \emptyset$ a.s. We may therefore choose n so large that

$$\int_{D_n} A_\tau \, d\mathbf{P} < \varepsilon.$$

Such a choice being made, we define the stopping times T_p inductively: First, $T_1 = 0$; then

$$T_{p+1}(\omega) = \inf\{r \leqq \tau : r \geqq T_p(\omega),$$
$$| Y_r(\omega) - Y_{T_p}(\omega) | > \varepsilon/2, \ A_r(\omega) - A_{T_p}(\omega) > \varepsilon/n\}.$$

The result is easily seen to be a (Y, ε)-chain. Now we have

$$\begin{aligned}\mathbf{E}[Y_\tau A_\tau \mid \mathfrak{F}_0] &= \mathbf{E}[\textstyle\sum_p Y_\tau(A_{T_{p+1}} - A_{T_p}) \mid \mathfrak{F}_0] \\ &= \mathbf{E}[\textstyle\sum_p \mathbf{E}[Y_\tau(A_{T_{p+1}} - A_{T_p}) \mid \mathfrak{F}_{T_{p+1}}] \mid \mathfrak{F}_0] \\ &= \mathbf{E}[\textstyle\sum_p Y_{T_{p+1}}(A_{T_{p+1}} - A_{T_p}) \mid \mathfrak{F}_0].\end{aligned}$$

On the other hand, the L^1 norm of the difference between (**) and the sum

$$(**1) \qquad \mathbf{E}[\textstyle\sum_p (Y_{T_{p+1}} - Y_{T_{p+1}}^-)(A_{T_{p+1}} - A_{T_{p+1}}^-) \mid \mathfrak{F}_0]$$

is smaller than $\varepsilon\mathbf{E}[A_\tau]$. It is therefore sufficient that we prove the property:

$$\begin{aligned}\mathbf{E}[|\textstyle\sum_p (Y_{T_{p+1}} - Y_{T_p})(A_{T_{p+1}} - A_{T_p}) \\ - \textstyle\sum_p (Y_{T_{p+1}} - Y_{T_{p+1}}^-)(A_{T_{p+1}} - A_{T_{p+1}}^-)|] \to 0,\end{aligned}$$

when ε goes to 0. This expectation is majorized by the sum of two integrals:

$$(2) \qquad \mathbf{E}[|\textstyle\sum_p (Y_{T_{p+1}}^- - Y_{T_p})(A_{T_{p+1}} - A_{T_p})|],$$

$$(3) \qquad \mathbf{E}[|\textstyle\sum_p (Y_{T_{p+1}} - Y_{T_{p+1}}^-)(A_{T_{p+1}}^- - A_{T_p})|].$$

The sum (2) is quite easily bounded: In fact, $| Y_{T_{p+1}}^- - Y_{T_p} |$ is smaller than ε, and $\sum_p (A_{T_{p+1}} - A_{T_p})$ than A_τ; (2) is therefore smaller than $\varepsilon\mathbf{E}[A_\tau]$. Things are less simple with (3): First, the sum

$$(4) \qquad \int_{\Omega - D_n} (\textstyle\sum_p | Y_{T_{p+1}} - Y_{T_{p+1}}^- | (A_{T_{p+1}}^- - A_{T_p}))\, d\mathbf{P}$$

can be split into two parts: the contribution of those terms $| Y_{T_{p+1}} - Y_{T_{p+1}}^- |$ which are greater than ε (a contribution which is itself smaller than $2K\varepsilon$, as there are at most n such terms by the definition of D_n, and the corresponding $(A_{T_{p+1}}^- - A_{T_p})$ are smaller than ε/n), and the contribution of the terms with $| Y_{T_{p+1}} - Y_{T_{p+1}}^- | \leqq \varepsilon$ (majorized by $\varepsilon \cdot \mathbf{E}[\sum_p (A_{T_{p+1}}^- - A_{T_p})] \leqq \varepsilon \cdot \mathbf{E}[A_\tau]$). The sum (4) is thus majorized by $\varepsilon \cdot (2K + \mathbf{E}[A_\tau])$.

The last integral we have to bound is

$$(5) \qquad \int_{D_n} (\textstyle\sum_p | Y_{T_{p+1}} - Y_{T_{p+1}}^- | (A_{T_{p+1}}^- - A_{T_p}))\, d\mathbf{P},$$

which is smaller than $2K \cdot \int_{D_n} A_\tau\, d\mathbf{P} \leqq 2K\varepsilon$.

The L^1 norm of the difference of (*) and (**) is thus majorized by $\varepsilon \cdot (4K + 5\mathbf{E}[A_\infty])$, for arbitrary intervals $[0, \tau]$ and (Y, ε)-chains. Lemma 7 is thus proved.[1]

We come now to the proof of our main uniqueness theorem.

THEOREM 5. *Let $\{X_t\}$ be a potential and belong to the class (D). There*

[1] A slight modification of this proof shows that the L^1 norm of the difference between (*) and (**) is smaller then $\varepsilon \cdot \mathbf{E}[A_\tau]$. We shall not need this result here.

is one, and only one, integrable right continuous increasing process $A = \{A_t\}$ *which generates* $\{X_t\}$ *and possesses the following property: The equality*

$$\mathbf{E}[\textstyle\sum_{t \leqq \tau} (A_t - A_t^-)(Y_t - Y_t^-)] = 0 \tag{5.1}$$

holds for every τ $(0 < \tau \leqq \infty)$ *and every bounded, right continuous martingale* $Y = \{Y_t\}$.

The right continuous increasing process A *will be called the natural increasing process which generates* $\{X_t\}$.

Proof of the uniqueness part. Let $\{A_t\}$ and $\{B_t\}$ be two integrable right continuous increasing processes which generate $\{X_t\}$ and satisfy condition (5.1) for every value of τ. Let Z be any bounded random variable, measurable with respect to $\mathfrak{F}_\tau$, and let $\{Y_t\}$ be a right continuous version of the martingale $\{\mathbf{E}[Z \mid \mathfrak{F}_t]\}$. According to Lemma 7, the integral $\mathbf{E}[Z \cdot A_\tau] = \mathbf{E}[Y_\tau \cdot A_\tau]$ and the sum

$$\begin{aligned}\mathbf{E}[\textstyle\sum_p Y_{T_p}(A_{T_{p+1}} - A_{T_p})] &= \mathbf{E}[\textstyle\sum_p \mathbf{E}[Y_{T_p}(A_{T_{p+1}} - A_{T_p}) \mid \mathfrak{F}_{T_p}]] \\ &= \mathbf{E}[\textstyle\sum_p Y_{T_p}(X_{T_p} - X_{T_{p+1}})],\end{aligned}$$

relative to one properly chosen chain of stopping times (T_p), differ by an arbitrarily small quantity; the same is true with $\{A_t\}$ replaced by $\{B_t\}$; the expectations $\mathbf{E}[Z \cdot A_\tau]$ and $\mathbf{E}[Z \cdot B_\tau]$ are thus equal. The random variable Z being arbitrary, A_τ and B_τ being $\mathfrak{F}_\tau$-measurable, this relation implies that A_τ and B_τ are a.s. equal. Using now the right continuity of the processes $\{A_t\}$ and $\{B_t\}$, we find that the set

$$\{\omega : \exists t,\ A_t(\omega) \neq B_t(\omega)\}$$

has probability 0; the two processes are thus essentially the same.

Proof of the existence part. We shall restate here, for the reader's convenience, some results which have been established in the existence paper.

There is a sequence of continuous increasing processes $\{A_t^n\}$, whose potentials $\{X_t^n\}$ increase to $\{X_t\}$, such that the random variables A_∞^n are uniformly integrable. Let $\{A_t'^n\}$ be any subsequence of $\{A_t^n\}$, such that the random variables $A_\infty'^n$ converge in the weak topology $w(L^1, L^\infty)$, and let $\{X_t'^n\}$ be the potential generated by $\{A_t'^n\}$; the relation $A_T'^n = \mathbf{E}[A_\infty'^n \mid \mathfrak{F}_T] - X_T'^n$ implies the weak convergence of the sequence $(A_T'^n)$ for arbitrary stopping times T. There is a right continuous increasing process $\{A_t'\}$, which generates $\{X_t\}$, such that the weak limit of the sequence $(A_\infty'^n)$ is A_∞'; the above relation shows that the sequence $(A_T'^n)$ converges to A_T' for every stopping time T. We shall first prove that $\{A_t'\}$ satisfies condition (5.1).

Let $Y = \{Y_t\}$ be a right continuous martingale, bounded by a constant K, and let (T_p) be a (Y, ε)-chain over $[0, s]$. We have just seen that the difference

$$\mathbf{E}[Y_s \cdot A_s'] - \mathbf{E}[\textstyle\sum_p Y_{T_p}(A_{T_{p+1}}' - A_{T_p}')] \tag{$*$}$$

is equal to

$$\mathbf{E}[Y_s \cdot A'_s] - \mathbf{E}[\textstyle\sum_p Y_{T_p}(X_{T_p} - X_{T_{p+1}})].$$

Similar relations hold for $\{A'^n_t\}$ and $\{X'^n_t\}$. As the processes $\{A'^n_t\}$ satisfy condition (5.1), Lemma 7 implies that the norm of the differences

$$(*n) \qquad \mathbf{E}[Y_s \cdot A'^n_s] - \mathbf{E}[\textstyle\sum_p Y_{T_p}(A'^n_{T_{p+1}} - A'^n_{T_p})]$$

is majorized by $\varepsilon \cdot (4K + 5 \sup_n \mathbf{E}[A'^n_\infty])$, a quantity which is finite, as the random variables A'^n_∞ are uniformly integrable. In order to establish (5.1) for $\{A'_t\}$, it is thus sufficient to prove that $(*n)$ tends to $(*)$ as n tends to infinity. According to the definition of weak convergence, the expectation $\mathbf{E}[Y_s \cdot A'^n_s]$ tends to $\mathbf{E}[Y_s \cdot A'_s]$, and it remains only to be shown that $\mathbf{E}[\sum_p Y_{T_p}(X'^n_{T_p} - X'^n_{T_{p+1}})]$ converges to $\mathbf{E}[\sum_p Y_{T_p}(X_{T_p} - X_{T_{p+1}})]$. Lebesgue's theorem implies that the expectation of each term in the first sum converges to that of the corresponding term in the second sum. We therefore have just to prove that the sums

$$\textstyle\sum_{p \geqq k} \mathbf{E}[Y_{T_p}(X'^n_{T_p} - X'^n_{T_{p+1}})]$$

tend to 0 as k tends to infinity, uniformly in n. We have

$$\begin{aligned} \mathbf{E}[Y_{T_p}(X'^n_{T_p} - X'^n_{T_{p+1}})] &= \mathbf{E}[Y_{T_p}(A'^n_{T_{p+1}} - A'^n_{T_p})] \\ &\leqq K \cdot \mathbf{E}[A'^n_{T_{p+1}} - A'^n_{T_p}] = K \cdot \mathbf{E}[X'^n_{T_p} - X'^n_{T_{p+1}}]. \end{aligned}$$

The sum of all these terms for $p \geqq k$ is thus majorized by $K \cdot \mathbf{E}[X'^n_{T_k}] \leqq K \cdot \mathbf{E}[X_{T_k}]$; $\{X_t\}$ belonging to the class (D), this quantity tends to 0, and the existence part of Theorem 5 is proved.

This reasoning gives some other results: Let $\{A''^n_t\}$ be another subsequence of the sequence $\{A^n_t\}$, with the property that the random variables A''^n_∞ converge in the weak topology; there is a right continuous increasing process $\{A''_t\}$ which generates $\{X_t\}$, such that the random variables A''^n_T converge to A''_T for any stopping time T. As $\{A'_t\}$ and $\{A''_t\}$ both are natural increasing processes, the uniqueness part of Theorem 5 shows that A'_T and A''_T are a.s. equal. A simple compactness argument may prove now that the whole sequence (A^n_T) must converge in the weak sense to A'_T.

The following theorem will be stated without proof. Similar results have been established in the paper [2] for additive functionals.

THEOREM 6. *Let $\{X_t\}$ be a potential of the class (D).*

(1) *Let $(A^i_t)_{i \in I}$ be the family of all natural increasing processes whose potentials $\{Z^i_t\}$ are dominated by $\{X_t\}$. The random variables $(A^i_\infty)_{i \in I}$ are uniformly integrable.*

(2) *Let $\{Z^n_t\}$, $\{Z_t\}$ be potentials dominated by $\{X_t\}$, generated by natural increasing processes $\{A^n_t\}$, $\{A_t\}$. Assume that Z^n_T converges weakly to Z_T for every stopping time T; then A^n_T converges weakly to A_T for every stopping time T.*

2. Property (5.1) seems somewhat difficult to handle. We are going now to characterize the natural increasing processes in a simpler way. A number of definitions concerning stopping times will be necessary for that purpose.

We shall adopt in the sequel the following convention, which will save us some artificial difficulties connected with the particular value $t = 0$. We shall define $\mathfrak{F}_t$ for $t < 0$ as the trivial σ-field $\{\emptyset, \Omega\}$; if $\{X_s\}$ is a supermartingale, we shall set $X_t = \mathbf{E}[X_0]$ for $t < 0$; if $\{A_s\}$ is a right continuous increasing process, we shall define $A_t = 0$ for $t < 0$. The time t will thus take its values in the interval $]-\infty, +\infty]$.

Definition 5. A stopping time T will be called a *time of discontinuity* for the family $\{\mathfrak{F}_t\}$ if there exists a sequence (S_n) of stopping times with the following properties:

(i) S_n a.s. increases to T.

(ii) The σ-field $\bigvee \mathfrak{F}_{S_n}$ generated by $\bigcup \mathfrak{F}_{S_n}$ is different from $\mathfrak{F}_T$.

Definition 6. A stopping time T will be said to be

(a) *totally inaccessible in the strong sense*, if it is not a.s. infinite and if, for every increasing sequence (S_n) of stopping times, which converges to $S \leqq T$, the event

$$\{\omega : \forall n,\ S_n(\omega) < S(\omega),\ S(\omega) = T(\omega) < \infty\}$$

has probability 0.

(b) *totally inaccessible in the weak sense*, if it is not a.s. infinite, and if, for every increasing sequence (S_n) of stopping times which converges to T, the event

$$\{\omega : \forall n,\ S_n(\omega) < T(\omega) < \infty\}$$

has probability 0.

(c) *inaccessible*, if there exists an event $A \in \mathfrak{F}_T$, with $\mathbf{P}[A] > 0$, such that the stopping time

$$\begin{aligned} T_A(\omega) &= T(\omega) \quad \text{for} \quad \omega \in A, \\ &= \infty \qquad\ \ \text{for} \quad \omega \notin A, \end{aligned}$$

is totally inaccessible in the weak sense.

(d) *accessible*, if T is not inaccessible. We emphasize that ∞ must be considered an accessible stopping time.

(e) *left approximable*, if there exists an increasing sequence (S_n) of stopping times, such that

$$\mathbf{P}\{\forall n,\ S_n < T;\ \lim_n S_n = T\} = 1.$$

Definition 7. Let $\{A_t\}$ be a right continuous increasing process. We shall say that $\{A_t\}$ *charges* a stopping time T if the event $\{A_T(\omega) \neq A_T^-(\omega)\}$ has a strictly positive probability.

Theorem 7. *A right continuous integrable increasing process $\{A_T\}$ is natural if and only if*

(1) *For every stopping time S, and every sequence (S_n) of stopping times which increases to S, the random variable A_S is measurable with respect to the σ-field $\bigvee_n \mathfrak{F}_{S_n}$.*

(2) *$\{A_t\}$ charges no stopping time T, totally inaccessible in the strong sense.*

Proof of the necessity. Let $\{A_t\}$ be a natural increasing process; we have seen that there is a sequence of continuous increasing processes $\{A_t^n\}$, such that A_S^n converges weakly to A_S. The random variables A_S^n being measurable with respect to $\bigvee_n \mathfrak{F}_{S_n}$, the same is true for A_S.

Before we establish assertion (2), we shall prove a lemma:

LEMMA 8. *Let T be a stopping time, totally inaccessible in the strong sense. There is a right continuous, uniformly integrable martingale $Y = \{Y_t\}$ whose only discontinuity is a unit jump at time T.*

Proof. Let $\{U_t\}$ be the right continuous increasing process

$$U_t(\omega) = 0 \quad \text{for} \quad t < T(\omega),$$

$$U_t(\omega) = 1 \quad \text{for} \quad t \geqq T(\omega).$$

The strong total inaccessibility of T is equivalent to the *regularity* of the potential generated by $\{U_t\}$; this potential is thus generated by another increasing process $\{V_t\}$, which is *continuous*. We therefore have

$$\mathbf{E}[V_\infty - U_\infty \mid \mathfrak{F}_t] = U_t - V_t.$$

The second member is the martingale $\{Y_t\}$ we are looking for.

Let us show now that assertion (2) is implied by this lemma. Let c be a positive constant, S^c the first time $|Y_t(\omega)|$ exceeds the value c, $\{Y_t^c\}$ the martingale obtained by stopping $\{Y_t\}$ at time S^c; the martingale $\{Y_t^c\}$ is bounded, and we have, using (5.1),

$$\mathbf{E}[(A_T - A_{T-})(Y_T^c - Y_{T-}^c)] = 0.$$

The result follows as c tends to infinity.

Proof of the sufficiency. We may obviously restrict ourselves to the case of a purely discontinuous increasing process $\{A_t\}$. We begin with an easy remark: Let $\{Y_t\}$ be a right continuous bounded martingale, and let (S_n) be a sequence of stopping times which increases to a stopping time S. If H is any event in $\bigvee_n \mathfrak{F}_{S_n}$, the limit

$$\lim_n \int_H (A_S - A_{S_n})(Y_S - Y_{S_n})\, d\mathbf{P}$$

is equal to 0; this follows from (1) and the martingale property.

We shall "extract" from $\{A_t\}$ a natural process in the following way: We choose a positive number ε, such that $\mathbf{P}\{\sup_t (A_t - A_t^-) \geqq \varepsilon\} > 0$. Let T be the instant of the first jump of $\{A_t\}$ whose size exceeds ε. The stopping time T cannot be totally inaccessible in the strong sense, and so we may find stopping times S_n and S, such that the sequence (S_n) increases to S, and the event

$$\{S_n(\omega) < S(\omega)\ \forall n,\ S(\omega) = T(\omega) < \infty\}$$

has a positive probability. We consider the following process:

$$\begin{aligned} B_t(\omega) &= 0 && \text{for} \quad t < S(\omega), \\ &= \lim_n [A_S(\omega) - A_{S_n}(\omega)] && \text{for} \quad t \geqq S(\omega). \end{aligned}$$

Let τ be any positive number. The expectation

$$\mathbf{E}[\textstyle\sum_{t \leqq \tau} (B_t - B_t^-)(Y_t - Y_t^-)]$$

is equal to

$$\lim_n \int_{\{S \leqq \tau\}} (A_S - A_{S_n})(Y_S - Y_{S_n})\, d\mathbf{P},$$

which is zero. Condition (5.1) is thus satisfied, and $\{B_t\}$ is a natural process; the process $\{A_t - B_t\}$ possesses properties (1) and (2), and so we may apply again the same procedure. If we take into account the fact that a series of positive increasing functions on $[0, +\infty]$, which converges to a finite value at infinity, converges uniformly on $[0, +\infty]$, we see that we may in fact go on transfinitely. The process $\{A_t\}$ must then be reached after a countable number of steps, and we find therefore that $\{A_t\}$ is a natural increasing process.

3. This section will contain a more detailed study of the notion of accessibility. Its results are independent of the preceding ones, and will not be used afterwards.

THEOREM 8. *A stopping time T is inaccessible if and only if there exists a right continuous, uniformly integrable martingale $\{Y_t\}$, such that*

$$\mathbf{P}[\{Y_T \neq Y_T^-\}] > 0.$$

Proof. If $\mathbf{P}[\{Y_T \neq Y_T^-\}] > 0$, there is an $\varepsilon > 0$ such that one at least of the events

$$A = \{Y_T(\omega) > Y_T^-(\omega) + \varepsilon\}, \qquad A' = \{Y_T(\omega) < Y_T^-(\omega) - \varepsilon\}$$

has a positive probability. Let us assume for instance that $\mathbf{P}[A] > 0$, and prove that T_A is weakly totally inaccessible. We consider a sequence of stopping times T_n which increase to T_A, and the sets

$$\begin{aligned} B_n &= \{\omega : T_n(\omega) < T_A(\omega)\}, \\ B &= \{\omega : \forall n,\ T_n(\omega) < T_A(\omega) < \infty\}. \end{aligned}$$

Using the martingale property, we get

$$\int_{B_n} Y_{T_n}\, d\mathbf{P} = \int_{B_n} Y_{T_A}\, d\mathbf{P},$$

Y_{T_A} being taken as Y_∞ when T_A is infinite. Passing to the limit and using uniform integrability, we obtain

$$\int_{\cap_n B_n} \lim Y_{T_n}\, d\mathbf{P} = \int_{\cap_n B_n} Y_{T_A}\, d\mathbf{P}$$

or

$$\int_{(\cap_n B_n)\cap\{T_A=\infty\}} Y_\infty\, d\mathbf{P} + \int_B Y_T^-\, d\mathbf{P} = \int_{(\cap_n B_n)\cap\{T_A=\infty\}} Y_\infty\, d\mathbf{P} + \int_B Y_{T_A}\, d\mathbf{P},$$

which contradicts the hypothesis that Y_T majorizes $Y_T^- + \varepsilon$ on B, unless we have $\mathbf{P}[B] = 0$; T_A is thus weakly totally inaccessible.

Let us suppose conversely that T is inaccessible. There is an event $A \in \mathfrak{F}_T$ such that the stopping time T_A is totally inaccessible in the weak sense. If it happens to be strongly totally inaccessible, then the existence of the martingale $\{Y_t\}$ follows from the proof of Theorem 7. If it is not strongly totally inaccessible, there exists a sequence of stopping times S_n, increasing to a stopping time S, such that the event

$$B = \{S_n < S\ \forall n,\ S = T_A < \infty\}$$

has a positive probability. We shall prove that B (which belongs to $\mathfrak{F}_S$) does not belong to the σ-field $\bigvee_n \mathfrak{F}_{S_n}$; this will imply the theorem, as we may take for $\{Y_t\}$ a right continuous version of the martingale $\{\mathbf{P}[B \mid \mathfrak{F}_t]\}$.

Assume indeed that B belongs to $\bigvee_n \mathfrak{F}_{S_n}$. It is possible to find an increasing sequence of integers n_k, and a sequence of events $B_k \in \mathfrak{F}_{S_{n_k}}$, such that

$$\mathbf{P}[(B \cap \complement B_k) \cup (B_k \cap \complement B)] < \mathbf{P}[B]/2^k.$$

Let C_k be the intersection $B_1 \cap B_2 \cap \cdots \cap B_k$, and let C be $\bigcap_k C_k$. C has a positive probability. Consider now the stopping times

$$R_k(\omega) = S_{n_k}(\omega) \quad \text{for } \omega \in C_k; \qquad R_k(\omega) = \infty \quad \text{for } \omega \notin C_k.$$

The sequence (R_k) increases to the stopping time T_C, which contradicts the weak total inaccessibility of T_A.

The proof of Theorem 8 gives the following:

COROLLARY. *Assume that the family $\{\mathfrak{F}_t\}$ has no time of discontinuity. Then total inaccessibility in the weak and the strong sense are equivalent.*

THEOREM 9. *A stopping time T is accessible if and only if it is possible to find an increasing sequence of stopping times (T_n) with the following properties:*

(i) $\lim_n T_n = T$; $T_n(\omega) < T(\omega)$ *a.s. for every* n,

(ii) $\mathfrak{F}_T = \bigvee_n \mathfrak{F}_{T_n}$.

Proof. The sufficiency follows immediately from Theorem 8 and from the well known martingale convergence theorems: no uniformly integrable martingale can have a discontinuity at time T under conditions (i) and (ii). Let us suppose conversely that T is an accessible stopping time, and let $\mathcal{G}$ be the family of all events $A \in \mathfrak{F}_T$ which possess the following property:

There is a sequence (S_n) of stopping times, which increases a.s. to T, such that $S_n(\omega) < T(\omega)$ a.s. on A for every n.

Let us prove that $\mathcal{G}$ contains every countable union of its events: We consider a sequence (A^p) of events which belong to $\mathcal{G}$, their associated sequences of stopping times (S_n^p), and their union A. For arbitrary integers m, p we choose an integer k_{mp} such that

$$\mathbf{P}[\{T(\omega) - S_{k_{mp}}^p(\omega) > 1/m\}] \leqq 1/2^{m+p}.$$

We may assume that k_{mp} increases with m for each p. We take then

$$S_m(\omega) = \inf_p S_{k_{mp}}^p(\omega).$$

The stopping times S_m are smaller than T, strictly smaller than T on A. It remains to be shown that $\lim_m S_m = T$ a.s. This follows from the inequality

$$\mathbf{P}[\{T(\omega) - S_m(\omega) > 1/m\}] \leqq \textstyle\sum_p 1/2^{m+p} = 1/2^m.$$

We may thus find an event A in $\mathcal{G}$ such that

$$\mathbf{P}[A] = \sup_{G\in\mathcal{G}} \mathbf{P}[G].$$

The stopping time T_{A^*} (where A^* is $\complement A$) must then be totally inaccessible in the weak sense; T being accessible, $\complement A$ must be a.s. empty, and this proves the existence of a sequence (T_n) which satisfies condition (i). Using accessibility again, we find that it satisfies condition (ii), and Theorem 9 is proved.

COROLLARY. *Assume that the family $\{\mathfrak{F}_t\}$ has no time of discontinuity. Then a stopping time T is accessible if and only if it is left approximable.*

If the family $\{\mathfrak{F}_t\}$ has no time of discontinuity, it is possible to give another simple characterization of the natural increasing process which generates a potential $\{X_t\}$ of the class (D). Let T be an accessible stopping time, B an event in $\mathfrak{F}_T$; the stopping time T_B being accessible, we choose a sequence (T_n) of stopping times which increases to T_B a.s. and takes values strictly smaller than T_B. Let $\{A_t\}$ be any right continuous increasing process which generates $\{X_t\}$. The relation

$$\mathbf{E}[X_{T_n} - X_{T_B}] = \mathbf{E}[A_{T_B} - A_{T_n}]$$

gives, through a passage to the limit,

$$\mathbf{E}[X_{T_B}^- - X_{T_B}] = \mathbf{E}[A_{T_B} - A_{T_B}^-]$$

or

$$\int_B (X_T^- - X_T)\, d\mathbf{P} = \int_B (A_T - A_T^-)\, d\mathbf{P}.$$

The event B being arbitrary in $\mathfrak{F}_T$, the integrated random variables must be a.s. equal. We may express this fact in the following manner: *The accessible discontinuities of $\{X_t\}$ a.s. are negative jumps.*

We construct now a right continuous increasing process in the following way: Let s be a number, and let $T_1^\varepsilon, T_2^\varepsilon, \cdots, T_k^\varepsilon$ be the instant of the first, second, $\cdots$, k^{th} jump of $\{X_t\}$ in the interval $[0, s]$, whose size exceeds ε. According to the proof of Theorem 9, each T_i^ε is the infimum of two stopping times $T_i'^\varepsilon$ and $T_i''^\varepsilon$, the first of which is accessible, the second being totally inaccessible. Then we define $\{A_s^\varepsilon\}$ as the sum

$$\sum_i (X_{T_i'^\varepsilon}^- - X_{T_i'^\varepsilon}).$$

We get a right continuous increasing process, which charges no totally inaccessible stopping time, and therefore is natural. We let ε tend to 0. The above computations show that the expectations $\mathbf{E}[A_\infty^\varepsilon]$ are majorized by $\mathbf{E}[A_\infty]$; these processes converge thus to a natural increasing process, which we shall call the *sum of all accessible discontinuities of* $\{X_t\}$. It is now easily seen that this process is the discontinuous part of the natural increasing process which generates $\{X_t\}$. Every right continuous increasing process which generates $\{X_t\}$ possesses these discontinuities; the natural process is thus the "smoothest" one among them.

4. The most interesting applications of our theorems occur in the theory of Markov processes. We are then given a compact metrizable space E and a strongly continuous Markov semigroup $\{P_t\}$ on the space $\mathcal{C}(E)$ of all real-valued continuous functions on E. We have explained at the beginning of [2] (after Hunt and Blumenthal) how to construct a canonical Markov process which admits that transition semigroup; we shall denote by $\{X_t\}$ its random variables, and by $\mathfrak{F}_t$ the completed σ-field generated by the random variables X_s, $s \leqq t$, as explained in [2].

THEOREM 10. (1) *The family* $\{\mathfrak{F}_t\}$ *has no time of discontinuity.*

(2) *A stopping time* T *relative to the family* $\{\mathfrak{F}_t\}$ *is accessible if and only if the process* $\{X_t\}$ *is a.s. sample continuous at time* T.

Proof. The well known theorem of Blumenthal asserts that, if T_n is an increasing sequence of stopping times with limit T, then X_{T_n} tends a.s. to X_T on the set where T is finite. It follows that a time T which bears discontinuities of the process $\{X_t\}$ cannot be left approximable.

Let us translate assertions (1) and (2) into the language of martingales: (1) means that, for every integrable random variable Y, every increasing sequence of stopping times S_n which converges to S, $\mathbf{E}[Y \mid \mathfrak{F}_{S_n}]$ converges a.s. to $\mathbf{E}[Y \mid \mathfrak{F}_S]$; (2) means that, if the sample functions of $\{X_t\}$ are continuous at time T, then the same is true for those of the martingale $\{\mathbf{E}[Y \mid \mathfrak{F}_t]\}$. Let H be a linear space of integrable random variables which is dense in L^1; it is sufficient to prove that these relations are true for random variables which belong to H. Let indeed (Z^n) be a sequence of random variables which converge to Y in the L^1 sense, and let $\{Z_t^n\}$, $\{Y_t\}$ be right continuous versions of the martingales $\mathbf{E}[Z^n \mid \mathfrak{F}_t]$, $\mathbf{E}[Y \mid \mathfrak{F}_t]$; using a well known inequality of Doob, we find that

$$\lambda \cdot \mathbf{P}[\sup_t \mid Y_t - Z_t^n \mid \geqq \lambda] \leqq \mathbf{E}[\mid Z^n - Y \mid].$$

If some continuity property holds for the sample functions of the $\{Z_t^n\}$ processes, it is thus also true for the sample functions of $\{Y_t\}$.

We choose now for the space H the linear space spanned by the random variables

$$Z = a_1 \circ X_{t_1} \cdot a_2 \circ X_{t_2} \cdots a_n \circ X_{t_n},$$

where $a_1, \cdots, a_n$ are functions in $\mathcal{C}(E)$. The random variables $Z_t = \mathbf{E}[Z \mid \mathfrak{F}_t]$ are easily computed; for instance, if t belongs to the interval $[t_i, t_{i+1}[$, we have

$$Z_t = a_1 \circ X_{t_1} \cdots a_i \circ X_{t_i} \cdot P_{t_{i+1}-t}(X_t, b),$$

where b denotes a function in $\mathcal{C}(E)$. The verification of the properties of this martingale is now standard, and will be left to the reader.

One particular consequence of Theorem 10 was used in the paper [2] (Lemma 2.1 of the second part); Dr. J. W. Woll pointed out to us that the proof given therein was wrong. No attempt will be made to fix it here, as the results we have proved are much more general.

5. We shall give some applications of the existence and uniqueness theorems we have proved to the theory of martingales. We consider only square integrable martingales, more precisely, martingales $\{Y_t\}$ such that

$$\sup_{t \in \mathbf{R}_+} \mathbf{E}[Y_t^2] < \infty.$$

This is only apparently stronger than square integrability, as any square integrable martingale satisfies this condition on every finite interval. The extension to general square integrable martingales will be obvious, and will be left to the reader.

We are going to prove that such a martingale may be decomposed into one "fixed discontinuity part", one sample continuous part, and a continuous sum of "pure jump type martingales." The analogy with Paul Lévy's decomposition of a process with independent increments is very striking. We shall need for that purpose the following definition:

DEFINITION 8. Let $\{X_t\}$ and $\{Y_t\}$ be two square integrable martingales; we shall say they are *orthogonal* if the process $\{X_t \cdot Y_t\}$ is a martingale.

This terminology is justified by the remark that, if one of the random variables X_0, Y_0 is equal to 0, then the expectation $\mathbf{E}[X_T \cdot Y_T]$ is equal to 0 for any stopping time T.

Let $\{X_t\}$ be a right continuous martingale with $\sup_t \mathbf{E}[X_t^2] < \infty$, and let (S_n) be any sequence of stopping times which increases to a stopping time S. The random variable $Z = \lim_n (X_S - X_{S_n})$ is easily seen to be square integrable; as it is $\mathfrak{F}_S$-measurable, and its conditional expectation with respect to $\bigvee_n \mathfrak{F}_{S_n}$ is equal to 0, any right continuous version $\{Z_t\}$ of the martingale $\{\mathbf{E}[Z \mid \mathfrak{F}_t]\}$ is equal to 0 up to time S, to Z after time S. Let $\{Y_t\}$ be the martingale $\{X_t - Z_t\}$; the martingales $\{Z_t\}$ and $\{Y_t\}$ are orthogonal. Let indeed A be an event in $\mathfrak{F}_t$; we have

$$\int_A Z \cdot Y_\infty \, d\mathbf{P} = \int_{A \cap \{t \geqq S\}} Z \cdot Y_\infty \, d\mathbf{P} + \int_{A \cap \{t < S\}} Z \cdot Y_\infty \, d\mathbf{P}.$$

The random variable $Z \cdot I_{\{t \geqq S\}}$ is $\mathfrak{F}_t$-measurable; the first integral is thus equal to

$$\int_{A \cap \{t \geqq S\}} Z \cdot Y_t \, d\mathbf{P}.$$

On the other hand, the event $A \cap \{t < S\}$ belongs to the σ-field $\bigvee_n \mathfrak{F}_{S_n}$, and thus to $\mathfrak{F}_S$. The second integral must be equal to

$$\int_{A \cap \{t < S\}} Z \cdot Y_S \, d\mathbf{P}$$

as Z is $\mathfrak{F}_S$-measurable; now the equality $Y_S = X_S - Z_S = \lim_n X_{S_n}$, and the fact that Z is orthogonal to any random variable which is measurable with respect to $\bigvee_n \mathfrak{F}_{S_n}$, imply that this integral is 0. Therefore, we have

$$\int_A Z \cdot Y_\infty \, d\mathbf{P} = \int_{A \cap \{t \geqq S\}} Z \cdot Y_t \, d\mathbf{P} = \int_A Z_t \cdot Y_t \, d\mathbf{P}.$$

The expectation $\mathbf{E}[Y_\infty^2]$ is smaller than $\mathbf{E}[X_\infty^2]$. Operating in the same way with $\{Y_t\}$ instead of $\{X_t\}$, and going on transfinitely, we split $\{X_t\}$ into a sum of two orthogonal martingales: $\{X'_t\}$, the "sum of all fixed discontinuities of $\{X_t\}$", and $\{X''_t\}$, a martingale which has no fixed discontinuities in the following sense: For any increasing sequence of stopping times S_n, which converges to S, $\lim_n X''_{S_n}$ is a.s. equal to X''_S.

The discontinuities of $\{X''_t\}$ are much more difficult to handle. We shall need some preliminary lemmas, one of which is an improvement of Lemma 7. It has been separated from Lemma 7 to avoid complicating too much the situation at that time.

Let $\{Y_s\}$ be a right continuous martingale with $\sup_s \mathbf{E}[Y_s^2] < \infty$, and let a, b be any numbers with $a < b$ and $0 \notin [a, b]$. We denote by $T_1, T_2, \cdots, T_n$ the first, second, $\cdots$, n^{th} instant in $[0, t]$ where a sample jump of $\{Y_s\}$ occurs, whose value lies in the interval $[a, b[$; the set of stopping times $T_1, \cdots, T_n, \cdots$ will be designated by $J_t(a, b)$. The union of all $J_t(a, b)$ for

$$(a, b) = (1, \infty), (\tfrac{1}{2}, 1), \cdots, (1/2^{n+1}, 1/2^n), \cdots$$

will be written J_t^+ ; we define in the same way J_t^- and $J_t = J_t^+ \cup J_t^-$.

LEMMA 9. *For every $\varepsilon > 0$, we have*

$$\mathbf{E}\left[\textstyle\sum_{T \in J_t(\varepsilon, \infty)} (Y_T - Y_{T^-})^2\right] \leqq \mathbf{E}[(Y_t - Y_0)^2].$$

Proof. Let (t_i) be any subdivision of the interval $[0, t]$; in the equality

$$\mathbf{E}[\textstyle\sum_i (Y_{t_{i+1}} - Y_{t_i})^2] = \mathbf{E}[(Y_t - Y_0)^2],$$

let us keep only the terms in the first member which correspond to intervals (t_i, t_{i+1}) such that $|Y_{t_{i+1}}(\omega) - Y_{t_i}(\omega)| \geqq \varepsilon$; taking a sequence of subdivisions whose step tends to 0 and applying Fatou's lemma, we obtain the above inequality.

Let ε decrease to 0; more and more stopping times are added to the above sum of the squares of jumps. As its expectation remains bounded, it increases a.s. to a random variable which we designate by $\sum_{T \epsilon J_t} (Y_T - Y_T^-)^2$; we may define in the same manner the sum extended to $T \epsilon J_t^+$ or J_t^-.

Let $\{A_t\}$ be a right continuous increasing process, such that $\mathbf{E}[A_\infty^2] < \infty$. We may define the sum

$$\textstyle\sum_{T \epsilon J_t^+(\varepsilon,\infty)} (A_T - A_T^-)(Y_T - Y_T^-).$$

As ε tends to 0, this random variable increases; Schwarz's inequality and the above lemma show that its expectation remains bounded; it therefore converges in the L^1 sense. Operating in the same manner with J_T^-, we define the integrable random variable

$$\textstyle\sum_{T \epsilon J_t} (A_T - A_T^-)(Y_T - Y_T^-).$$

LEMMA 10. *Let $\{Y_t\}$ be a martingale, with $\sup_t \mathbf{E}[Y_t^2] < \infty$; (T_m) a (Y, ε)-chain over $[0, \infty]$; $\{A_t\}$ a right continuous increasing process with $\mathbf{E}[A_\infty^2] < \infty$. When ε tends to 0, the conditional expectation*

$$(*) \qquad \mathbf{E}[A_\infty Y_\infty - \textstyle\sum_m Y_{T_m}(A_{T_{m+1}} - A_{T_m}) \mid \mathfrak{F}_0]$$

converges in the L^1 norm to the conditional expectation

$$(**) \qquad \mathbf{E}[\textstyle\sum_{T \epsilon J_\infty} (Y_T - Y_T^-)(A_T - A_T^-) \mid \mathfrak{F}_0].$$

Proof. We shall only indicate the differences from the proof of Lemma 7, which arise from the fact that $\{Y_t\}$ is no longer assumed to be bounded. In fact, this boundedness was not used before we had to majorize (4) and (5). To deal with (5), for instance, we may split it into one innocent part,

$$\int_{D_n} \textstyle\sum' |Y_{T_{p+1}} - Y_{T_{p+1}}^-| \cdot (A_{T_{p+1}}^- - A_{T_p})\, d\mathbf{P} \leqq \varepsilon \cdot \mathbf{E}[A_\infty],$$

(in the integrand, the sum $\sum'$ is carried over the values of p satisfying $|Y_{T_{p+1}} - Y_{T_{p+1}}^-| \leqq \varepsilon$), and a disturbing one, the corresponding sum extended over the terms such that $|Y_{T_{p+1}} - Y_{T_{p+1}}^-|$ is greater than ε. As $A_{T_{p+1}} - A_{T_{p+1}}^-$ is smaller than ε, this sum is majorized by the integral, over D_n, of the integrable random variable $\sum_{T \epsilon J_\infty} (Y_T - Y_T^-)^2$. If n has been taken initially large enough, the integral

$$\int_{D_n} (\textstyle\sum_T (Y_T - Y_T^-)^2)\, d\mathbf{P}$$

is arbitrarily small. The integral (4) can be handled in a similar way, and the lemma follows.

The result of Lemma 10 may be easily carried over to an interval $[t, \infty]$. The (Y, ε)-chains being now relative to this interval, the sum

$$(*) \qquad \mathbf{E}[A_\infty Y_\infty - A_t Y_t - \textstyle\sum_m Y_{T_m}(A_{T_{m+1}} - A_{T_m}) \mid \mathfrak{F}_t]$$

converges in the L^1 norm to

$$(**) \qquad \mathbf{E}[\textstyle\sum_{T \in J_\infty, T(\omega) > t} (Y_T - Y_{\bar{T}})(A_T - A_{\bar{T}}) \mid \mathfrak{F}_t].$$

We are coming back to the case of a right continuous, square integrable martingale $Y = \{Y_t\}$ without fixed discontinuities. We begin with the following lemma:

LEMMA 11. *For every $\varepsilon > 0$, the process*

$$S_t^\varepsilon = \textstyle\sum_{T \in J_t(\varepsilon, \infty)} (Y_T - Y_{\bar{T}})$$

is an integrable, right continuous increasing process.

Proof. The only assertion which is not quite obvious is that about integrability; it follows from Lemma 9 and the inequality, if T belongs to $J_t(\varepsilon, \infty)$,

$$Y_T - Y_{\bar{T}} \leqq (Y_T - Y_{\bar{T}})^2/\varepsilon.$$

Let now ε' be a positive number smaller than ε. We obviously have

$$S_t^{\varepsilon'} - S_t^\varepsilon = \textstyle\sum_{T \in J_t(\varepsilon', \varepsilon)} (Y_T - Y_{\bar{T}}).$$

This difference, which we shall designate by $S_t^{\varepsilon'\varepsilon}$, is a right continuous increasing process. As its discontinuities are jumps of $\{Y_t\}$, and $\{Y_t\}$ has no fixed discontinuity, they must be totally inaccessible in the strong sense. The potential $\{S_t^{\varepsilon'\varepsilon}\}$ is thus regular, and Theorem 3 from the existence paper implies that $\{S_t^{\varepsilon'\varepsilon}\}$ is generated also by one continuous increasing process $\{G_t^{\varepsilon'\varepsilon}\}$. We may define in the same manner processes $\{G_t^\varepsilon\}$. The uniqueness theorem implies that

$$G_t^{\varepsilon'} = G_t^{\varepsilon'\varepsilon} + G_t^\varepsilon .$$

We denote by $\{C_t^{\varepsilon'\varepsilon}\}$, $\{C_t^\varepsilon\}$ the martingales $\{S_t^{\varepsilon'\varepsilon} - G_t^{\varepsilon'\varepsilon}\}$, $\{S_t^\varepsilon - G_t^\varepsilon\}$.

LEMMA 12. *The random variables $C_\infty^{\varepsilon'\varepsilon}$ are square integrable, and the following properties hold:*

(i) $\mathbf{E}[(C_\infty^{\varepsilon'\varepsilon} - C_t^{\varepsilon'\varepsilon})^2 \mid \mathfrak{F}_t] = \mathbf{E}[\sum_{T \in J_\infty(\varepsilon', \varepsilon), T(\omega) > t} (Y_T - Y_{\bar{T}})^2 \mid \mathfrak{F}_t]$.

(ii) *The martingale $\{C_t^{\varepsilon'\varepsilon}\}$ is orthogonal to every square integrable martingale $\{X_t\}$ which has no common jump with it (which means that $A_T - A_{\bar{T}} = 0$ a.s. for every $T \in J_\infty(\varepsilon', \varepsilon)$).*

Proof of (i). We shall establish together the integrability of $(C_\infty^{\varepsilon'\varepsilon})^2$ and the relation (i). Though the proof would work with any stopping time T instead of t, we shall give it only for the case $t = 0$. Let η be a positive number, and let (T_i) be a $(\{G_t^{\varepsilon'\varepsilon}\}, \eta)$-chain of stopping times over $[0, \infty]$. We already know that $S_{T_{i+1}}^{\varepsilon'\varepsilon} - S_{T_i}^{\varepsilon'\varepsilon}$ is square integrable, and the same is true for $G_{T_{i+1}}^{\varepsilon'\varepsilon} - G_{T_i}^{\varepsilon'\varepsilon}$, which is bounded by η because $\{G_t^{\varepsilon'\varepsilon}\}$ is a continuous increasing process. We may therefore write, using the martingale property,

$$\begin{aligned}\mathbf{E}[(C_\infty^{\varepsilon'\varepsilon})^2 \mid \mathfrak{F}_0] &= \mathbf{E}[\textstyle\sum_i (C_{T_{i+1}}^{\varepsilon'\varepsilon} - C_{T_i}^{\varepsilon'\varepsilon})^2 \mid \mathfrak{F}_0]\\ &= \mathbf{E}[\textstyle\sum_i (S_{T_{i+1}}^{\varepsilon'\varepsilon} - S_{T_i}^{\varepsilon'\varepsilon})^2 \mid \mathfrak{F}_0] + \mathbf{E}[\textstyle\sum_i (G_{T_{i+1}}^{\varepsilon'\varepsilon} - G_{T_i}^{\varepsilon'\varepsilon})^2 \mid \mathfrak{F}_0]\\ &\qquad + 2\mathbf{E}[\textstyle\sum_i (S_{T_{i+1}}^{\varepsilon'\varepsilon} - S_{T_i}^{\varepsilon'\varepsilon})(G_{T_{i+1}}^{\varepsilon'\varepsilon} - G_{T_i}^{\varepsilon'\varepsilon}) \mid \mathfrak{F}_0].\end{aligned}$$

When η tends to 0, the first sum in the second member tends to the second member of (i), while the two last sums converge to 0 in the L^1 sense (their expectations being respectively bounded by $\eta\cdot\mathbf{E}[G_\infty^{\varepsilon'\varepsilon}]$ and $2\eta\cdot\mathbf{E}[S_\infty^{\varepsilon'\varepsilon}]$.

Proof of (ii). We shall apply Lemma 10 (or rather the generalization which has been pointed out after Lemma 10), replacing in its statement $\{Y_t\}$ by $\{X_t\}$, and $\{A_t\}$ by $\{C_t^{\varepsilon'\varepsilon}\}$, which is a difference of right continuous integrable increasing processes. The hypothesis that $\{X_t\}$ and $\{C_t^{\varepsilon'\varepsilon}\}$ have no common jumps implies that (**) is equal to 0, and so the same must be true for (*). On the other hand, $\{C_t^{\varepsilon'\varepsilon}\}$ being a martingale, each expectation

$$\mathbf{E}[X_{T_m}(C_{T_{m+1}}^{\varepsilon'\varepsilon} - C_{T_m}^{\varepsilon'\varepsilon}) \mid \mathfrak{F}_t]$$

is equal to 0. It follows that $\mathbf{E}[X_\infty C_\infty^{\varepsilon'\varepsilon} - X_t C_t^{\varepsilon'\varepsilon} \mid \mathfrak{F}_t] = 0$, and this is the definition of orthogonality.

THEOREM 11. *Let $\{Y_t\}$ be a right continuous martingale without fixed discontinuities, such that $\mathbf{E}[Y_\infty^2] < \infty$. There exists one, and only one, right continuous martingale $\{C_t^+\}$ with the following properties:*

(i) *The jumps of $\{C_t^+\}$ are exactly the positive jumps of $\{Y_t\}$.*

(ii) *$\{C_t^+\}$ is orthogonal to every martingale which has no jump in common with it.*

The martingale $\{C_t^+\}$ will be called the compensated sum of the positive jumps of $\{Y_t\}$.

Proof. We shall show that the random variables $C_\infty^\varepsilon = C_\infty^{\varepsilon\infty}$ converge in the L^2 sense to a random variable C_∞^+; let $\{C_t^+\}$ be a right continuous version of the martingale $\{\mathbf{E}[C_\infty^+ \mid \mathfrak{F}_t]\}$; as the conditional expectation operator diminishes norms, it will follow that C_T^ε converges to C_T^+ for every stopping time T.

It is sufficient to remark that

$$\mathbf{E}[(C_\infty^{\varepsilon'} - C_\infty^\varepsilon)^2] = \mathbf{E}[(C_\infty^{\varepsilon'\varepsilon})^2] = \mathbf{E}[\textstyle\sum_{T\in J_\infty(\varepsilon',\varepsilon)} (Y_T - Y_T^-)^2]$$

(Lemma 12); Lemma 9 implies that (C_∞^ε) satisfies the Cauchy condition, and we thus have convergence in L^2.

Let us prove now that, if T is a stopping time which belongs to $J_\infty(\varepsilon, \infty)$, then $\{C_t^+\}$ has at time T a jump equal to $Y_T - Y_T^-$. Indeed, if this were not true, the martingale $\{C_t^+ - C_t^{\varepsilon'}\}$ would have a jump equal to

$$(C_T^+ - C_T^{+-}) - (Y_T - Y_T^-)$$

at time T for $\varepsilon' < \varepsilon$. According to Lemma 9, we would have

$$\mathbf{E}[(C_\infty^+ - C_\infty^{\varepsilon'})^2] \geqq \mathbf{E}[(C_T^+ - C_T^{+-}) - (Y_T - Y_T^-))^2],$$

and this would contradict the convergence in norm which has been established.

On the other hand, the relation

$$\mathbf{E}[(C_\infty^+)^2] = \mathbf{E}[\textstyle\sum_{T \epsilon J_\infty^+} (Y_T - Y_T^-)^2]$$

which follows from Lemma 12(i), implies together with Lemma 9 and the above remarks, that $\{C_t^+\}$ cannot have any jump besides the positive jumps of $\{Y_t\}$.

If a martingale $\{X_t\}$ has no jump in common with $\{C_t^+\}$, it has no jump in common with any $\{C_t^\varepsilon\}$; it is thus orthogonal to $\{C_t^\varepsilon\}$ (by Lemma 12(ii)), and a passage to the limit gives (ii).

Only the uniqueness remains to be shown. Let $\{D_t\}$ be any martingale which has the same jumps as $\{C_t^+\}$; $\{D_t\}$ may be written $\{X_t + C_t^+\}$, with a sample continuous martingale $\{X_t\}$. If $\{D_t\}$ has property (ii), $\{X_t\}$ must be orthogonal both to $\{D_t\}$ and $\{C_t^+\}$, and thus to itself; this proves that $\{D_t\} = \{C_t^+\}$.

All that has been done for positive jumps could have been done for negative ones; this would have led to processes $\{C_t^{-\varepsilon}\}$, $\{C_t^-\}$. The martingale $C_t = C_t^+ - C_t^-$ is the compensated jumps of all jumps of $\{Y_t\}$.

Some emphasis must be put on condition (ii) of Theorem 11: It characterizes the "pure jump type" martingales, i.e., those martingales which possess no continuous part. The proof at the beginning of Section 5 amounts to saying that this property is shared by the fixed discontinuity part of any martingale.

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