AN EXTENSION OF RAUCH'S METRIC COMPARISON THEOREM AND SOME APPLICATIONS[1]

BY

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1. Introduction

In [8] Toponogov proved a theorem relating the angles of a triangle in a Riemannian manifold V to those of a triangle having the same lengths of sides in the simply connected two-dimensional space which has constant curvature equal to the lower bound of sectional curvatures of V. Toponogov's proof used a theorem of Alexandrov for surfaces. But for triangles whose side-lengths are not too big in comparison to the upper bound of sectional curvatures of V, Toponogov's theorem is equivalent to Rauch's metric comparison theorem [6, p. 36]. In this article we want to give a new proof of Toponogov's theorem, a proof using only Rauch's metric comparison theorem. Strictly speaking the proof will use too a slight extension of Rauch's theorem; this extension will be proved in §2 as Theorem 1. In itself, this extension is of interest; we give in §3 a first application of it as Proposition 1. In §4 another application of the extension is a very short proof of a theorem of Toponogov concerning manifolds of maximum diameter: Theorem 2 below. And in §5 we give the new proof of Toponogov's theorem.

2. The extension

Definitions and notations are those of [1], [2], [3]. Moreover by $S_n(\delta)$ we shall denote the simply connected n-dimensional manifold whose curvature is constant and equal to δ (and $S_2(\delta) = S(\delta)$); that is, if $\delta > 0$, a sphere; if $\delta = 0$, a euclidean space; if $\delta < 0$, a hyperbolic space. In this paper V will always be a complete Riemannian manifold of dimension n whose sectional curvatures form a set $\mathrm{curv}(V)$ satisfying $\delta \leqq \mathrm{curv}(V) \leqq 1$. Rauch's metric comparison theorem works with a one-parameter family of geodesics of V issuing from a fixed point $p \in V$ and asserts (if some nonconjugacy hypothesis is verified) that the length of the curve of V built up by the extremities of the geodesics of the family is less than or equal to the length of the curve built up by the extremities of the one-parameter family of geodesics in $S_n(\delta)$ associated in a natural way with the starting family in V. Now it can be helpful to have an analogous theorem in which the family of geodesics one works with is formed by geodesics whose starting points run through a given geodesic, and which are orthogonal at these points to the given geodesic. We shall now write down in a more precise way the material for the theorem we anticipate.

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Let $\Gamma = \{\gamma(s)\}$ $(0 \leqq s \leqq l)$ be a geodesic of V, and $\Lambda(s) = \{\lambda(t, s)\}$ a one-parameter family of geodesics of V such that (a) $0 \leqq s \leqq l$; (b) $0 \leqq t \leqq m(s)$; (c) for any s, $\lambda(0, s) = \gamma(s)$; (d) for any s, $\langle \lambda_t'(0, s), \gamma'(s) \rangle = 0$; (e) for any s, $\nabla_{\gamma'(s)}(\lambda_t'(0, s)) = 0$. Build up in $S(\delta)$ the natural associated situation in the following way. First let $\tilde{\Gamma} = \{\tilde{\gamma}(s)\}$ $(0 \leqq s \leqq l)$ be any fixed geodesic of $S(\delta)$ of length equal to l, and let $X \in T_{\tilde{\gamma}(0)}$ be any fixed unit vector tangent to $S(\delta)$ at the origin $\tilde{\gamma}(0)$ of $\tilde{\Gamma}$ and normal to $\tilde{\Gamma}$. Let $\{X(s)\}$ $(0 \leqq s \leqq l)$ be the field of vectors tangent to $S(\delta)$ along $\tilde{\Gamma}$ defined by the conditions: (a) $X(s)$ is continuous in s; (b) $X(0) = X$; (c) for any s, $X(s)$ is normal to $\tilde{\Gamma}$ at $\tilde{\gamma}(s)$. Now one can define uniquely a one-parameter family $\tilde{\Lambda}(s) = \{\tilde{\lambda}(t, s)\}$ of geodesics in $S(\delta)$ by the following conditions: (a) $0 \leqq s \leqq l$; (b) $0 \leqq t \leqq m(s)$; (c) $\tilde{\lambda}(0, s) = \tilde{\gamma}(s)$ for any s; (d) for any s, $\tilde{\lambda}_t'(0, s) = X(s)$. The extension of Rauch's theorem concerns the curves

$$\Omega = \{\omega(s) = \lambda(m(s), s)\} \quad \text{and} \quad \tilde{\Omega} = \{\varpi(s) = \tilde{\lambda}(m(s), s)\} \quad (0 \leqq s \leqq l)$$

which are the loci of the extremities of the geodesics $\Lambda(t)$ and $\tilde{\Lambda}(t)$, respectively.

THEOREM 1. *If for any s, $m(s) \leqq \pi/2$, one has for the lengths of Ω and $\tilde{\Omega}$ the following relation*: $l(\Omega) \leqq l(\tilde{\Omega})$.

The proof is that given in [6, pp. 36–39]; we shall only indicate the differences due to the fact that one is working with a family of geodesics which are no longer issuing from a fixed point. What corresponds to the nonexistence of a point conjugate to p on the geodesics issuing from p is now the nonexistence of a focal point for the set of geodesics normal to Γ. Four differences are now to be noted. First, in $S_n(1)$ a first focal point is always at distance $\pi/2$. Second, the fundamental lemma [6, p. 32] is still valid when the nonconjugacy is replaced by the nonfocal hypothesis, with the difference that one has to replace the curve $\mu(s)$ joining the endpoints by a curve $\mu(s)$ normal to Γ and ending at $\eta(s_2)$ (notations are those of [6]); the validity of the fundamental lemma, used for $S_n(1)$ and V, implies first that there are no focal points in V at distance less than $\pi/2$, so that the hypothesis $m(s) < \pi/2$ will assure us of the nonfocal-points-in-V hypothesis; now the fundamental lemma can be used at its place (p. 38) in the proof of the metric comparison theorem. Third, in line 5 from the bottom of page 38 in [6], one has now $2\eta_\alpha'(0) \cdot \eta_\alpha(0) = 0$, no longer because $\eta_\alpha(0) = 0$, but now because $\eta_\alpha'(0) = 0$; this is due to the condition $\nabla_{\gamma'(s)}(\lambda_t'(0, s)) = 0$ for the family $\Lambda(s)$. Fourth, the passage to the limit in the relation (62) of page 39 is not necessary because here one can apply (61) directly since $\eta_\alpha(0)\eta_\alpha(0) = \tilde{\eta}_\alpha(0)\tilde{\eta}_\alpha(0) = 1$. Remark also that the above proof works only for $m(s) < \pi/2$ for any s, but if one knows only that $m(s) \leqq \pi/2$, one can use a trivial limit argument to conclude the proof.

It is of interest for §2 of this paper to know when $l(\Omega) = l(\tilde{\Omega})$. The answer is easy; looking at Rauch's proof, one sees that $l(\Omega) = l(\tilde{\Omega})$ is equivalent to the fact that the two-dimensional submanifold of V formed by the union of the $\Lambda(s)$ is totally geodesic and of curvature everywhere equal to δ.

3. About a lemma of Klingenberg

KLINGENBERG'S LEMMA [4, Theorem 1, p. 655]. *Let V be a compact Riemannian orientable manifold of even dimension such that $0 < \operatorname{curv}(V) \leqq 1$; let $C(p)$ denote the cut-locus of p in V. Then, for any p in V and q in $C(p)$, one has $d(p, q) \geqq \pi$.*

One can ask about the validity of Klingenberg's Lemma when the hypothesis is weakened to $0 \leqq \operatorname{curv}(V) \leqq 1$. From [5] one knows that the answer is still yes when the dimension of V is equal to 2. We shall not prove the desired result but only the following weaker result:

PROPOSITION 1. *Let V be a compact manifold, Riemannian, orientable, of even dimension, such that $0 \leqq \operatorname{curv}(V) \leqq 1$. Then if there exist two points p, q of V such that* (a) $q \in C(p)$; (b) $d(p, q) < \pi$, *then there exists a one-parameter family $\Gamma(t)$ of closed geodesics of V such that* (a) $-\infty < t < +\infty$; (b) *for any t, $l(\Gamma(t)) = k < 2\pi$;* (c) *the union of the $\Gamma(t)$ is a totally geodesic submanifold of V of dimension 2 whose curvature is everywhere zero.*

In fact Klingenberg's proof of his lemma is based on this: If there exist $p, q \in V$ such that $d(p, q) < \pi$ and $q \in C(p)$, then there exists a closed geodesic $\Gamma = \{\gamma(s)\}$ $(0 \leqq s \leqq l)$ of length $l \leqq 2\pi$, enjoying the property that there does not exist a sequence of curves of lengths $< l(\Gamma) = l$ and converging toward Γ. Now by an old trick of Synge [7], there exists a field $\{Y(s)\}$ $(0 \leqq s \leqq l; Y(0) = Y(l))$ of unit vectors such that (a) for any s, $Y(s) \in T_{\gamma(s)}$; (b) for any s, $\langle Y(s), \gamma'(s)\rangle = 0$; (c) for any s, $\nabla_{\gamma'(s)} Y(s) = 0$. Define now a one-parameter family of geodesics of V: $\{\Lambda(s)\}$ $(\Lambda(s) = \{\lambda(t, s)\}$; $0 \leqq t \leqq \pi/2; 0 \leqq s \leqq l)$ by the conditions: (a) for any s, $\lambda(0, s) = \gamma(s)$; (b) for any s, $\lambda'_t(0, s) = Y(s)$. Put $\Omega_t = \{\lambda(t, s)\}$ (t fixed; $0 \leqq s \leqq l$). Let now $\hat{\Gamma}$ be a line of length equal to 1 in the euclidean plane $S(0)$, and along $\hat{\Gamma}$ let $\hat{\Lambda}$ be the continuous family of lines of length t and orthogonal to $\hat{\Gamma}$; call $\hat{\Omega}_t$ the locus corresponding in $S(0)$ to Ω_t in V. One has, for any t,

$$l(\hat{\Omega}_t) = l(\hat{\Gamma}) = l.$$

But Theorem 1 yields

$$l(\Omega_t) \leqq l(\hat{\Omega}_t) = l.$$

What one said above about Klingenberg's argument implies that there exists an $\varepsilon > 0$ such that, for any t such that $0 \leqq t \leqq \varepsilon$, one should have $l(\Omega_t) = l$. So equality has to be attained in Theorem 1, and we saw after the proof of the theorem that this implies that the union $\bigcup_s \Lambda(s)$ $(0 \leqq s \leqq l; 0 \leqq t \leqq \varepsilon)$ is a totally geodesic submanifold of V, of curvature zero. One can write $\bigcup_s \Lambda(s) = \bigcup_{0 \leqq t \leqq \varepsilon} \Omega_t$; and so $\Omega_t = \Gamma(t)$ $(0 \leqq t \leqq \varepsilon)$ is a family of geodesics having the property required in the conclusion of Proposition 1 except that

t ranges only over $[0, \varepsilon]$. But changing now the field $\{Y(s)\}$ into the field $\{-Y(s)\}$ will give the same property for t running over $[\eta, \varepsilon]$ with $\eta < 0$ and $\varepsilon > 0$; one can repeat the above argument with $\Gamma(\eta)$ and $\Gamma(\varepsilon)$; one knows, moreover, that the limit of closed geodesics of the same length is a closed geodesic of the same length. One thus gets Proposition 1.

Remark. Looking for Klingenberg's lemma for an odd-dimensional simply connected manifold of strictly positive curvature, one might think, as was pointed out to us by L. W. Green, of constructing the Riemannian product $V \times V$, which verifies the hypothesis of Proposition 1. But we want to remark that this proposition will not help; in fact the cut-locus of $(p, q) \in V \times W$ is easily verified to be

$$C((p, q)) = (C(p) \times W) \cup (V \times C(q)),$$

where of course $C(p)$ (resp. $C(q)$) means the cut-locus of p (resp. q) in V (resp. W). And so the minimum distance of (p, q) to its cut-locus (which is used in Klingenberg's argument) will be attained exactly for points (r, s) where $r = p$ and s minimizes the distance between q and $C(q)$, or $s = q$ and r minimizes the distance between p and $C(p)$. In one of these situations the existence of a totally geodesic submanifold asserted by Proposition 1 is trivial. See also M. Berger, *On the diameter of some Riemannian manifolds*, Department of Mathematics, University of California, 1962.

4. Manifolds with maximum diameter

Let V be a complete Riemannian manifold such that $0 < \delta \leqq \operatorname{curv}(V)$. According to Bonnet's lemma, V is compact and of diameter $d(V) \leqq \pi/\sqrt{\delta}$. In [8] Toponogov proved the following:

Theorem 2 (Toponogov). *If $d(V) = \pi/\sqrt{\delta}$, then V is isometric to $S_n(\delta)$.*

We want to give a proof of this result using only Theorem 1. One reason is that it is a very quick one. Another, essential, reason is that we shall use Theorem 2 to prove Toponogov's theorem (Theorem 3 below), whereas Toponogov's proof of Theorem 2 used Theorem 3.

Let p, q be two points of V such that $d(p, q) = \pi/\sqrt{\delta}$, and fix a shortest geodesic $\Gamma = \{\gamma(s)\}$ $(-\pi/2\sqrt{\delta} \leqq s \leqq \pi/2\sqrt{\delta})$ from p to q, $p = \gamma(-\pi/2\sqrt{\delta})$ and $q = \gamma(\pi/2\sqrt{\delta})$. Put $r = \gamma(0)$, and pick any X such that (a) $\| X \| = 1$; (b) $X \in T_r$; (c) $\langle X, \gamma'(0)\rangle = 0$. Define a field

$$\{X(s)\} \quad (-\pi/2\sqrt{\delta} \leqq s \leqq \pi/2\sqrt{\delta})$$

of vectors along Γ by the conditions: (a) $X(0) = X$; (b) for any s, $\nabla_{\gamma'(s)} X(s) = 0$. Define a one-parameter family of geodesics of V by $\Lambda(s) = \{\lambda(t, s)\}$ such that (a) $-\pi/2\sqrt{\delta} \leqq s \leqq \pi/2\sqrt{\delta}$; (b) $\lambda(0, s) = \gamma(s)$; (c) $\lambda_t'(0, s) = X(s)$; (d) $0 \leqq t \leqq f_k(s)$, where $f_k(s)$ is a function of s which has to be such that, if one builds up, as explained in §1, the situation with $\hat{p}$, $\hat{q}$, $\hat{r}$, $\hat{\Gamma}$, $\hat{\Lambda}(s)$ in $S(\delta)$ ($\hat{}$ instead of $\tilde{}$), then the curve $\hat{\Omega}$ corresponding to

$\Omega_{k,X} = \{\lambda(f_k(s), s)\}$ is a shortest geodesic in $S(\delta)$ from $\hat{p}$ to $\hat{q}$ whose midpoint is at distance $k\pi/2$ from $\hat{r}$. By Theorem 1, one then has $l(\Omega) \leqq l(\hat{\Omega}) = \pi/\sqrt{\delta}$. But Ω connects p, q, so $l(\Omega) \geqq d(p, q) = \pi/\sqrt{\delta}$; so one has to have $l(\Omega) = l(\hat{\Omega})$. From what we said after the proof of Theorem 1, this implies that all curves $\Omega_{k,X}$, for all X in T_r as above and all $k \in [0, 1]$, are shortest geodesics from p to q; moreover, all Jacobi fields along Γ and vanishing at p are the same as in $S(\delta)$. One can repeat the same argument replacing Γ by any of the $\Omega_{k,X}$; from this one deduces easily that all geodesics starting from p in V reach q at length $\pi/\sqrt{\delta}$, and that all Jacobi fields along them are the same as in $S_n(\delta)$. This (see [6, p. 21, (26)]) implies an isometry between $S_n(\delta) - \hat{q}$ and $V - q$; but the angles between geodesics starting from p are the same when they meet again in q; so one has the desired isometry.

5. Toponogov's theorem

We want now to give a proof of Theorem 3 below, which is almost equivalent to a theorem of Toponogov [8, Theorem 1, p. 719]. Toponogov's proof rests on a theorem of Alexandrov for surfaces; ours will rest on Rauch's metric comparison theorem and Theorem 1 above.

THEOREM 3 (Toponogov). *Let V be a complete Riemannian manifold whose sectional curvature set* $\text{curv}(V)$ *satisfies* $\delta \leqq \text{curv}(V) \leqq 1$ (*where δ is any real number* $\leqq 1$). *Let p, q, r be three points of V, and let* $\Gamma = \{\gamma(s)\}$ $(0 \leqq s \leqq d(p, q); \gamma(0) = p)$ (*resp.* $\Lambda = \{\lambda(s)\}$ $(0 \leqq s \leqq d(p, r); \lambda(0) = p)$) *be a shortest geodesic segment of V from p to q (resp. from p to r). Let in $S(\delta)$ three points $\hat{p}$, $\hat{q}$, $\hat{r}$ and two geodesics* $\hat{\Gamma} = \{\hat{\gamma}(s)\}$ $(0 \leqq s \leqq d(p, q); \hat{\gamma}(0) = \hat{p})$, $\hat{\Lambda} = \{\hat{\lambda}(s)\}$ $(0 \leqq s \leqq d(p, r); \hat{\lambda}(0) = \hat{p})$ *be such that* (a) $\hat{d}(\hat{p}, \hat{q}) = d(p, q)$ *and* $\hat{d}(\hat{p}, \hat{r}) = d(p, r)$; (b) $\langle\hat{\gamma}'(0), \hat{\lambda}'(0)\rangle = \langle\gamma'(0), \lambda'(0)\rangle$; (c) $\hat{\Gamma}$ (*resp.* $\hat{\Lambda}$) *is a shortest geodesic from $\hat{p}$ to $\hat{q}$ (resp. from $\hat{p}$ to $\hat{r}$). Then one has*

$$d(q, r) \leqq \hat{d}(\hat{q}, \hat{r}).$$

Remark that the condition $\text{curv}(V) \leqq 1$ is not a restriction but merely a normalization of the upper bound (if it exists) of the set $\text{curv}(V)$; because in the following we shall always work in compact subsets of V, such a normalization can be always assumed.

An outline of the proof can be the following one: According to Theorem 2, Theorem 3 is trivial if $d(V) = \pi/\sqrt{\delta}$; hence, one can assume $d(V) < \pi/\sqrt{\delta}$. Theorem 3 is proved first for triangles such that $d(p, q) < \pi/2$ and $d(p, r) < \pi/2$; this is a direct consequence of Rauch's metric comparison theorem (see Lemma 1). Then Theorem 3 is proved (Lemma 5) for triangles such that $d(p, r)$ is little enough in comparison to $d(p, q)$ and $\langle\gamma'(0), \lambda'(0)\rangle \leqq 0$; the proof uses Theorem 1 and Lemma 1. Then Theorem 3 is proved (Lemma 6) for triangles such that $d(p, r)$ is little enough in comparison to $d(p, q)$ (no further condition); the proof is a reduction to Lemma 5. Finally one proves Theorem 3 in general by putting points $p_1, p_2, \cdots, p_{k-1}$ on Λ such that Lemma 6 applies to all triangles p_i, p_{i+1}, q and using a device to go from p_i, p_{i+1}, q to p_{i+1}, p_{i+2}, q.

In the remainder of the paper, notations and hypotheses are tacitly assumed to be those of Theorem 3. As done in [1, p. 96, Theorem 6], we remark first that, from Rauch's metric comparison theorem, one deduces immediately the following:

LEMMA 1 (Rauch). *In the circle of unit tangent vectors to $S(\delta)$ at $\hat{p}$, there exists a unique shortest arc connecting $\hat{\gamma}'(0)$ and $\hat{\lambda}'(0)$ if $\hat{\gamma}'(0) \neq -\hat{\lambda}'(0)$ (or two if $\hat{\gamma}'(0) = -\hat{\lambda}'(0)$); call it ω (or either one of the two). Then Theorem 3 is true under the following additional condition: There exists a shortest geodesic segment $\hat{\Phi}$ of $S(\delta)$ from $\hat{q}$ to $\hat{r}$ such that every geodesic of $S(\delta)$ which starts at $\hat{p}$ with a tangent vector belonging to ω and ends at $\hat{\Phi}$, is of length $\leqq \pi$. Moreover, this condition is always fullfilled if $d(p, q) \leqq \pi/2$ and $d(p, r) \leqq \pi/2$.*

The last assertion can follow from a look at $S(\delta)$ for $\delta \leqq 1$; it is a convexity property on $S(\delta)$.

In the following, when two different points $\hat{p}$, $\hat{q}$ of $S(\delta)$ are given with, moreover, a shortest geodesic $\hat{\Gamma}$ from $\hat{p}$ to $\hat{q}$, by $S(\delta)/2$ one will always mean the closed half of $S(\delta)$ built up by the points of $S(\delta)$ which lie to the right of the full geodesic which covers $\hat{\Gamma}$.

LEMMA 2. *Let $\hat{p}$, $\hat{q}$ be two points of $S(\delta)$, and $\hat{\Gamma}$ a shortest geodesic in $S(\delta)$ from $\hat{p}$ to $\hat{q}$. If $\delta > 0$, suppose moreover that $\hat{d}(\hat{p}, \hat{q}) < \pi/\sqrt{\delta}$. Let*

$$\hat{\Sigma} = \{\hat{r} \in S(\delta)/2 \mid \hat{d}(p, \hat{r}) = \alpha\}$$

(with $\alpha < \pi/\sqrt{\delta}$ if $\delta > 0$) be the semicircle of $S(\delta)/2$ of center $\hat{p}$ and of radius α, and take for parametrization $\hat{\Sigma} = \{\hat{\sigma}(t)\}$ $(0 \leqq t \leqq \pi)$ of $\hat{\Sigma}$ the angle t at $\hat{p}$ between $\hat{\Gamma}$ and the unique shortest geodesic from $\hat{p}$ to $\hat{\sigma}(t)$. Then, when t grows from 0 to π, the function $\hat{d}(\hat{q}, \hat{\sigma}(t))$ is strictly increasing.

Put $\hat{r} = \hat{\Gamma} \cap \hat{\Sigma}$ and call $\hat{s}$ the point other than $\hat{r}$ where $\hat{\Sigma}$ meets the geodesic of $S(\delta)$ which covers $\hat{\Gamma}$; then $\hat{d}(\hat{q}, \hat{r}) < \hat{d}(\hat{q}, \hat{s})$, because $\hat{\Gamma}$ is the unique shortest geodesic from $\hat{p}$ to $\hat{q}$. Suppose first, for any $t \in [0, \pi]$, that there is a unique shortest geodesic $\hat{\Phi}(t)$ from $\hat{q}$ to $\hat{\sigma}(t)$; then the exponential map $T_{\hat{p}} \to S(\delta)$ is regular on $\hat{\Sigma}$, so $f(t) = \hat{d}(\hat{q}, \hat{\sigma}(t))$ is a differentiable function of t. If this function were not strictly increasing in t, from $f(\pi) > f(0)$ it would follow that there would exist a $t_0 \in]0, \pi[$ such that $f(t_0)$ is a critical value and one would have the geodesic $\hat{\Phi}(t_0)$ meeting $\hat{\Sigma}$ at right angles at $\hat{\sigma}(t_0)$. Then the union of $\hat{\Phi}(t_0)$ with the shortest geodesic from $\hat{p}$ to $\hat{\sigma}(t_0)$ would be a geodesic from $\hat{p}$ to $\hat{q}$ making an angle $\in]0, \pi[$ with $\hat{\Gamma}$ at $\hat{p}$; such a thing never happens on an $S(\delta)$ except when $\delta > 0$ and $\hat{p}$ and $\hat{q}$ are antipodal, but one had assumed $\hat{d}(\hat{p}, \hat{q}) < \pi/\sqrt{\delta}$; so the lemma is proved in this first case. If now the exponential map $T_{\hat{p}} \to S(\delta)$ is not regular on $\hat{\Sigma}$, it can only happen if $\hat{q}$ and $\hat{s}$ are antipodal; but then $\hat{d}(\hat{\sigma}(t), \hat{q}) = \pi/\sqrt{\delta} - \hat{d}(\hat{s}, \hat{\sigma}(t))$. Because, for any t, $\hat{d}(\hat{s}, \hat{\sigma}(t)) < \pi/\sqrt{\delta}$, one can apply the proof above to $\hat{s}$ and $\hat{\Sigma}$; replacing t by $\pi - t$, one gets the lemma in this case.

LEMMA 3. *Let K be a compact subset of V. Then there exists a strictly positive real number η_K with the following property: Let p, q, r be any three distinct points in K such that $d(p, q) = d(p, r) < \eta_K$. Then, if $\Phi = \{\varphi(t)\}$ $(0 \leqq t \leqq d(p, q); \varphi(0) = q)$ (resp. $\Psi = \{\psi(t)\}$ $(0 \leqq t \leqq d(q, r); \psi(0) = q)$) s any shortest geodesic from q to p (resp. from q to r), one has $\langle \varphi'(0), \psi'(0) \rangle > 0$.*

One knows [9] that there exists, for any $x \in V$, a real strictly positive number α_x such that $d(x, y) \geqq \alpha_x$ for any $y \in C(x)$. Put $\alpha = \inf_{x \in K}(\alpha_x)$; because of the compactness of K, one has $\alpha > 0$. Let $\eta_K = \inf(\alpha/2, \pi/2)$; then $\eta_K > 0$. We prove now that η_K satisfies the requirements of the lemma. The idea is to use Rauch's metric comparison theorem for V and $S_n(1)$; notations will be those of [1, p. 96]. Let $\dot{q}$, $\dot{p}$, $\dot{r}$, $\dot{\Phi} = \{\dot{\varphi}(t)\}$ $(0 \leqq t \leqq d(q, p); \dot{\varphi}(0) = \dot{q})$, $\dot{\Psi} = \{\dot{\psi}(t)\}$ $(0 \leqq t \leqq d(q, r); \dot{\psi}(0) = \dot{q})$, be the elements of $S_n(1)$ corresponding to q, p, r, Φ, Ψ. Call $\Sigma = \{\sigma(t)\}$ $(0 \leqq t \leqq d(p, r))$ a shortest geodesic of V from p to r; from $d(p, q) = d(q, r) < \eta_K$, one deduces

$$d(q, \sigma(t)) \leqq d(q, p) + d(p, \sigma(t)) \leqq d(q, p) + d(p, r) < 2\eta_K \leqq \alpha_q ;$$

so the exponential map $T_q \to V$ is regular on Σ, and so there arises the one-parameter family $\{\Theta(t)\}$ formed by the unique shortest geodesic $\Theta(t)$ from q to $\sigma(t)$ $(0 \leqq t \leqq d(p, r);\ \Theta(0) = \Phi;\ \Theta(d(p, r)) = \Psi)$; one can apply Theorem 6 of [1, p. 96], because for any t, $d(q, \sigma(t)) < 2\eta_K \leqq \pi$. So for the curves Σ, $\dot{\Sigma}$ of this theorem, one gets $l(\dot{\Sigma}) \leqq l(\Sigma)$. But $\dot{\Sigma}$ has $\dot{p}$ and $\dot{q}$ as end points in $S_n(1)$, so

$$l(\Sigma) = d(p, r) = d(p, q) \geqq \dot{d}(\dot{p}, \dot{r}).$$

So, on $S_n(1)$, $\dot{d}(\dot{p}, \dot{r}) \leqq \dot{d}(\dot{p}, \dot{q}) < \pi/2$ (by the choice of η_K); a look at $S_n(1)$ shows that this implies $\langle \dot{\varphi}'(0), \dot{\psi}'(0) \rangle > 0$. But

$$\langle \dot{\varphi}'(0), \dot{\psi}'(0) \rangle = \langle \varphi'(0), \psi'(0) \rangle,$$

which proves the lemma.

For the moment, we confine our attention to $S(\delta)$ only, with $\hat{p}$, $\hat{q}$ being points on $S(\delta)$, and $\hat{\Gamma}$ a shortest geodesic on $S(\delta)$ from $\hat{p}$ to $\hat{q}$, and consider, too, the corresponding $S(\delta)/2$; if $\delta > 0$, suppose, moreover, that $m = \hat{d}(\hat{p}, \hat{q}) < \pi\sqrt{\delta}$. Call $\hat{\mathbf{\Gamma}}$ the complete geodesic of $S(\delta)$ which covers $\hat{\Gamma}$.

LEMMA 4. *There exists a strictly real positive number $r(m)$ having the following property: For any $\hat{r}$ such that $\hat{d}(\hat{p}, \hat{r}) \leqq r(m)$, there is a unique shortest geodesic $\hat{\Lambda}$ from $\hat{q}$ to $\hat{r}$, which meets $\hat{\Gamma}$ at $\hat{q}$ with an angle $< \pi/2$ and has the property that every point $z \in \hat{\Lambda}$ verifies $\hat{d}(z, \hat{\mathbf{\Gamma}}) \leqq \pi/2$ (where $\hat{d}(z, \hat{\mathbf{\Gamma}})$ is the infimum of the distance of z to any points of $\hat{\mathbf{\Gamma}}$).*

If $\delta > 0$, one can find $r(m)$ in the following way: Let $\hat{\Theta}$ be the geodesic of $S(\delta)/2$ which starts from $\hat{q}$ and whose maximal distance to $\hat{\mathbf{\Gamma}}$ is exactly

$\pi/2$. Draw then the semicircle $\hat{\Sigma}$ of $S(\delta)/2$ which has $\hat{p}$ as center and is tangent to $\hat{\Theta}$. Clearly the radius $r(m)$ of $\hat{\Sigma}$ fulfills the requirements of the lemma in this case. If $\delta \leqq 0$, put the point $\hat{w}$ on $\hat{\Gamma}$ so that $\hat{p}$ is between $\hat{q}$ and $\hat{w}$ and $\hat{d}(\hat{p}, \hat{w}) = k$, where k is a given strictly positive constant. Let $\hat{v}$ be the point of $S(\delta)/2$ which, on the perpendicular to $\hat{\Gamma}$ at $\hat{w}$, verifies $\hat{d}(\hat{v}, \hat{q}) = \pi/2$. Draw the shortest geodesic $\hat{\Theta}$ from $\hat{q}$ to $\hat{v}$; then draw the semicircle of $S(\delta)/2$ of center $\hat{p}$ and tangent to $\hat{\Theta}$; clearly its radius $r(m)$ fulfills the requirements of the lemma.

One can refine the lemma by means of the following remarks:

(A) $\delta \leqq 1$ and $d(z, \hat{\Gamma}) < \pi/2$ implies that there exists a unique geodesic starting from z and meeting $\hat{\Gamma}$ orthogonally at a distance $< \pi/2$.

(B) As chosen in the proof of the lemma, the function $r(m)$ is continuous in m.

(C) From Remark (B) one sees that there exists, for any m such that $0 < m < \pi/\sqrt{\delta}$, a real number $s(m)$, which is strictly positive, continuous in m, such that $x < s(m)$ implies $2x < r(m - x)$.

(D) From Remark (C), one sees that there exists for any k, d such that $0 < k \leqq d < \pi/\sqrt{\delta}$ (if $\delta > 0$) a strictly positive real number $\varepsilon(k, d)$ such that, for any m verifying $k \leqq m \leqq d$, one has $s(m) \geqq \varepsilon(k, d)$.

LEMMA 5. *Theorem 3 is true under the following additional conditions*: (a) $d(p, q) < \pi/\sqrt{\delta}$; (b) $\langle\gamma'(0), \lambda'(0)\rangle < 0$; (c) $d(p, r) < r(d(p, q))$ (*where* $r(d(p, q))$ *is the function defined in Lemma* 4).

Let $\hat{\Omega}$ be the unique shortest geodesic in $S(\delta)/2$ from $\hat{q}$ to $\hat{r}$: $\hat{\Omega} = \{\hat{\omega}(t)\}$ $(0 \leqq t \leqq \hat{d}(\hat{r}, \hat{q});\ \hat{\omega}(0) = \hat{r})$. From

$$d(p, r) = \hat{d}(\hat{p}, \hat{r}) < r(d(p, q)) = r(\hat{d}(\hat{p}, \hat{q}))$$

and from remark (A), one knows that there exists a unique geodesic from a point $\hat{\omega}(t) \in \hat{\Omega}$ orthogonal to $\hat{\Gamma}$ and of length $< \pi/2$; call it $\hat{\Lambda}(t)$, and call $\hat{\psi}(t)$ its foot on $\hat{\Gamma}$. Because of the acute angle conclusion in Lemma 4, there exists a well defined $t_0 \in [0, \hat{d}(\hat{r}, \hat{q})[$ such that $\hat{\psi}(t_0) = \hat{p}$; and one has, for any $t \geqq t_0$, $\hat{\psi}(t) \in \hat{\Gamma}$. Call $\hat{\Omega}_1$ (resp. $\hat{\Omega}_2$) the restriction of $\hat{\Omega}$ from $\hat{r}$ to $\hat{s} = \hat{\omega}(t_0)$ (resp. from $\hat{s}$ to $\hat{q}$). One has $\hat{d}(\hat{p}, \hat{\omega}(t)) < \pi/2$ for any $t \in [0, t_0]$ because $\hat{d}(\hat{p}, \hat{r}) < r(m) \leqq \pi/2$ and $\hat{d}(\hat{p}, \hat{\omega}(t_0)) < \pi/2$ (see Lemma 1).

Now build up in V a one-parameter family of geodesics $\{\Lambda(t)\}$ $(t_0 \leqq t \leqq \hat{d}(\hat{r}, \hat{q}))$ defined as corresponding to the family $\{\hat{\Lambda}(t)\}$ $(t_0 \leqq t \leqq \hat{d}(\hat{r}, \hat{q}))$ in $S(\delta)$ in order to apply Theorem 1. This can be done more precisely as follows: Define first a unit vector $Y(t_0) \in T_p$ belonging to the two-dimensional plane of T_p generated by $\gamma'(0)$ and $\lambda'(0)$ and such that

$$\langle\hat{\lambda}'(0), \hat{\lambda}'_t(0, t_0)\rangle = \langle\lambda'(0), Y(t_0)\rangle \quad \text{and} \quad \langle\gamma'(0), Y(t_0)\rangle = 0$$

(in the case where $\gamma'(0) = -\lambda'(0)$ this has no meaning; take then any unit vector $Y(t_0)$ orthogonal to $\gamma'(0)$). Then define $\{Y(t)\}$ $(t_0 \leqq t \leqq \hat{d}(\hat{r}, \hat{q}))$ by the condition $\nabla_{\psi(t)} Y(t) = 0$ for any $t \in [t_0, \hat{d}(\hat{r}, \hat{q})]$. Then define $\Lambda(t)$ as starting at $\psi(t)$, having at $\psi(t)$ the above-defined $Y(t)$ as tangent vector and the same length as $\hat{\Lambda}(t)$. Call s the end of $\Lambda(t_0)$.

From Theorem 1, one has $d(s, q) \leqq l(\Omega_2) \leqq l(\hat{\Omega}_2) = \hat{d}(\hat{s}, \hat{q})$. One saw above that it is possible to apply Lemma 1 to $p, r, s, \Lambda, \Lambda(t_0)$, from which it follows that $d(r, s) \leqq \hat{d}(\hat{r}, \hat{s})$. By adding we obtain

$$d(r, q) \leqq d(r, s) + d(s, q) \leqq \hat{d}(\hat{r}, \hat{s}) + \hat{d}(\hat{s}, \hat{q}) = \hat{d}(\hat{r}, \hat{q}).$$

LEMMA 6. *Let p, q be two points of V such that, if $\delta > 0$, $d(p, q) < \pi/\sqrt{\delta}$. Let $K = \{x \in V \mid d(x, p) \leqq r(d(p, q))\}$, and let η_K be the number associated with K in Lemma 3. Then Theorem 3 is true under the following condition: r is such that $d(p, r) < \inf(\eta_K/2, s(d(p, q))$ (where $s(d(p, q))$ is the function defined in Remark* (C) *above*).

Define a point s of V by (a) $s \in \Gamma$; (b) $d(p, s) = d(p, r)$. One has $p, r, s \in K$, and one can apply Lemma 3 (note if $r = s$, Theorem 3 is trivial, so one can always assume $r \neq s$; and then Lemma 3 is applied to the set p, s, r instead of p, q, r). Use the corresponding notations of Lemma 3, so that $\Phi = \{\varphi(t) = \gamma(d(p, s) - t)\}$ $(0 \leqq t \leqq d(p, s))$. One has a shortest geodesic $\Psi = \{\psi(t)\}$ $(0 \leqq t \leqq d(s, r))$ from s to r such that

$$\langle\varphi'(0), \psi'(0)\rangle = -\langle\gamma'(d(p, s), \psi'(0)\rangle > 0.$$

Moreover, by the definition of $s(m)$ in Remark (C), one has

$$d(s, r) \leqq d(s, p) + d(p, r) = 2d(p, s) < r(d(p, q) - d(p, s)) = r(d(s, q)).$$

So the conditions of Lemma 5 are fulfilled for the set s, q, r, Γ_1, Ψ (where Γ_1 means the restriction of Γ from s to q). But one has to be careful to define corresponding elements in $S(\delta)/2$; there is no problem for $\hat{s}, \hat{q}, \hat{\Gamma}_1 \subset \hat{\Gamma}$. Define $\hat{\Psi} \subset S(\delta)/2$ as a geodesic starting from $\hat{s}$ and such that

$$\langle\hat{\psi}'(0), \hat{\gamma}'(d(p, s))\rangle = \langle\psi'(0), \gamma'(d(p, s))\rangle;$$

then define $\hat{r}_1$ as $\hat{r}_1 = \hat{\psi}(d(s, r))$. Lemma 5 asserts that

$$d(q, r) \leqq \hat{d}(\hat{q}, \hat{r}_1). \tag{1}$$

One needs now to compare $\hat{d}(\hat{q}, \hat{r}_1)$ with $\hat{d}(\hat{q}, \hat{r})$ (where $\hat{r}$ is the point defined in Theorem 3). Do that, defining first a point $\hat{r}_2 \in S(\delta)/2$ by the two conditions: $\hat{d}(\hat{p}, \hat{r}_2) = d(p, r) = \hat{d}(\hat{p}, \hat{r})$ and $\hat{d}(\hat{r}_2, \hat{s}) = d(r, s) = \hat{d}(\hat{r}_1, \hat{s})$. One can apply Lemma 1 to the set $p, s, r, \Lambda, \Gamma_2$ (where Γ_2 is the restriction from p to s of Γ) and the corresponding set in $S(\delta)$: $\hat{p}, \hat{s}, \hat{r}, \hat{\Lambda}, \hat{\Gamma}_2$; this is possible because $d(p, s) < \pi/2$ and $d(p, r) < \pi/2$. One gets

$$d(r, s) \leqq \hat{d}(\hat{r}, \hat{s}). \tag{2}$$

Call α (resp. β) the angle at $\hat{p}$ between $\hat{\Gamma}_2$ and $\hat{\Lambda}$ (resp. between $\hat{\Gamma}$ and the shortest geodesic from $\hat{p}$ to $\hat{r}_2$); apply Lemma 2 to the semicircle of center

$\hat{p}$ and radius equal to $d(p, r)$ and the point $\hat{s}$; one gets from (2) that $\alpha \geqq \beta$. Apply (in the other logical sense) Lemma 2 to the semicircle of center $\hat{p}$ and radius equal to $d(p, r)$ but now for point $\hat{q}$; one gets $\hat{d}(\hat{q}, \hat{r}_2) \leqq \hat{d}(\hat{q}, \hat{r})$.

Call γ (resp. δ) the angle at $\hat{s}$ between $\hat{\Phi}$ and the shortest geodesic from $\hat{s}$ to $\hat{r}_2$ (resp. between $\hat{\Phi}$ and $\hat{\Psi}$); we claim that $\delta \geqq \gamma$. In fact, apply Lemma 1 to s, p, r, Φ, Ψ and the corresponding set $\hat{s}$, $\hat{p}$, $\hat{r}_1$, $\hat{\Phi}$, $\hat{\Psi}$ in $S(\delta)$; this is possible because, from the definition of $s(m)$, $d(p, s) < \pi/2$ and $d(s, r) < \pi/2$. Lemma 1 yields $d(p, r) \leqq \hat{d}(\hat{p}, \hat{r}_1) = \hat{d}(\hat{p}, \hat{r}_2)$. Apply this inequality to Lemma 2 for the semicircle of center $\hat{s}$ and radius equal to $d(r, s)$ and for the point $\hat{p}$; one gets the claim $\delta \geqq \gamma$. But now apply, in the other sense, Lemma 2 to the semicircle of center $\hat{s}$ of radius $d(r, s)$ but for the point $\hat{q}$; one gets $\hat{d}(\hat{q}, \hat{r}_1) \leqq \hat{d}(\hat{q}, \hat{r}_2)$ (Note, in fact, that the angle at $\hat{s}$ between $\hat{\Phi}$ and any geodesic is equal to π minus the angle between this geodesic and $\hat{\Gamma}_2$.) Finally, from (1), one deduces

$$d(q, r) \leqq \hat{d}(\hat{q}, \hat{r}_1) \leqq \hat{d}(\hat{q}, \hat{r}_2) \leqq \hat{d}(\hat{q}, \hat{r}).$$

Proof of Theorem 2. Let p, q, r be any three points of V. If $d(V) = \pi/\sqrt{\delta}$, according to Theorem 2, one knows that V is isometric to $S_n(\delta)$; so Theorem 3 is fulfilled with equality. Henceforth, assume $d(V) = d < \pi/\sqrt{\delta}$. Define

$$k = \inf_{z \in \Lambda}(d(z, q)).$$

If $k = 0$, then $q \in \Lambda$, and so Λ covers Γ, and then the theorem is trivial. Henceforth, $0 < k \leqq d < \pi/\sqrt{\delta}$. Let $\varepsilon = \varepsilon(k, d)$ be the corresponding number introduced in Remark (D) above. Let, for $z \in \Lambda$,

$$B_z = \{x \in V \mid d(x, z) \leqq \varepsilon\} \quad \text{and} \quad K = \bigcup_{z \in \Lambda} B_z .$$

Let η be the strictly positive real number associated in Lemma 3 with the compact subset K of V. Put $\zeta = \min(\varepsilon, \eta)$. And put points

$$p = p_0, \quad p_1, \quad \cdots, \quad p_i, \quad p_{i+1}, \quad \cdots, \quad p_{k-1}, \quad p_k = r$$

in finite number on Λ so that, for any $i = 0, 1, \cdots, k-1$, one has $d(p_i, p_{i+1}) < \zeta$. Let Γ_i be a shortest geodesic from p_i to q, and call Λ_i the restriction of Λ from p_i to p_{i+1}. Then remark that the choice of the p_i assures us that each set p_i, q, p_{i+1}, Γ_i, Λ_i fulfills the hypothesis of Lemma 6. In fact, for any $i = 0, 1, \cdots, k-1$,

$$d(p_i, p_{i+1}) < \zeta = \min(\varepsilon, \eta) \leqq \min(\eta, s(d(p_i, q))$$

by the choice of remark (D) above and the remark that $K \supset B_{p_i}$.

An outline of the proof is the following: One will build up in $S(\delta)/2$ by induction, points $\hat{r}'_i$, $\hat{r}_i$ ($i = 0, 1, \cdots, k-1, k$) which will satisfy

$$\hat{d}(\hat{q}, \hat{r}'_{i+1}) \leqq \hat{d}(\hat{q}, \hat{r}_{i+1}) \quad \text{and} \quad \hat{d}(\hat{q}, \hat{r}_{i+1}) \leqq \hat{d}(\hat{q}, \hat{r}'_i).$$

In the last step, one will get $d(q, r) \leqq \hat{d}(\hat{q}, \hat{r}'_k)$; and the beginning being $\hat{r}'_0 = \hat{r}$, there will follow the required

$$d(q, r) \leqq \hat{d}(\hat{q}, \hat{r}'_k) \leqq \hat{d}(\hat{q}, \hat{r}'_0) = \hat{d}(\hat{q}, \hat{r}).$$

Construct first $\hat{r}_1$ and $\hat{r}'_1$ to see how things work. In $S(\delta)$ let, for the beginning, $\hat{p}$, $\hat{q}$, $\hat{r}$, $\hat{\Gamma}$, $\hat{\Lambda}$ be defined as in Theorem 3. Let $\hat{p}_1$ on $\hat{\Lambda}$ be such that $\hat{d}(\hat{p}, \hat{p}_1) = d(p, p_1)$. Then define a point $\hat{p}'_1$ in $S(\delta)/2$ by the conditions

$$\hat{d}(\hat{p}, \hat{p}'_1) = \hat{d}(\hat{p}, \hat{p}_1) = d(p, p_1) \quad \text{and} \quad \hat{d}(\hat{q}, \hat{p}'_1) = d(q, p_1),$$

in order that the triangle $\hat{p}$, $\hat{q}$, $\hat{p}'_1$ in $S(\delta)/2$ have the same side-lengths as the triangle p, q, p_1 in V. Call $\hat{\Gamma}_1$ the unique shortest geodesic from $\hat{q}$ to $\hat{p}'_1$ (uniqueness follows from the choice of $r(m)$ and $d < \pi/\sqrt{\delta}$). Call $\hat{\Phi}_1$ the unique shortest geodesic in $S(\delta)$ from $\hat{p}$ to $\hat{p}'_1$, and call $\hat{r}_1$ the point of $S(\delta)$ which is, on the geodesic starting from $\hat{p}$ and covering $\hat{\Phi}_1$, at the distance $\hat{d}(\hat{p}, \hat{r}) = d(p, r)$ from $\hat{p}$. Define $\hat{\Phi}'_1$ as the geodesic which, in the half space $S(\delta)/2_1$ associated in $S(\delta)$ with the triple $\hat{p}'_1$, $\hat{q}$, $\hat{\Gamma}_1$, has length $l(\hat{\Phi}'_1) = d(p_1, r)$ and meets at $\hat{p}'_1$ the geodesic $\hat{\Gamma}_1$ with the same angle as Λ_1 does with Γ_1. Call $\hat{r}'_1$ the end of $\hat{\Phi}'_1$; note that $\hat{d}(\hat{p}'_1, \hat{r}'_1) = \hat{d}(\hat{p}'_1, \hat{r}_1) = \hat{d}(\hat{p}_1, \hat{r}) = d(p_1, r)$. (In this situation, one can prove that $\hat{d}(\hat{q}, \hat{r}'_1) \leqq \hat{d}(\hat{q}(\hat{q}, \hat{r}_1) \leqq \hat{d}(\hat{q}, \hat{r})$. But it will be a particular case of the following induction.)

Now such a process can be continued inductively; suppose one has defined $\hat{p}'_i$, $\hat{q}$, $\hat{\Gamma}_i$, $\hat{\Phi}'_i$ (and additionally, $\hat{p}_i$, $\hat{r}_i$, $\hat{r}'_i$, $\hat{\Phi}_i$) for $i = 1, \cdots, k-1$. One defines the next set as follows: The point $\hat{p}_{i+1}$ is on $\hat{\Phi}'_i$ with the condition $\hat{d}(\hat{p}'_i, \hat{p}_{i+1}) = d(p_i, p_{i+1})$. Then $\hat{p}'_{i+1}$ is in the half space $S(\delta)/2_i$ which is associated in $S(\delta)$ with the triple $\hat{p}'_i$, $\hat{q}$, $\hat{\Gamma}_i$, and subject to the two distance conditions

$$\hat{d}(\hat{p}'_i, \hat{p}'_{i+1}) = d(p_i, p_{i+1}) \quad \text{and} \quad \hat{d}(\hat{p}'_{i+1}, \hat{q}) = d(p_{i+1}, q),$$

which express that the triangle $\hat{p}'_i$, $\hat{p}'_{i+1}$, $\hat{q}$ of $S(\delta)/2_i$ has the same side-lengths as the triangle p_i, p_{i+1}, q of V. Then define $\hat{\Gamma}_{i+1}$ as the unique shortest geodesic from $\hat{q}$ to $\hat{p}'_{i+1}$; and after, define $\hat{\Phi}_{i+1}$ as the geodesic which, starting from $\hat{p}'_i$, covers the unique shortest geodesic from $\hat{p}'_i$ to $\hat{p}'_{i+1}$ and whose length is equal to $d(p_i, r)$; call its end $\hat{r}_{i+1}$. Denote now by $\hat{\Phi}'_{i+1}$ the geodesic in $S(\delta)/2_i$ which has length $l(\hat{\Phi}'_{i+1}) = d(p_{i+1}, r)$ and meets in $\hat{p}'_{i+1}$ the geodesic $\hat{\Gamma}_{i+1}$ with the same angle as Λ_{i+1} does with Γ_{i+1}; and denote the end $\hat{\Phi}'_{i+1}$ by $\hat{r}'_{i+1}$. Remark that

$$\hat{d}(\hat{p}'_{i+1}, \hat{r}'_{i+1}) = \hat{d}(\hat{p}'_{i+1}, \hat{r}_{i+1}) = \hat{d}(\hat{p}_{i+1}, \hat{r}_i) = d(p_{i+1}, r).$$

One claims now, for each $i = 0, 1, \cdots, k-1$, the inequalities

$$\hat{d}(\hat{q}, \hat{r}_{i+1}) \leqq \hat{d}(\hat{q}, \hat{r}'_i), \tag{3}$$

$$\hat{d}(\hat{q}, \hat{r}'_{i+1}) \leqq \hat{d}(\hat{q}, \hat{r}_{i+1}). \tag{4}$$

Devices here are quite similar to the proof of Lemma 1. First apply Lemma 6 to the set p_i, q, p_{i+1}, Γ_i, Λ_i; the corresponding set in $S(\delta)$ is $\hat{p}'_i$, $\hat{q}$, $\hat{p}_{i+1}$, $\hat{\Gamma}_i$, and the restriction of $\hat{\Phi}'_i$ from $\hat{p}'_i$ to $\hat{p}_{i+1}$. We saw above that this is legitimate; one gets $d(q, p_{i+1}) \leqq \hat{d}(\hat{q}, \hat{p}_{i+1})$. But $d(q, p_{i+1}) = \hat{d}(\hat{p}'_{i+1}, \hat{q})$ by construction of $\hat{p}'_{i+1}$, and so $\hat{d}(\hat{q}, \hat{p}'_{i+1}) \leqq \hat{d}(\hat{q}, \hat{p}_{i+1})$. Call α_i (resp. β_i) the angle at $\hat{p}'_i$ between $\hat{\Gamma}_i$ and $\hat{\Phi}'_i$ (resp. between $\hat{\Gamma}_i$ and $\hat{\Phi}_{i+1}$); apply Lemma 2 to the semicircle in $S(\delta)/2_i$ of center $\hat{p}'_i$, radius $d(p_i, p_{i+1})$ and for the point $\hat{q}$. From $\hat{d}(\hat{q}, \hat{p}'_{i+1}) \leqq \hat{d}(\hat{q}, \hat{p}_{i+1})$, Lemma 2 yields $\alpha_i \geqq \beta_i$. Again apply Lemma 2 in the other logical sense for the semicircle in $S(\delta)/2_i$ of center $\hat{p}'_i$, radius $d(p_i, r)$ and point $\hat{q}$; one gets the above inequality (3).

One proves now the inequality (4). Call γ_i (resp. δ_i) the angle at p_{i+1} (resp. at $\hat{p}'_{i+1}$) between Γ_{i+1} and $-\Lambda_i$ (reversed sense on Λ_i) (resp. between $\hat{\Gamma}_{i+1}$ and the restriction from $\hat{p}'_{i+1}$ to $\hat{p}'_i$ of $-\hat{\Phi}_{i+1}$ (reversed sense)); remark, by construction, that γ_i is equal to the angle at $\hat{p}'_{i+1}$ between $\hat{\Gamma}_{i+1}$ and $-\hat{\Phi}'_{i+1}$ (this denotes a geodesic starting from $\hat{p}'_{i+1}$, with direction opposite to that of $\hat{\Phi}'_{i+1}$ and of length $d(p_i, p_{i+1})$; the end of $-\hat{\Phi}'_{i+1}$ will be called $\hat{s}_i$). Apply now Lemma 6 to the set $p_{i+1}, q, p_i, \Gamma_{i+1}, -\Lambda_i$ in V, and the corresponding set in $S(\delta)$, $\hat{p}'_{i+1}, \hat{q}, \hat{s}_i, \hat{\Gamma}_{i+1}, -\hat{\Phi}'_{i+1}$; one gets $d(q, p_i) \leqq \hat{d}(\hat{q}, \hat{s}_i)$. But $d(q, p_i) = \hat{d}(\hat{q}, \hat{p}'_i)$; apply now Lemma 2, using this inequality, to the semicircle of center $\hat{p}'_{i+1}$, radius $d(p_i, p_{i+1})$ and for the point $\hat{q}$; one gets $\gamma_i \geqq \delta_i$. Remark now that $\pi - \gamma_i \leqq \pi - \delta_i$ and that $\pi - \gamma_i$ (resp. $\pi - \delta_i$) is the angle at $\hat{p}'_{i+1}$ between $\hat{\Gamma}_{i+1}$ and $\hat{\Phi}'_{i+1}$ (resp. between $\hat{\Gamma}_{i+1}$ and $\hat{\Phi}_{i+1}$); and apply then Lemma 2 to the semicircle of center $\hat{p}'_{i+1}$, radius $d(p_{i+1}, r)$ and for the point $\hat{q}$; one gets exactly (4).

From (3) and (4) and a trivial induction, it follows that

$$\hat{d}(\hat{q}, \hat{r}'_k) \leqq \hat{d}(\hat{q}, \hat{r}'_1) \leqq \hat{d}(\hat{q}, \hat{r}). \tag{5}$$

But apply Lemma 6 to the set $p_{k-1}, q, p_k = r, \Gamma_{k-1}, \Lambda_{k-1}$ in V, and the corresponding set $\hat{p}'_{k-1}, \hat{q}, \hat{p}'_k = \hat{r}_k = \hat{r}'_k, \hat{\Gamma}_{k-1}, \hat{\Phi}'_{k-1}$ in $S(\delta)$; one gets

$$d(q, r) \leqq \hat{d}(\hat{q}, \hat{r}'_k).$$

Now Theorem 3 follows from (5).

Bibliography

1. M. Berger, *Les variétés riemanniennes à courbure positive*, Bull. Soc. Math. Belgique, vol. 10 (1958), pp. 89–104.
2. ———, *Sur quelques variétés riemanniennes suffisamment pincées*, Bull. Soc. Math. France, vol. 88 (1960), pp. 57–71.
3. ———, *Les variétés riemanniennes* ($\frac{1}{4}$)*-pincées*, Ann. Scuola Norm. Sup. Pisa (3), vol. 14 (1960), pp. 161–170.
4. W. Klingenberg, *Contribution to Riemannian geometry in the large*, Ann. of Math. (2), vol. 69 (1959), pp. 654–666.
5. A. Pogorelov, *A theorem regarding geodesics on closed convex surfaces*, Mat. Sbornik (N.S.), vol. 18 (1946), pp. 181–183 (in Russian with English summary).
6. H. E. Rauch, *Geodesics and curvature in differential geometry in the large*, New York, Yeshiva University, 1959.
7. J. L. Synge, *On the connectivity of spaces of positive curvature*, Quart. J. Math. (Oxford Ser.), vol. 7 (1936), pp. 316–320.

8. V. A. TOPONOGOV, *On Riemannian spaces whose curvature has a positive lower bound*, Dokl. Akad. Nauk SSSR, vol. 120 (1958), pp. 719–721 (in Russian).
9. J. H. C. WHITEHEAD, *On the covering of a complete space by the geodesics through a point*, Ann. of Math. (2), vol. 36 (1935), pp. 679–704.

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