LOCALLY FREE MODULES AND A PROBLEM OF WHITEHEAD

BY
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1. Introduction

The problem of the title, proposed by J. H. C. Whitehead in 1952, asks for a characterization of those abelian groups A for which $\operatorname{Ext}(A, Z) = 0$, where Z is the ring of rational integers.[1] In this paper we add to the information obtained recently by R. Nunke [7], J. Rotman [11], and others concerning this and related problems.

The starting point of our discussion is the observation of Rotman [11, p. 251] that a group satisfying the afore-mentioned condition is locally free. Recall that a torsion-free abelian group A is called *locally free* if every pure subgroup of A of finite rank is a free direct summand of A. (This terminology is due to Nunke. Observe that a group is locally free if and only if it is $\aleph_1$-free and separable.) We first show that to every abelian group A there corresponds in a canonical fashion a locally free group $A^\#$ and a homomorphism $j_A : A \to A^\#$. It then turns out that useful information regarding the size of the kernel and cokernel of j_A can be obtained simply from a knowledge of the structure of $\operatorname{Ext}(A, Z)$. Hence, speaking loosely, the group $\operatorname{Ext}(A, Z)$ determines the extent to which A deviates from being locally free. The primary task of this paper is a detailed analysis of this situation.

In Section 2 we develop several convenient properties of the functor $(\cdot)^\#$ mentioned above. The main theorems of the following section contain all of the information we have been able to gather by homological methods concerning the relationship between the two groups A and $A^\#$.

We then exhibit several applications of our results. For example, we see easily that, if A is a reduced abelian group, then $\operatorname{Ext}(A, Z)$ is torsion-free if and only if A and certain of its subgroups are locally free. A part of this result has been obtained independently by Nunke and Ti Yen. We also show that, if $\operatorname{Ext}(A, Z) = 0$, $B \subseteq A$, and $B''/B = \ker(j_{A/B})$, then B and B'' have equal rank;[2] this fact provides a generalization of that part of Rotman's Density Lemma [11, p. 249] which applies to the problem of Whitehead. In addition, it becomes apparent that our principal results contain, as a special

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[1] Throughout this paper we shall use freely the concepts and techniques of homological algebra. For a systematic exposition of the subject we refer the reader to [2]. A more concise account of that portion of the theory which we shall use can be found in [7] and [11].

When we write $\operatorname{Hom}(A, B)$, $\operatorname{Ext}(A, B)$, etc., we shall omit the superfluous subscripts which refer to the coefficient ring.

[2] If A is a module over an integral domain R, and Q is the quotient field of R, the rank of A is defined to be the Q-dimension of the vector space $Q \otimes A$.

case, a part of a recent theorem of Nunke and Rotman [10] concerning the structure of singular integral cohomology groups.

In the final section of the paper we show that, if A has no p-divisible subgroups (p a prime) and the p-primary component of $\mathrm{Ext}(A, Z)$ is trivial, then A is slender; this generalizes a theorem of Rotman [11, p. 251]. We also prove that, if $\mathrm{Ext}(A, Z) = 0$, then the cardinality of $\mathrm{Hom}(A, Z)$ is greater than that of A.

Our results will be phrased for principal ideal domains. Hence throughout the paper R will denote a principal ideal domain with quotient field Q. All R-modules considered will be unitary. If A is an R-module, tA will represent the torsion submodule of A. If p is a prime in R, A_p and $t_p A$ will denote, respectively, the localization of A at p and the p-primary component of A.

2. The functor $(\cdot)^\#$

Before proceeding to an analysis of locally free modules, we review briefly some relevant properties of the functor $\mathrm{Hom}(\cdot, R)$. We refer the reader to [1] for a detailed treatment of this duality theory in a more general setting.

If A is an R-module, we shall write $A^* = \mathrm{Hom}(A, R)$; A^* is called the *dual* of A. If B is another module and $f : A \to B$ is a homomorphism, then the induced homomorphism $f^* : B^* \to A^*$ is called the *adjoint* of f. The natural homomorphism of A into A^{**} will be denoted by i_A. If A, B, and f are as above, then the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i_A"] & B \arrow[d, "i_B"] \\ A^{**} \arrow[r, "f^{**}"'] & B^{**} \end{tikzcd}$$

is commutative.

A will be called *torsionless* if i_A is a monomorphism. It is an easy exercise to prove that a module is torsionless if and only if it is isomorphic to a submodule of a direct product of copies of R. Hence a torsionless module is $\aleph_1$-free [4, p. 168]. In addition, it is apparent from the definition of i_A that A is torsionless if and only if, for any $x \neq 0$ in A, there exists $g \in A^*$ such that $g(x) \neq 0$. From this it follows without difficulty that a locally free module is torsionless.

To show that the converse is not true, we shall present an unpublished example due to Nunke of a torsionless abelian group which is not locally free.[3] Let $\prod$ be the direct product, and $\sum$ the direct sum of a countably infinite family of infinite cyclic groups; then $\sum$ is embedded in $\prod$ in an obvious way. Let $A = \sum + 2\prod$, a subgroup of $\prod$. That A is torsionless follows from the preceding paragraph. However, it is not difficult to show (using [4, Theorem

[3] It follows from this example that the assertion of Exercise 42(a) of [4, p. 183] is false. Additional information concerning this situation is provided in Theorem 4.2.

47.3, p. 171]) that the element $x = (2, 2, 2, \cdots)$ generates a pure subgroup of A which is not a direct summand, and hence A is not locally free.

The following two propositions seem to be well known, but to our knowledge have not appeared in the literature.

PROPOSITION 2.1. *Let A be an R-module. Then any pure submodule of A^* is locally free.*

Proof. Let $0 \to K \to F \to A \to 0$ be an exact sequence, where F is a free R-module. Since the functor $(\cdot)^*$ is left exact, we obtain the exact sequence $0 \to A^* \to F^* \to K^*$. Now, F^* is isomorphic to a direct product of copies of R, and is therefore locally free by [4, Theorem 47.1] (see the first paragraph of the proof, p. 168). Since K^* is torsion-free, A^* is isomorphic to a pure submodule of F^*. If B is a pure submodule of A^*, then B is likewise isomorphic to a pure submodule of F^*. We may then apply Theorem 49.3 of [4, p. 178] to conclude that B is locally free, completing the proof of the proposition.

PROPOSITION 2.2. *An R-module A is locally free if and only if i_A is a monomorphism of A onto a pure submodule of A^{**}.*

Proof. If the latter condition holds, then A is isomorphic to a pure submodule of A^{**}, and is hence locally free by Proposition 2.1.

Conversely, assume that A is locally free. Then, by a previous remark, A is torsionless, and so i_A is a monomorphism. For notational convenience we shall identify A with its image in A^{**}. Let $\langle\ \rangle : A^* \times A^{**} \to R$ denote the natural pairing of A^* and A^{**}. Since A^{**} is $\aleph_1$-free, every pure submodule of A^{**} of rank one is cyclic. Let y generate such a submodule, and let $A \cap (Ry) = Rx$; then $x = ay$ for some $a \in R$, and Rx is a pure submodule of A. If $x \neq 0$, then, since A is locally free, Rx is a direct summand of A, and hence there exists $z \in A^*$ such that $\langle z, x\rangle = 1$. If $\langle z, y\rangle = b$, then $ab = 1$, and so $Rx = Ry$; i.e., $Ry \subseteq A$. We have shown that any pure submodule of A^{**} of rank one is either contained in A or has trivial intersection with A. It then follows that A is a pure submodule of A^{**}, completing the proof.

DEFINITION 2.3. If A is any R-module, denote by $A^{\#}$ the intersection of all pure submodules of A^{**} which contain $i_A(A)$. Let $j_A : A \to A^{\#}$ be the homomorphism defined by i_A (i.e., j_A differs from i_A only by contraction of the range).

It is easy to see that, since A^{**} is torsion-free, $A^{\#}$ is itself a pure submodule of A^{**}. Furthermore, it is clear from Proposition 2.1 that $A^{\#}$ is locally free. Finally, $\ker(j_A) = \ker(i_A)$ and $\operatorname{coker}(j_A) = t\{A^{**}/i_A(A)\}$.

PROPOSITION 2.4. *Let A, B be R-modules, and let $f : A \to B$ be a homomorphism. Then there exists a homomorphism $f^{\#} : A^{\#} \to B^{\#}$ such that the diagram*

(1)
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j_A"] & B \arrow[d, "j_B"] \\ A^{\#} \arrow[r, "f^{\#}"'] & B^{\#} \end{tikzcd}$$

is commutative.

Proof. Let $f^{\#}$ be the restriction to $A^{\#}$ of the mapping $f^{**} : A^{**} \to B^{**}$. If $x \in A^{\#}$, then ax is in $i_A(A)$ for some $a \in R$. Since the diagram

(2)
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i_A"] & B \arrow[d, "i_B"] \\ A^{**} \arrow[r, "f^{**}"'] & B^{**} \end{tikzcd}$$

is commutative, we obtain that $af^{\#}(x) = f^{**}(ax) \in i_B(B)$, and so $f^{\#}(x) \in B^{\#}$. Hence $f^{\#}(A^{\#}) \subseteq B^{\#}$, and so $f^{\#}$ is a homomorphism of $A^{\#}$ into $B^{\#}$. The commutativity of diagram (1) follows easily from the commutativity of diagram (2). This completes the proof.

It is a routine exercise to verify that the mapping defined on the category of R-modules and R-homomorphisms which assigns to each module A the module $A^{\#}$, and to each homomorphism f the homomorphism $f^{\#}$, is an additive functor. Furthermore, it follows easily from Proposition 2.4 that j is a natural transformation of the identity functor into the functor $(\cdot)^{\#}$.

PROPOSITION 2.5. *Let A be an R-module. Then A is torsionless if and only if j_A is a monomorphism, and A is locally free if and only if j_A is an isomorphism.*

Proof. The first statement follows from our previous remark that $\ker(i_A) = \ker(j_A)$. The second statement follows immediately from Proposition 2.2 and the definition of j_A.

The next proposition provides a characterization of $A^{\#}$ among locally free modules in terms of a universal property.

PROPOSITION 2.6. *Let A and B be R-modules, and let B be locally free. If $f : A \to B$ is a homomorphism, then there exists a unique homomorphism $g : A^{\#} \to B$ such that $gj_A = f$. If f is a monomorphism, then g is also a monomorphism.*

Proof. Since B is locally free, j_B is an isomorphism, by Proposition 2.5. Now set $g = j_B^{-1}f^{\#}$; then g is a homomorphism of $A^{\#}$ into B, and we obtain easily from Proposition 2.4 that $gj_A = f$. To show uniqueness, suppose $h : A^{\#} \to B$ is another homomorphism such that $hj_A = f$. If $x \in A^{\#}$, then $ax \in j_A(A)$ for some $a \in R$; i.e., $ax = j_A(u)$, $u \in A$. Then $ag(x) = g(ax) = gj_A(u) = f(u)$, and $ah(x) = f(u)$ by a similar argument. Since B is torsion-free, it follows that $g(x) = h(x)$. This is true for all $x \in A^{\#}$, and so $h = g$. This completes the proof of uniqueness of g.

Finally, let us assume that f is a monomorphism. If $x \in A^{\#}$, then $ax \in j_A(A)$ for some $a \in R$, and so $ax = j_A(u)$, $u \in A$. We may repeat the above argument to conclude that $ag(x) = f(u)$. If $g(x) = 0$, then $f(u) = 0$, and so $u = 0$, since f is a monomorphism. Then $ax = j_A(u) = 0$, and so $x = 0$, since $A^{\#}$ is torsion-free. It then follows that g is a monomorphism, and the proof is complete.

We end this section with a proposition which will be useful in our later work.

PROPOSITION 2.7. *Given an R-module A, let*

$$\mu : \ker(j_A) \to A \quad \textit{and} \quad \nu : \operatorname{Im}(j_A) \to A^{\#}$$

be the inclusion mappings, and let $\pi : A \to \operatorname{Im}(j_A)$ be the canonical mapping (so π differs from j_A only in contraction of the range). Then $\mu^* = 0$; and

$$\nu^* : (A^{\#})^* \to (\operatorname{Im}(j_A))^*, \qquad \pi^* : (\operatorname{Im}(j_A))^* \to A^*, \quad \textit{and} \quad j_A^* : (A^{\#})^* \to A^*$$

are isomorphisms.

Proof. If $f \in A^*$ and $x \in \ker(j_A)$, then $(\mu^* f)x = f(x)$. (Here we are identifying $\ker(j_A)$ with its image in A.) But $\ker(j_A) = \ker(i_A)$, and it follows from the definition of i_A that $\ker(i_A)$ consists of all $x \in A$ such that $f(x) = 0$ for all $f \in A^*$. We then obtain immediately that $\mu^*(f) = 0$ for all $f \in A^*$; i.e., $\mu^* = 0$.

Now, the exact sequences

$$\operatorname{Im}(j_A) \xrightarrow{\ \nu\ } A^{\#} \to \operatorname{coker}(j_A) \to 0, \qquad A \xrightarrow{\ \pi\ } \operatorname{Im}(j_A) \to 0$$

give rise to the exact sequences

$$0 \to (\operatorname{coker}(j_A))^* \to (A^{\#})^* \xrightarrow{\ \nu^*\ } (\operatorname{Im}(j_A))^*, \qquad 0 \to (\operatorname{Im}(j_A))^* \xrightarrow{\ \pi^*\ } A^*.$$

But $\operatorname{coker}(j_A)$ is a torsion module, and so $(\operatorname{coker}(j_A))^* = 0$; hence ν^* and π^* are monomorphisms. But $\nu\pi = j_A$, and so $j_A^* = \pi^*\nu^*$, from which it follows that j_A^* is a monomorphism, too.

Now let $\theta : A^{\#} \to A^{**}$ be the inclusion mapping. Then $\theta j_A = i_A$, and so $i_A^* = j_A^*\theta^*$. We may then apply Theorem 1.4 of [5] to conclude that i_A^* is an epimorphism, from which it follows that j_A^* is likewise an epimorphism. Since $j_A^* = \pi^*\nu^*$ and all three mappings are monomorphisms, we then obtain readily that all three are indeed isomorphisms.

3. The main theorems

From now on we shall assume that R is a principal ideal domain which is not a field. In this section we shall compare the R-modules A and $A^{\#}$ by homological methods.

If p is a prime in R and A is an R-module, we shall write

$$s_p A = \{x \in A \mid px = 0\};$$

$s_p A$ is sometimes called the *p-socle* of A. $s_p A$ is a vector space over the field $R/(p)$, and we shall denote its dimension by $r_p(A)$. If $A \subseteq B$, then $s_p A \subseteq s_p B$, and so $r_p(A) \leqq r_p(B)$. It is also possible to show without much difficulty that $r_p(B) \leqq r_p(A) + r_p(B/A)$. If A is p-primary, then $s_q A = 0$ for any prime q distinct from p; in this case we shall omit the superfluous subscripts and write sA, $r(A)$ for $s_p A$, $r_p(A)$, respectively.

LEMMA 3.1. *Let A be an R-module, and M a divisible module containing A. Then $r(A/pA) \leqq r_p(M/A)$.*

Proof. Let $\{x_\alpha + pA\}$ be an $R/(p)$-basis of A/pA, where $\{x_\alpha\} \subseteq A$. Since M is divisible, there exist $\{y_\alpha\} \subseteq M$ such that $py_\alpha = x_\alpha$. Observe that $\{y_\alpha + A\} \subseteq s_p(M/A)$. If $c_1 y_{\alpha_1} + \cdots + c_n y_{\alpha_n} \equiv 0 \pmod{A}$, where $c_i \in R$, then

$$c_1 x_{\alpha_1} + \cdots + c_n x_{\alpha_n} = p(c_1 y_{\alpha_1} + \cdots + c_n y_{\alpha_n}) \equiv 0 \pmod{pA}.$$

In this case $c_i \equiv 0 \pmod{p}$ for $i \leqq n$, since $\{x_\alpha + pA\}$ is an $R/(p)$-basis of A/pA. Hence $\{y_\alpha + A\}$ is an $R/(p)$-linearly independent subset of $s_p(M/A)$, from which it follows that $r(A/pA) \leqq r_p(M/A)$.

LEMMA 3.2. *For any R-module A we have the following exact sequences:*

$$\begin{aligned} (3)\quad & 0 \to (\ker(j_A))^* \to \operatorname{Ext}(\operatorname{Im}(j_A), R) \\ & \qquad \to \operatorname{Ext}(A, R) \to \operatorname{Ext}(\ker(j_A), R) \to 0, \end{aligned}$$

$$(4)\quad 0 \to \operatorname{Ext}(\operatorname{coker}(j_A), R) \to \operatorname{Ext}(A^\#, R) \to \operatorname{Ext}(\operatorname{Im}(j_A), R) \to 0.$$

Proof. Consider the exact sequence $0 \to \ker(j_A) \xrightarrow{\mu} A \to \operatorname{Im}(j_A) \to 0$, where μ is the inclusion mapping. Since $\mu^* = 0$, by Proposition 2.7, (3) is simply a portion of the resulting cohomology sequence.

Now consider the exact sequence $0 \to \operatorname{Im}(j_A) \xrightarrow{\nu} A^\# \to \operatorname{coker}(j_A) \to 0$, where ν is the inclusion mapping. By Proposition 2.7, $\nu^* : (A^\#)^* \approx (\operatorname{Im}(j_A))^*$ is an isomorphism. Therefore the resulting cohomology sequence gives rise to (4), and the proof is complete.

THEOREM 3.3. *The following conditions hold for any R-module A and any prime p in R:*

(a) $r\{\operatorname{Hom}(\ker(j_A), R/pR)\} \leqq r_p\{\operatorname{Ext}(A, R)\}$.

(b) $r\{\operatorname{Hom}[s_p(\operatorname{coker}(j_A)), R/pR]\} \leqq r_p\{\operatorname{Ext}(A, R)\}$.

Proof. For notational convenience we shall set $B = \ker(j_A)$, $C = \operatorname{coker}(j_A)$.

Consider the exact sequence $0 \to R \xrightarrow{m_p} R \xrightarrow{f} R/pR \to 0$, where f is the canonical mapping and $m_p(a) = pa$ for $a \in R$. This gives rise to the exact cohomology sequence

$$A^* \xrightarrow{f_*} \operatorname{Hom}(A, R/pR) \to \operatorname{Ext}(A, R) \xrightarrow{m_{p*}} \operatorname{Ext}(A, R).$$

But $m_{p*}(u) = pu$ for u in $\operatorname{Ext}(A, R)$, and so we may shorten the above to the exact sequence

$$A^* \xrightarrow{f_*} \operatorname{Hom}(A, R/pR) \xrightarrow{\delta} s_p\{\operatorname{Ext}(A, R)\}.$$

Now let $\mu : B \to A$ be the inclusion mapping, and set $X = \operatorname{coker}(\mu)$. Then the exact sequence

$$0 \to B \xrightarrow{\mu} A \to X \to 0$$

gives rise to the exact cohomology sequence

$$\operatorname{Hom}(A, R/pR) \xrightarrow{\bar{\mu}} \operatorname{Hom}(B, R/pR) \to \operatorname{Ext}(X, R/pR).$$

But $X \approx \operatorname{Im}(j_A)$ and is hence torsion-free, and so it follows from Corollary 7.8 of [7, p. 237] that $\operatorname{Ext}(X, R/pR) = 0$. Thus $\bar{\mu}$ is an epimorphism. (This fact can also be derived by an easy nonhomological argument.) Since all modules in the exact sequence just mentioned can be viewed also as vector spaces over the field $R/(p)$, we then see easily that there exists a homomorphism $\alpha : \operatorname{Hom}(B, R/pR) \to \operatorname{Hom}(A, R/pR)$ such that $\bar{\mu}\alpha$ is the identity mapping on $\operatorname{Hom}(B, R/pR)$. We now assemble the information which we have so far collected into the following master diagram:

$$\begin{tikzcd}
A^* \arrow[rr, "\mu^*"] \arrow[d, "f_*"'] & & B^* \arrow[d] \\
\operatorname{Hom}(A, R/pR) \arrow[rr, shift left, "\bar{\mu}"] \arrow[d, "\delta"'] & & \operatorname{Hom}(B, R/pR) \arrow[ll, shift left, "\alpha"] \arrow[dll, dotted, "\sigma"] \\
s_p \operatorname{Ext}(A, R) & &
\end{tikzcd}$$

where $\sigma = \delta\alpha$ by definition, and the mapping $B^* \to \operatorname{Hom}(B, R/pR)$ is induced by $f : R \to R/pR$. The upper square and the lower triangle of the diagram are commutative, and the vertical sequence on the left is exact. But, recalling that $\mu^* = 0$ by Proposition 2.7, we obtain from routine diagram-chasing that σ is a monomorphism. (a) then follows immediately.

Turning now to (b), we observe that the exact sequence

$$0 \to s_p C \to C \to C/s_p C \to 0$$

gives rise to the exact cohomology sequence

$$\mathrm{Ext}(C, R) \to \mathrm{Ext}(s_p C, R) \to 0.$$

But $p\{\mathrm{Ext}(s_p C, R)\} = 0$; hence, setting $E = \mathrm{Ext}(C, R)$, we see that

$$r\{\mathrm{Ext}(s_p C, R)\} \leqq r(E/pE).$$

Furthermore, setting $M = \mathrm{Ext}(A^{\#}, R)$, we obtain from the exact sequence (4) of Lemma 3.2 that $M/E \approx \mathrm{Ext}(\mathrm{Im}(j_A), R)$. But M is divisible, since $A^{\#}$ is torsion-free [2, Proposition 5.3, p. 135]. Hence, utilizing Lemma 3.1 and the above inequality, we see that $r\{\mathrm{Ext}(s_p C, R)\} \leqq r_p\{\mathrm{Ext}(\mathrm{Im}(j_A), R)\}$. But since $(\ker(j_A))^*$ is torsion-free, it follows easily from the exact sequence (3) of Lemma 3.2 that $r_p\{\mathrm{Ext}(\mathrm{Im}(j_A), R)\} \leqq r_p\{\mathrm{Ext}(A, R)\}$, and so we may conclude that $r\{\mathrm{Ext}(s_p C, R)\} \leqq r_p(\mathrm{Ext}(A, R)\}$.

Now $s_p C = \sum \oplus_i C_i$, where $C_i \approx R/pR$ and i traces an index set of cardinality equal to $r_p(C)$. But $\mathrm{Ext}(C_i, R) \approx C_i$ (Exercise 1 of [2, p. 139]); hence $\mathrm{Ext}(s_p C, R) \approx \prod_i C_i \approx \mathrm{Hom}(s_p C, R/pR)$. Combining this isomorphism with the final inequality of the preceding paragraph, we obtain at last that $r\{\mathrm{Hom}(s_p C, R/pR)\} \leqq r_p\{\mathrm{Ext}(A, R)\}$. This establishes (b) and completes the proof of the theorem.

In the remainder of the discussion we shall sometimes, for notational convenience, write $r^p(A)$ for $r(A/pA)$, A being an R-module and p a prime in R. If α is a cardinal number, we shall often denote 2^α by $\exp(\alpha)$.

COROLLARY 3.4. *If A is an R-module and p is a prime in R, then $r^p(\ker(j_A))$ and $r_p(\mathrm{coker}(j_A))$ are less than or equal to $r_p\{\mathrm{Ext}(A, R)\}$. Furthermore, the following conditions hold if R has a countable number of elements*:

(a) *If $r_p\{\mathrm{Ext}(A, R)\} < 2^{\aleph_0}$, then $r^p(\ker(j_A))$ and $r_p(\mathrm{coker}(j_A))$ are both finite.*

(b) *If $r_p\{\mathrm{Ext}(A, R)\} \geqq 2^{\aleph_0}$, then $\exp[r^p(\ker(j_A))] \leqq r_p\{\mathrm{Ext}(A, R)\}$, and $\exp[r_p(\mathrm{coker}(j_A))] \leqq r_p\{\mathrm{Ext}(A, R)\}$.*

Proof. These estimates follow in routine fashion from the statements of Theorem 3.3. The details of the argument will be omitted.

It is unfortunate that Theorem 3.3 and its corollary exhaust the information we have at present regarding $\mathrm{coker}(j_A)$. However, a more penetrating result concerning $\ker(j_A)$ can be obtained if we assume that R has a countable number of elements; this assumption will hence be in effect throughout the remainder of this section.

In the following discussion, the cardinality of a set X will be denoted by $|X|$.

LEMMA 3.5. *Let A be a torsion R-module. Then*:

(a) *If $|\mathrm{Ext}(A, R)| < 2^{\aleph_0}$, then A is finitely generated.*

(b) *If $|\mathrm{Ext}(A, R)| \geqq 2^{\aleph_0}$, then $2^{|A|} \leqq |\mathrm{Ext}(A, R)|$.*

Proof. We may apply Theorem 29.2 of [4, p. 98] to obtain the existence of an exact sequence $0 \to C \to A \to M \to 0$, where C is a direct sum of cyclic modules and M is divisible. Then $\mathrm{Hom}(C, R) = 0$ because C is a torsion module, and so we obtain the exact cohomology sequence

$$0 \to \mathrm{Ext}(M, R) \to \mathrm{Ext}(A, R) \to \mathrm{Ext}(C, R) \to 0.$$

Now $M = \sum \oplus_i M_i$, where M_i is indecomposable and primary and i traces an index set of cardinality α, say. Then $\mathrm{Ext}(M, R) \approx \prod_i \mathrm{Ext}(M_i, R)$. But it follows from [4, p. 211] that for each i there exists a prime $p = p(i)$ in R such that $\mathrm{Hom}(M_i, M_i) \approx \bar{R}_p$, the completion of the valuation ring R_p. From this we obtain by a routine computation that $\mathrm{Ext}(M_i, R) \approx \bar{R}_p$. Since $|\bar{R}_p| = 2^{\aleph_0}$, we see that if $|\mathrm{Ext}(M, R)| < 2^{\aleph_0}$, then $M = 0$. If $|\mathrm{Ext}(M, R)| \geqq 2^{\aleph_0}$, then $2^\alpha \leqq |\mathrm{Ext}(M, R)|$.

Let us now write $C = \sum \oplus_j C_j$, where each C_j is cyclic and primary and j traces an index set of cardinality β. We obtain from Exercise 1 of [2, p. 139] that $\mathrm{Ext}(C_j, R) \approx C_j$, and so

$$\mathrm{Ext}(C, R) \approx \prod_j C_j.$$

If $|\mathrm{Ext}(C, R)| < 2^{\aleph_0}$, we see that β is finite; in this case C is clearly finitely generated. If $|\mathrm{Ext}(C, R)| \geqq 2^{\aleph_0}$, we obtain that $2^\beta \leqq |\mathrm{Ext}(C, R)|$.

Assume now that $|\mathrm{Ext}(A, R)| < 2^{\aleph_0}$; then $|\mathrm{Ext}(M, R)|$ and $|\mathrm{Ext}(C, R)|$ are likewise. It then follows from the above remarks that $M = 0$ and C is finitely generated. In this case $A = C$, and (a) is established.

On the other hand, suppose $|\mathrm{Ext}(A, R)| \geqq 2^{\aleph_0}$. Then, since

$$|\mathrm{Ext}(A, R)| = |\mathrm{Ext}(C, R)| \; |\mathrm{Ext}(M, R)|,$$

we obtain easily from the preceding remarks that $2^\alpha \leqq |\mathrm{Ext}(A, R)|$, $2^\beta \leqq |\mathrm{Ext}(A, R)|$. But since $|A| \leqq \aleph_0 + \max(\alpha, \beta)$, it follows that

$$2^{|A|} \leqq 2^{\aleph_0} |\mathrm{Ext}(A, R)| = |\mathrm{Ext}(A, R)|.$$

This establishes (b) and completes the proof of the lemma.

LEMMA 3.6. *If A is an R-module, then $|A^*| \leqq 2^{|A|}$. If A has finite rank, then* $\mathrm{rank}(A^*) \leqq \mathrm{rank}(A)$.

Proof. Let α be the rank of A; then there exists an exact sequence $F \to A \to C \to 0$, where F is a free R-module of rank α and C is a torsion module. This gives rise to the exact sequence $0 \to C^* \to A^* \to F^*$. But since C is a torsion module, $C^* = 0$, and so A^* is isomorphic to a submodule of F^*. But F^* is a direct product of α copies of R, and so $|F^*| \leqq 2^\alpha$, since R is countable. Since $\alpha \leqq |A|$, we then get that $|A^*| \leqq |F^*| \leqq 2^\alpha \leqq 2^{|A|}$.

If A has finite rank, then $\mathrm{rank}(A^*) \leqq \mathrm{rank}(F^*) = \mathrm{rank}(F) = \mathrm{rank}(A)$, completing the proof of the lemma.

If p is a prime in R, we shall denote by $Q(p)$ the submodule of Q generated by all elements which have denominator a power of p. This module plays an important role in the following lemma.

Recall that an R-module A is *p-divisible* if $pA = A$.

LEMMA 3.7. *Let A be a nontrivial torsion-free p-divisible R-module, where p is a prime in R. Then*

$$|\operatorname{Ext}(A, R)| \geqq 2^{\aleph_0} \quad \textit{and} \quad \exp[\operatorname{rank}(A)] \leqq |\operatorname{Ext}(A, R)|.$$

Proof. We see that A contains a free submodule F which has rank equal to that of A. But since A is torsion-free and p-divisible, it then follows easily that A contains a submodule $E = \sum \oplus_i E_i$, where i traces an index set of cardinality equal to rank(A), and $E_i \approx Q(p)$. We obtain from a routine computation that $\operatorname{Ext}(E_i, R) \approx \bar{R}_p/R$, where $\bar{R}_p$ is, as before, the completion of the valuation ring R_p. Then $\operatorname{Ext}(E, R) \approx \prod_i \operatorname{Ext}(E_i, R)$, and is the direct product of a family of copies of $\bar{R}_p/R$, the cardinality of the family being equal to the rank of A. Since $|\bar{R}_p/R| = 2^{\aleph_0}$ and $A \neq 0$, it is clear that $|\operatorname{Ext}(E, R)| \geqq 2^{\aleph_0}$ and $\exp[\operatorname{rank}(A)] \leqq |\operatorname{Ext}(E, R)|$.

Now, the exact sequence $0 \to E \to A \to A/E \to 0$ gives rise to the exact cohomology sequence $\operatorname{Ext}(A, R) \to \operatorname{Ext}(E, R) \to 0$. From this, together with the preceding remarks, it follows that $|\operatorname{Ext}(A, R)| \geqq 2^{\aleph_0}$ and $\exp[\operatorname{rank}(A)] \leqq |\operatorname{Ext}(A, R)|$, which was to be proved.

The following theorem completes our analysis of the kernel of j_A.

THEOREM 3.8. *Let R be a countable principal ideal domain. Then the following conditions hold for any R-module A:*

(a) *If* $|\operatorname{Ext}(A, R)| < 2^{\aleph_0}$, *then* $\ker(j_A)$ *is finitely generated.*

(b) *If* $|\operatorname{Ext}(A, R)| \geqq 2^{\aleph_0}$, *then* $\exp(|\ker(j_A)|) \leqq |\operatorname{Ext}(A, R)|$.

Proof. Set $B = \ker(j_A)$, and let B' be any submodule of B. Then the exact sequence $0 \to B' \to A$ induces the exact cohomology sequence

$$\operatorname{Ext}(A, R) \to \operatorname{Ext}(B', R) \to 0,$$

and so $|\operatorname{Ext}(B', R)| \leqq |\operatorname{Ext}(A, R)|$. In particular,

$$|\operatorname{Ext}(B, R)| \leqq |\operatorname{Ext}(A, R)| \quad \text{and} \quad |\operatorname{Ext}(tB, R)| \leqq |\operatorname{Ext}(A, R)|.$$

Now set $F = B/tB$. Since $(tB)^* = 0$, the exact sequence

$$0 \to tB \to B \to F \to 0$$

gives rise to the exact sequence

$$0 \to \operatorname{Ext}(F, R) \to \operatorname{Ext}(B, R),$$

and so $|\operatorname{Ext}(F, R)| \leqq |\operatorname{Ext}(B, R)| \leqq |\operatorname{Ext}(A, R)|$. Furthermore, if p is a prime in R, F/pF is an epimorphic image of B/pB. From this fact and Corollary 3.4 we obtain that

$$\exp[r(F/pF)] \leqq \exp[r(B/pB)] \leqq r_p\{\operatorname{Ext}(A, R)\} \leqq |\operatorname{Ext}(A, R)|.$$

Let $\{x_\alpha + pF\}$ be an $R/(p)$-basis of F/pF, where α traces an index set of cardinality equal to $r(F/pF)$. Let K be the pure submodule of F generated by all x_α (i.e., the smallest pure submodule of F containing all x_α). Set

$M = F/K$. Since K is pure in F, M is torsion-free. Furthermore, we see that $K + pF = F$, and so $pM = M$; i.e., M is p-divisible. Observe finally that the exact sequence $0 \to K \to F \to M \to 0$ gives rise to the exact cohomology sequence

$$K^* \to \mathrm{Ext}(M, R) \to \mathrm{Ext}(F, R). \tag{5}$$

Suppose now that $|\mathrm{Ext}(A, R)| < 2^{\aleph_0}$. We then obtain from our previous remarks that $|\mathrm{Ext}(F, R)| \leqq |\mathrm{Ext}(B, R)| < 2^{\aleph_0}$ and $r(F/pF)$ is finite. Since $\mathrm{rank}(K) \leqq r(F/pF)$ and R is countable, we see from Lemma 3.6 that K^* has finite rank, and so $|K^*| \leqq \aleph_0$. Hence, by (5) above, $|\mathrm{Ext}(M, R)| < 2^{\aleph_0}$, and so we may apply Lemma 3.7 to conclude that $M = 0$. Thus $F = K$, a module of finite rank.

Since $F = B/tB$, we may then construct an exact sequence

$$0 \to F_0 \to B \to T \to 0,$$

where F_0 is free of the same (finite) rank as F, and T is a torsion module. This gives rise to the exact cohomology sequence $F_0^* \to \mathrm{Ext}(T, R) \to \mathrm{Ext}(B, R)$. Since F_0 is free, F_0^* is also free with same rank as F_0, and so $|F_0^*| \leqq \aleph_0$. Then, since $|\mathrm{Ext}(B, R)| < 2^{\aleph_0}$, it follows that $|\mathrm{Ext}(T, R)| < 2^{\aleph_0}$. Therefore, by Lemma 3.5, T is finitely generated. Since F_0 is finitely generated, we obtain finally that B is finitely generated. This establishes (a).

Turning now to (b), we assume that $|\mathrm{Ext}(A, R)| \geqq 2^{\aleph_0}$. Since $\mathrm{rank}(K) \leqq r(F/pF)$, we have that $|K| \leqq \aleph_0 + r(F/pF)$, and so, by Lemma 3.6 and a preceding remark,

$$|K^*| \leqq 2^{\aleph_0} \exp[r(F/pF)] \leqq 2^{\aleph_0} |\mathrm{Ext}(A, R)| = |\mathrm{Ext}(A, R)|.$$

Since $|\mathrm{Ext}(F, R)| \leqq |\mathrm{Ext}(A, R)|$, we then obtain from (5) that $|\mathrm{Ext}(M, R)| \leqq |\mathrm{Ext}(A, R)|$. We may then apply Lemma 3.7 to conclude that $\exp[\mathrm{rank}(M)] \leqq |\mathrm{Ext}(A, R)|$. But $\mathrm{rank}(F) = \mathrm{rank}(K) + \mathrm{rank}(M)$, and so $\exp[\mathrm{rank}(F)] \leqq \exp[\mathrm{rank}(K)] \exp[\mathrm{rank}(M)] \leqq |\mathrm{Ext}(A, R)|$. Since $|F| \leqq \aleph_0 + \mathrm{rank}(F)$, we also obtain the inequality

$$2^{|F|} \leqq |\mathrm{Ext}(A, R)|.$$

Now we have seen that $|\mathrm{Ext}(tB, R)| \leqq |\mathrm{Ext}(A, R)|$, and so, by Lemma 3.5, $2^{|tB|} \leqq |\mathrm{Ext}(A, R)|$. But since $F = B/tB$,

$$|F| \leqq \aleph_0 + \max(|F|, |tB|),$$

and so $2^{|F|} \leqq 2^{\aleph_0} |\mathrm{Ext}(A, R)| = |\mathrm{Ext}(A, R)|$. This establishes (b) and completes the proof of the theorem.

4. Applications

We shall now apply the results of the preceding section to the problem of Whitehead and to related problems. For a while we may assume that R is any principal ideal domain which is not a field.

THEOREM 4.1. *Let p be a prime in R, and A an R-module. Assume that $t_p\{\text{Ext}(A, R)\} = 0$. Then A possesses a p-divisible submodule B such that A/B is torsionless.*

Proof. Set $B = \ker(j_A)$; then $A/B \approx \text{Im}(j_A)$, which we know is torsionless because it is a submodule of the locally free module $A^{\#}$. Hence we need only show that B is p-divisible. But we have from Corollary 3.4 that $r(B/pB) \leqq r_p\{\text{Ext}(A, R)\} = 0$, since $t_p\{\text{Ext}(A, R)\} = 0$. Therefore $B/pB = 0$, and so $pB = B$; i.e., B is p-divisible. This completes the proof of the theorem.

THEOREM 4.2. *The following statements are equivalent for any reduced R-module A:*

(a) $\text{Ext}(A, R)$ *is torsion-free.*

(b) *If $B \subseteq A$ and A/B has bounded order, then B is locally free.*

(c) *If $B \subseteq A$ and $A/B \approx R/pR$, where p is either a unit or a prime, then B is locally free.*

If any (and hence all) of these conditions hold, then $\text{Ext}(B, R)$ *is torsion-free whenever $B \subseteq A$ and A/B has bounded order.*

Proof. (a) $\Rightarrow$ (b). Let $B \subseteq A$ and $T = A/B$. We then get the exact cohomology sequence

$$\text{Ext}(T, R) \to \text{Ext}(A, R) \to \text{Ext}(B, R) \to 0.$$

If $aT = 0$ for some $a \in R$, then $a\{\text{Ext}(T, R)\} = 0$; hence, if $\text{Ext}(A, R)$ is torsion-free, the mapping $\text{Ext}(T, R) \to \text{Ext}(A, R)$ is trivial. In this case $\text{Ext}(B, R) \approx \text{Ext}(A, R)$, and hence $\text{Ext}(B, R)$ is torsion-free; this, incidentally, establishes the last statement of the theorem. Also, we see that in order to establish (b) we need only show that if $\text{Ext}(A, R)$ is torsion-free, then A itself is locally free.

Now if $\text{Ext}(A, R)$ is torsion-free, then $r_p\{\text{Ext}(A, R)\} = 0$ for all primes p in R, and so we may apply Corollary 3.4 to conclude that

$$r^p(\ker(j_A)) = r_p(\text{coker}(j_A)) = 0.$$

Hence $t_p(\text{coker}(j_A)) = 0$ for all p, and so $\text{coker}(j_A) = 0$, since it is a torsion module. On the other hand, we have that $\ker(j_A)/p(\ker(j_A)) = 0$, and so $p(\ker(j_A)) = \ker(j_A)$ for all primes p in R. It then follows easily that $a(\ker(j_A)) = \ker(j_A)$ for all $a \neq 0$ in R; i.e., $\ker(j_A)$ is divisible. Since A is reduced, it follows that $\ker(j_A) = 0$. Thus we have shown that $j_A : A \approx A^{\#}$ is an isomorphism, and so A is locally free.

(b) $\Rightarrow$ (c). Assume (b) holds; then A itself is certainly locally free. Let B be a submodule of A such that $A/B \approx R/pR$ for some prime p. Then A/B has bounded order, and so B is locally free.

(c) $\Rightarrow$ (a). If (c) holds, then A is certainly locally free, and therefore possesses a direct summand isomorphic to R. Let us write $A \approx R \oplus A_1$;

then $\operatorname{Ext}(A, R) \approx \operatorname{Ext}(A_1, R)$, and so we need only prove that $\operatorname{Ext}(A_1, R)$ is torsion-free.

We shall follow the standard practice of identifying $\operatorname{Ext}(A_1, R)$ with a set of equivalence classes of exact sequences of the form $0 \to R \to E \to A_1 \to 0$; a detailed description of this correspondence is to be found in [2, Chapter XIV]. Let the exact sequence $0 \to R \to E \to A_1 \to 0$ represent the element u of $\operatorname{Ext}(A_1, R)$, and assume that $pu = 0$ for some prime p in R; this means that there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \longrightarrow & E & \longrightarrow & A_1 & \longrightarrow & 0 \\ & & {\scriptstyle m_p}\downarrow & & {\scriptstyle f}\downarrow & & \| & & \\ 0 & \longrightarrow & R & \rightarrow & R \oplus A_1 & \longrightarrow & A_1 & \longrightarrow & 0 \end{array}$$

where the vertical mapping on the right is the identity and $m_p : R \to R$ is defined by $m_p(a) = pa$. It then follows from routine diagram-chasing that f is a monomorphism; furthermore, $\operatorname{coker}(f) \approx \operatorname{coker}(m_p) \approx R/pR$. Since $R \oplus A_1 \approx A$, we may then apply (c) to conclude that E is locally free, in which case the exact sequence $0 \to R \to E \to A_1 \to 0$ splits. This means simply that $u = 0$. We have thus shown that $s_p\{\operatorname{Ext}(A_1, R)\} = 0$ for all primes p in R, from which it follows immediately that $\operatorname{Ext}(A_1, R) \approx \operatorname{Ext}(A, R)$ is torsion-free. This establishes (a) and completes the proof of the theorem.

The fact that A reduced and $\operatorname{Ext}(A, R)$ torsion-free implies A locally free was proved independently, in unpublished work, by Nunke and Ti Yen.

In the remainder of this section we shall once again assume that R has a countable number of elements. We may then use Theorem 3.8 to derive information concerning modules A for which $\operatorname{Ext}(A, R)$ is countable. First we prove the following necessary lemma.

LEMMA 4.3. *Let A be an R-module, and B a submodule of A. If B and A/B are $\aleph_1$-free, then A is likewise $\aleph_1$-free.*

Proof. Let F be a submodule of A of countable rank; then $F \cap B$ is a submodule of B of countable rank, and is therefore free, since B is $\aleph_1$-free. Furthermore, $F/(F \cap B)$ has countable rank and is isomorphic to a submodule of A/B; hence $F/(F \cap B)$ is free, since A/B is $\aleph_1$-free. It then follows that $F \approx (F \cap B) \oplus (F/(F \cap B))$ and is therefore free. This completes the proof that A is $\aleph_1$-free.

THEOREM 4.4. *Let R be a principal ideal domain with a countable number of elements, and let A be an R-module such that $|\operatorname{Ext}(A, R)| < 2^{\aleph_0}$. Then $A = T \oplus B$, where T is a finitely generated torsion module and B is $\aleph_1$-free. Furthermore, B contains a free submodule F of finite rank such that B/F is torsionless.*

Proof. We have from Theorem 3.8 that $\ker(j_A)$ is finitely generated, in which case $\ker(j_A) = T \oplus F$, where T is a finitely generated torsion module and F is free of finite rank. Since $tA \subseteq \ker(j_A)$, we see that $T = tA$. It then follows from Theorem 8 of [6, p. 18] that $A = T \oplus B$ for some torsion-free submodule B of A. Replacing F by an isomorphic image, if necessary, we may assume that $F \subseteq B$. Then $B/F \approx A/\ker(j_A) \approx \operatorname{Im}(j_A)$, which is torsionless. In particular, B/F is $\aleph_1$-free. We may then apply Lemma 4.3 to conclude that B is $\aleph_1$-free, completing the proof of the theorem.

The following corollary to Theorem 4.4 is related to a recent result of Nunke and Rotman [10] concerning the structure of singular integral cohomology groups.

COROLLARY 4.5. *Let A be a countable R-module such that* $\operatorname{Ext}(A, R)$ *is countable. Then $A \approx T \oplus F$, where T is a finitely generated torsion module and F is free. In particular,* $\operatorname{Ext}(A, R) \approx \operatorname{Ext}(T, R)$ *and is finitely generated.*

Proof. We have from Theorem 4.4 that $A = T \oplus B$, where T is a finitely generated torsion module and B is $\aleph_1$-free. Now, however, B has countable rank, and so $B = F$ is free. The rest of the corollary follows without difficulty.

We now return to the problem of Whitehead. Following Rotman [11], let us call an R-module A a *W-module* if $\operatorname{Ext}(A, R) = 0$. We shall derive a property of submodules of W-modules.[4]

If A is any R-module and $B \subseteq A$, let $B''/B = \ker(j_{A/B})$. It is not difficult to verify that B'' is simply the "double annihilator" of B with respect to the duality theory discussed earlier; for further information regarding this construction we refer the reader to [1, p. 476]. The two most relevant properties of B'' are the following: (a) A/B'' is torsionless, (b) if $B \subseteq A_1 \subseteq A$ and A/A_1 is torsionless, then $B'' \subseteq A_1$.

THEOREM 4.6. *Let R be a principal ideal domain with a countable number of elements. Let A be a W-module over R, and let $B \subseteq A$. Then*

$$\operatorname{rank}(B'') = \operatorname{rank}(B).$$

Proof. Observe first that A is locally free, by Theorem 4.2.

Replacing B by the smallest pure submodule of A containing it, we may assume that B is pure in A. Hence, if B has finite rank, B is a direct summand of A, in which case clearly $B'' \subseteq B$.

Suppose now that B has infinite rank. Then, since A is a W-module, the exact sequence $0 \to B \to A \to A/B \to 0$ gives rise to the exact cohomology sequence $B^* \to \operatorname{Ext}(A/B, R) \to 0$. Hence, by Lemma 3.6, $|\operatorname{Ext}(A/B, R)| \leqq |B^*| \leqq 2^{|B|}$. We may then apply Theorem 3.8 to conclude that $\exp(|B''/B|) = \exp(|\ker(j_{A/B})|) \leqq 2^{|B|}$, and therefore, by the Generalized Continuum Hypothesis, $|B''/B| \leqq |B|$. Since $|B|$ is infinite, it follows that $|B''| = |B|\,|B''/B| = |B|$, in which case $\operatorname{rank}(B'') = |B''| = |B| = \operatorname{rank}(B)$. This completes the proof.

[4] But, unfortunately, we must use the Generalized Continuum Hypothesis in the argument.

Observe that, if $\bar{B}/B$ is the maximal divisible submodule of A/B, then $\bar{B} \subseteq B''$. Hence, by Theorem 4.6, $\operatorname{rank}(\bar{B}) = \operatorname{rank}(B)$. This fact forms a part of Rotman's Density Lemma [11, p. 249].

We know of no locally free but nonfree module over any principal ideal domain which satisfies the conclusion of Theorem 4.6. However, there exist locally free abelian groups which satisfy the above formulation of the Density Lemma, but which are not free.

5. Concluding remarks

We shall now present two results which, while intimately related to the previous theorems, do not follow from them as immediate corollaries. For a while we may assume that R is any principal ideal domain.

LEMMA 5.1. *The following statements are equivalent for any R-module A and prime p in R:*

(a) $t_p\{\operatorname{Ext}(A, R)\} = 0$.

(b) *The canonical mapping $f : R \to R/pR$ induces an epimorphism*

$$f_* : A^* \to \operatorname{Hom}(A, R/pR).$$

Proof. Define the mapping $m_p : R \to R$ by $m_p(a) = pa$. Then the exact sequence

$$0 \to R \xrightarrow{m_p} R \xrightarrow{f} R/pR \to 0$$

induces the exact cohomology sequence

$$A^* \xrightarrow{f_*} \operatorname{Hom}(A, R/pR) \to \operatorname{Ext}(A, R) \xrightarrow{m_{p*}} \operatorname{Ext}(A, R).$$

But, if u is in $\operatorname{Ext}(A, R)$, then $m_{p*}(u) = pu$. The lemma follows easily.

LEMMA 5.2. *Let K be a field, V a vector space over K, and $\{x_n\}$ an infinite sequence of nonzero elements of K. Then there exists a K-homomorphism $f : V \to K$ such that $f(x_n) \neq 0$ for infinitely many n.*

Proof. Assume first that the subspace W of V generated by the $\{x_n\}$ has infinite dimension. Then, passing to a subsequence, if necessary, we may assume that the $\{x_n\}$ are linearly independent. In this case it is easy to construct a homomorphism $f : V \to K$ such that $f(x_n) = 1$ for all n.

Suppose now that W has finite dimension. Let $u_1, \cdots, u_m$ be a basis of W, and write $x_n = \sum_{i=1}^{m} a_{in} u_i$. Then, for some $i \leqq m$, $a_{in} \neq 0$ for infinitely many n. Define a homomorphism $\bar{f} : W \to K$ by the conditions $\bar{f}(u_i) = 1$, $\bar{f}(u_j) = 0$ for $j \neq i$. Let f be any extension of $\bar{f}$ to V; then it is easily verified that f has the desired properties.

THEOREM 5.3. *Let p be a prime in R, and A an R-module with no p-divisible submodules. Let $\{x_n\}$ be an infinite sequence of nonzero elements of A. If $t_p\{\operatorname{Ext}(A, R)\} = 0$, then there exists $g \in A^*$ such that $g(x_n) \neq 0$ for infinitely many n.*

Proof. Observe first that, since A has no p-divisible submodules, A is torsionless by Theorem 4.1. It then follows easily that for each n there exist $y_n \in A$ and an integer $\alpha_n \geqq 0$ such that $p^{\alpha_n} y_n = x_n$ and $y_n \notin pA$. Let $\bar{y}_n$ be the image of y_n in A/pA; then $\bar{y}_n \neq 0$. A/pA may be viewed as a vector space over the field $K = R/(p)$. Therefore, by Lemma 5.2, there exists a K-homomorphism $\bar{f} : A/pA \rightarrow R/pR$ such that $\bar{f}(\bar{y}_n) \neq 0$ for infinitely many n. $\bar{f}$ may be "lifted" in trivial fashion to an R-homomorphism $f : A \rightarrow R/pR$ such that $f(y_n) = \bar{f}(\bar{y}_n) \neq 0$ for infinitely many n.

Since $t_p\{\mathrm{Ext}(A, R)\} = 0$, we may apply Lemma 5.1 to conclude that the natural mapping $A^* \rightarrow \mathrm{Hom}(A, R/pR)$ is an epimorphism. Hence there exists $g \in A^*$ such that $g(y_n) + pR = f(y_n)$, and so $g(y_n) \neq 0$ for infinitely many n. Then $g(x_n) = p^{\alpha_n} g(y_n) \neq 0$ for infinitely many n, completing the proof of the theorem.[5]

Let $P = \prod_{i=1}^{\infty} R_i$, where $R_i \approx R$, and let e_n be the element of P with i^{th} coordinate equal to δ_{in}. Recall that an R-module is *slender* if any homomorphism $f : P \rightarrow A$ has the property that $f(e_n) = 0$ for all but finitely many n [4, p. 169].

COROLLARY 5.4. *Assume that R has infinitely many primes. Then, if A satisfies the hypotheses of Theorem* 5.3, *A is slender.*

Proof. Assume the statement is false; then there exists a homomorphism $f : P \rightarrow A$ such that $f(e_n) = x_n \neq 0$ for infinitely many n. We may then apply Theorem 5.3 to conclude that there exists $g \in A^*$ such that $g(x_n) \neq 0$ for infinitely many n. Set $h = gf$; then $h \in P^*$ and $h(e_n) = g(x_n) \neq 0$ for infinitely many n. But, since R has infinitely many primes, this is a contradiction to Theorem 47.3 of [4, p. 171]. (The theorem is stated only for abelian groups, but the same proof works in the slightly more general situation described above.) It then follows that A is slender, completing the proof.

One may give a topological interpretation of sorts to Theorem 5.3. If A is an R-module, we may provide A with the weak topology generated by the elements of A^* relative to the discrete topology on R; this topology has been studied by Nunke [8]. Then, if A satisfies the hypotheses of Theorem 5.3, the conclusion states simply that a sequence of elements of A converges in this topology if and only if it is ultimately constant.

In our final result we compare the cardinality of a W-module with that of its dual. First we need a lemma which is of some interest in itself.

LEMMA 5.5. *Let A be a W-module over the countable principal ideal domain R. If p is a prime in R, then* $\exp[\mathrm{rank}(A)] \leqq \exp[r(A/pA)]$. *Hence, if the Generalized Continuum Hypothesis holds,* $\mathrm{rank}(A) \leqq r(A/pA)$.

[5] The referee has kindly pointed out the similarity between the argument used here and a part of the proof of the principal result of [3], in which it is shown essentially that a W-group is torsionless.

Proof. We note in passing that A, being a W-module, is locally free (Theorem 4.2) and thus certainly torsion-free.

Let $\{x_\alpha + pA\}$ be an $R/(p)$-basis of A/pA, where α traces an index set of cardinality equal to $r(A/pA)$. Let B be the pure submodule of A generated by the $\{x_\alpha\}$; then $\operatorname{rank}(B) \leqq r(A/pA)$, and A/B is torsion-free. Furthermore, $A = B + pA$, and so $p(A/B) = A/B$; i.e., A/B is p-divisible. Then, since $\operatorname{Ext}(A, R) = 0$, the exact sequence $0 \to B \to A \to A/B \to 0$ gives rise to the exact cohomology sequence

$$B^* \to \operatorname{Ext}(A/B, R) \to 0,$$

from which it follows that $|\operatorname{Ext}(A/B, R)| \leqq |B^*|$.

Suppose now that $r(A/pA)$ is finite. Then B has finite rank, and so, by Lemma 3.6, $\operatorname{rank}(B^*) \leqq \operatorname{rank}(B)$. Then $|B^*| \leqq \aleph_0 + \operatorname{rank}(B^*) = \aleph_0$, and it follows from Lemma 3.7 that $A/B = 0$. Hence $A = B$, and $\operatorname{rank}(A) = \operatorname{rank}(B) = r(A/pA)$.

Assume now that $r(A/pA)$ is infinite. Then $|B| = \operatorname{rank}(B)$, and so, by Lemma 3.6 and a previous remark,

$$|\operatorname{Ext}(A/B, R)| \leqq |B^*| \leqq 2^{|B|} = \exp[\operatorname{rank}(B)] \leqq \exp[r(A/pA)].$$

We may then apply Lemma 3.7 to conclude that $\exp[\operatorname{rank}(A/B)] \leqq \exp[r(A/pA)]$. Since $\operatorname{rank}(A) = \operatorname{rank}(B) + \operatorname{rank}(A/B)$ and all cardinalities involved are infinite, it follows that

$$\exp[\operatorname{rank}(A)] \leqq \exp[\operatorname{rank}(B)] \exp[\operatorname{rank}(A/B)] \leqq \exp[r(A/pA)],$$

completing the proof of the lemma.

It is not difficult to show that $r(A/pA) \leqq \operatorname{rank}(A)$ for any torsion-free R-module A, and so equality holds if A is a W-module. However, we shall not need this stronger result.

THEOREM 5.6. *Let A be a W-module of infinite rank over a countable principal ideal domain R. Then $|A^*| = 2^{|A|}$.*

Proof. Let p be a prime in R; then since $\operatorname{Ext}(A, R) = 0$, it follows easily from Lemma 5.1 that $r\{\operatorname{Hom}(A, R/pR)\} \leqq |\operatorname{Hom}(A, R/pR)| \leqq |A^*|$. Since A has infinite rank, $|A| = \operatorname{rank}(A)$, and so we may apply Lemma 5.5 to conclude that

$$2^{|A|} = \exp[\operatorname{rank}(A)] \leqq \exp[r(A/pA)] = r\{\operatorname{Hom}(A, R/pR)\} \leqq |A^*|.$$

But $|A^*| \leqq 2^{|A|}$ by Lemma 3.6, and so $|A^*| = 2^{|A|}$. This completes the proof of the theorem.

REFERENCES

1. H. BASS, *Finitistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc., vol. 95 (1960), pp. 466–488.
2. H. CARTAN AND S. EILENBERG, *Homological algebra*, Princeton University Press, 1956.

3. A. Ehrenfeucht, *On a problem of J. H. C. Whitehead concerning Abelian groups*, Bull. Acad. Polon. Sci. Cl. III, vol. 3 (1955), pp. 127–128.
4. L. Fuchs, *Abelian groups*, Budapest, Publishing House of the Hungarian Academy of Sciences, 1958.
5. J. P. Jans, *Duality in Noetherian rings*, Proc. Amer. Math. Soc., vol. 12 (1961), pp. 829–835.
6. I. Kaplansky, *Infinite abelian groups*, University of Michigan Press, 1954.
7. R. J. Nunke, *Modules of extensions over Dedekind rings*, Illinois J. Math., vol. 3 (1959), pp. 222–241.
8. ———, *On direct products of infinite cyclic groups*, Proc. Amer. Math. Soc., vol. 13 (1962), pp. 66–71.
9. ———, *Slender groups*, Bull. Amer. Math. Soc., vol. 67 (1961), pp. 274–275.
10. R. J. Nunke and J. J. Rotman, *Singular cohomology groups*, J. London Math. Soc., vol. 37 (1962), pp. 301–306.
11. J. Rotman, *On a problem of Baer and a problem of Whitehead in abelian groups*, Acta Math. Acad. Sci. Hungar., vol. 12 (1961), pp. 245–254.

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