ALGEBRAIC HOMOTOPY GROUPS AND FROBENIUS ALGEBRAS

BY

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1. Introduction

In [6], injective and projective homotopy groups are defined in an additive category with abelian structure in the sense of Heller [5], and two exact relative homotopy sequences of a pair, dual to each other, are described, which become absolute sequences if the pair is a fibration. For the category of R-modules, where R is an associative algebra over a commutative ring K, there is an important (not exact) abelian structure, called the "φ-relative structure", given by the natural embedding $\varphi : K \to R$. The corresponding homotopy concepts and groups are called in [6] *weak* homotopies and *weak* homotopy groups. In general, there are two dual concepts of weak homotopy, and thus two different kinds of weak homotopy groups. However, for an important class of algebras, namely Frobenius algebras, it is shown in the present paper that the two concepts of weak homotopy coincide. From this, we further show that the n^{th} injective and projective weak homotopy groups are naturally isomorphic. Thus, as in the topological case, there is only one kind of homotopy groups, but two dual relative homotopy sequences.

The result that if the two dual homotopy concepts coincide, then the corresponding n^{th} homotopy groups are isomorphic, can be obtained in an arbitrary exact category with abelian structure having enough injectives and projectives. There is another interesting consequence of this coincidence of the two dual homotopy concepts: The relative homotopy sequences can be extended to "complete" exact sequences of positive-, 0- and negative-dimensional homotopy groups. For fibrations of a special type, these sequences become absolute sequences. Moreover, the negative-dimensional homotopy functors appear to be a sort of functors Ext^n. It should be noted that these extended exact sequences could also be obtained by the standard methods of homological algebra carried to the "relative" case (or to arbitrary abelian structures). However, our different approach to these results might be of some interest in itself, and gives an example for the use of the methods of algebraic homotopy theory.

These general results can be applied to the homotopy groups of various abelian structures: the weak homotopy groups in the category of R-modules, when R is a Frobenius algebra; the ordinary homotopy groups, when R is a quasi-Frobenius ring; the chain homotopy groups of R-module complexes.

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We confine ourselves to the first of these applications. More precisely, we apply the general results to the *weak homotopy groups in the category of R-modules, R being an associative algebra over an arbitrary commutative ring*, and then proceed to the *comparison of one of our complete exact sequences to the complete exact sequences in the cohomology theory of a finite group* (*Tate*) *and that of a Frobenius algebra* (*Nakayama*).

In Section 2, we summarize the results of algebraic homotopy theory we need in the sequel. In Sections 3 and 4 the general results mentioned above are established.

In Section 5, for the category of R-modules, where R is an associative algebra over a commutative ring K, we deal with one of the extended exact homotopy sequences for a fibration of the kind considered here. For any two R-modules A and B, let $\Pi_n^{(R,K)}(A, B)$ denote the n^{th} weak homotopy group. If we choose for A a K-projective R-module Q, then the negative-dimensional groups $\Pi_{-n}^{(R,K)}(Q, B)$ are the ordinary groups $\operatorname{Ext}_R^n(Q, B)$, and every R-epimorphism is a fibration of the kind considered. This leads to complete exact sequences consisting of weak homotopy groups and of cohomology groups (Theorem 5.5).

Section 6 contains the application of Theorem 5.5 to the special cases (a) where R is the enveloping algebra $F \otimes_K F^*$ of a Frobenius K-algebra F and $Q = F$, (b) where R is the group algebra $Z(\mathfrak{G})$ of a finite group $\mathfrak{G}$ over the ring of integers Z, and $Q = I$ (I being the additive group of Z with trivial $\mathfrak{G}$-module structure). In analogy with the definition of the Hochschild cohomology groups of an associative algebra, for any two-sided F-module B, we introduce the weak homotopy groups $\pi_n(F, B) = \Pi_n^{(F \otimes_K F^*, K)}(F, B)$ of a Frobenius K-algebra F; in analogy with the usual definition of the cohomology groups of a group, for any $\mathfrak{G}$-module B, we introduce the weak homotopy groups $\pi_n(\mathfrak{G}; B) = \Pi_n^{(Z(\mathfrak{G}), Z)}(I, B)$ of a finite group $\mathfrak{G}$. Thus, we obtain complete exact sequences consisting of weak homotopy groups and cohomology groups of a Frobenius algebra and of a finite group respectively. In the case of a finite group $\mathfrak{G}$, we show that the weak homotopy groups $\pi_n(\mathfrak{G}; B)$ are essentially the homology groups $H_n(\mathfrak{G}; B)$, and that our complete sequence coincides with the well-known complete derived sequence of $\mathfrak{G}$ (cf. [2, Chapter XII]). By an analogous argument, it can be shown that in the case of a Frobenius K-algebra F (being free over K), the weak homotopy groups $\pi_n(F, B)$ are essentially the modified homology groups $H_n^*(F, B)$ of F in the sense of Nakayama (cf. [7]).

2. Review of algebraic homotopy theory in abelian categories

In this section, we first recall some of the definitions and results given in [6]. We shall only consider abelian categories with enough injectives and projectives. Then, the homotopy functors $\bar{\Pi}_n$ and $\underline{\Pi}_n$ are functors of two variables in the same category.[2] Furthermore, we shall confine ourselves to exact categories, i.e., to additive categories where in particular every map has kernel and cokernel.

Let $\mathcal{K}$ be an exact category with an (additional) abelian structure $\mathcal{S}$ having enough injectives and projectives with respect to $\mathcal{S}$. Taking $\mathcal{S}$ as homotopy structure, the i-homotopy groups $\bar{\Pi}(A, B)$ are introduced as follows (cf. [6]). Let ι be a proper monomorphism from A to an injective object $\bar{A}$, and let $\iota^* \operatorname{Hom}(\bar{A}, B)$ denote the image of the homomorphism

$$\iota^*: \operatorname{Hom}(\bar{A}, B) \to \operatorname{Hom}(A, B) \quad \text{induced by } \iota: A \to \bar{A}.$$

Then $\bar{\Pi}(A, B)$ is the factor group $\operatorname{Hom}(A, B)/\iota^* \operatorname{Hom}(\bar{A}, B)$. This group is independent of the choice of the map $\iota: A \to \bar{A}$.

For every map $\alpha: A \to A'$ and $\beta: B \to B'$, there are induced homomorphisms $\bar{\alpha}^*: \bar{\Pi}(A', B) \to \bar{\Pi}(A, B)$ and $\bar{\beta}_*: \bar{\Pi}(A, B) \to \bar{\Pi}(A, B')$ defined in an obvious way. Clearly, the triple of functions $\bar{\alpha}^*$, $\bar{\beta}_*$, and $\bar{\Pi}(A, B)$ define a functor of two variables in $\mathcal{K}$ with values in the category $\mathcal{G}$ of abelian groups, denoted by $\bar{\Pi} : \mathcal{K} \times \mathcal{K} \to \mathcal{G}$.

The p-homotopy groups $\underline{\Pi}(A, B)$ and the functor $\underline{\Pi} : \mathcal{K} \times \mathcal{K} \to \mathcal{G}$ are introduced dually.

A map $\varphi : A \to A'$ is an i-equivalence if there exists an i-homotopy inverse; A and A' are then said to be of the same i-type ($A \sim_i A'$). A suspension of A is an object ΣA which can be embedded into a proper s.e.s. (short exact sequence) $O \to A \to \bar{A} \to \Sigma A \to O$ with $\bar{A}$ injective. Since there are enough injectives, for each A, a suspension exists; its i-type is uniquely determined by the i-type of A. Furthermore, to each map $\varphi : A \to A'$, a map

$$\Sigma\varphi : \Sigma A \to \Sigma A',$$

whose i-homotopy class is uniquely determined by the i-homotopy class of φ, is assigned in an obvious way. Thus, we obtain a functor $\Sigma : \mathcal{K} \to \mathcal{K}$ defined up to i-equivalence and i-homotopy. The definition of the functor Σ can be iterated as follows: $\Sigma^n = \Sigma\Sigma^{n-1}$ for $n > 1$. The functors $\bar{\Pi}_n$ are introduced as composite functors $\bar{\Pi}(\Sigma^n \times \mathrm{Id}) : \mathcal{K} \times \mathcal{K} \to \mathcal{G}$, and are determined up to natural isomorphisms. The groups $\bar{\Pi}_n(A, B) = \bar{\Pi}(\Sigma^n A, B)$ are the n^{th} i-homotopy groups.

The concepts of p-equivalence, p-type, dual suspension ΩB of B, such as the functors $\underline{\Pi}_n : \mathcal{K} \times \mathcal{K} \to \mathcal{G}$ and the n^{th} p-homotopy groups

$$\underline{\Pi}_n(A, B) = \underline{\Pi}(A, \Omega^n B)$$

are introduced dually.

In the category $P\mathcal{K}$ of pairs, i.e., the category whose objects are maps $\alpha : A_1 \to A_2$ of $\mathcal{K}$, whose maps $f: \alpha \to \alpha'$ are pairs (φ_1, φ_2) of maps of $\mathcal{K}$ such that $\alpha'\varphi_1 = \varphi_2\,\alpha$, and where composition and sum are defined in terms of the corre-

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[2] In [6], the functors $\bar{\Pi}_n$ and $\underline{\Pi}_n$ are denoted by $\Pi_n{}^i$ and $\Pi_n{}^p$, respectively.

sponding operations in $\mathcal{K}$, we take, as in [6], the structure $P^*\mathcal{S}$ as homotopy structure. Since $\mathcal{K}$ is an exact category, the class of proper s.e.s. with respect to $P^*\mathcal{S}$ can be characterized as follows. For every map $f = (\varphi_1, \varphi_2) : \alpha \to \alpha'$, we denote by φ_0 and φ_3 the induced maps of the kernels and cokernels of α, α' respectively, and we shall refer to the four maps φ_0, φ_1, φ_2, and φ_3 as the components of f. It can be shown in a straightforward way that an s.e.s. in $P\mathcal{K}$ is a proper s.e.s. with respect to $P^*\mathcal{S}$ if and only if all components form proper s.e.s. in $\mathcal{K}$. Moreover, it is easy to see that a pair $\xi : X_1 \to X_2$ is injective if and only if ξ splits, i.e., if the s.e.s. $O \to \operatorname{Ker} \xi \to X_1 \to \operatorname{Im} \xi \to O$ and $O \to \operatorname{Im} \xi \to X_2 \to \operatorname{Coker} \xi \to O$ are splitting s.e.s., and if X_1 and X_2 are injective. Dually, a pair $\eta : Y_1 \to Y_2$ is projective if and only if η splits, and Y_1 and Y_2 are projective.

It should be noted that in $P\mathcal{K}$ we have in general not enough injectives (and projectives) with respect to $P^*\mathcal{S}$; therefore, we cannot introduce $\bar{\Pi}_n$ as functor of two variables in $P\mathcal{K}$. However, we shall only consider mixed i-homotopy functors $\bar{\mathrm{P}}_n : \mathcal{K} \times P\mathcal{K} \to \mathcal{G}$, the so-called "relative" i-homotopy functors (cf. [6]), which are given as follows. To each object A of $\mathcal{K}$, we choose a proper monomorphism $\iota(A)$ from A to an injective object $\bar{A}$, and to each map $\varphi : A \to A'$ of $\mathcal{K}$ a map $\iota(\varphi) : \iota(A) \to \iota(A')$ of $P\mathcal{K}$ such that $\varphi_1 = \varphi$. Thus, we obtain a functor $\iota : \mathcal{K} \to P\mathcal{K}$ defined up to i-equivalence and i-homotopy. The functors $\bar{\mathrm{P}}_n$ are introduced as composite functors $\bar{\Pi}(\iota\Sigma^{n-1} \times \mathrm{Id}) : \mathcal{K} \times P\mathcal{K} \to \mathcal{G}$, $n > 0$, and are determined up to natural isomorphisms. Clearly, we have $\bar{\mathrm{P}}_n = \bar{\mathrm{P}}_1(\Sigma^{n-1} \times \mathrm{Id})$.

The "relative" p-homotopy functors $\underline{\mathrm{P}}_n : P\mathcal{K} \times \mathcal{K} \to \mathcal{G}$ are introduced dually.

For any object A of $\mathcal{K}$ and any pair $\beta : B_1 \to B_2$ in $\mathcal{K}$, there are exact sequences

$$S_*(A, \beta) : \cdots \to \bar{\Pi}_n(A, B_1) \xrightarrow{\bar{\beta}_*} \bar{\Pi}_n(A, B_2) \xrightarrow{\bar{\jmath}_*} \bar{\mathrm{P}}_n(A, \beta) \xrightarrow{\bar{\partial}_*} \bar{\Pi}_{n-1}(A, B_1) \to \cdots \to \bar{\Pi}(A, B_2),$$

and

$$S^*(\beta, A) : \cdots \to \underline{\Pi}_n(B_2, A) \xrightarrow{\underline{\beta}^*} \underline{\Pi}_n(B_1, A) \xrightarrow{\underline{j}^*} \underline{\mathrm{P}}_n(\beta, A) \xrightarrow{\underline{\partial}^*} \underline{\Pi}_{n-1}(B_2, A) \to \cdots \to \underline{\Pi}(B_1, A),$$

respectively called *injective and projective homotopy sequences of the pair β with respect to A.* The homomorphisms in $S_*(A, \beta)$ are defined as follows:

(i) $\bar{\beta}_*$ is the homomorphism induced by the map $\beta : B_1 \to B_2$.

(ii) For the pair $\tilde{\omega}B_2 : O \to B_2$, one identifies $\bar{\Pi}_n(A, B_2)$ with $\bar{\mathrm{P}}_n(A\, \tilde{\omega}B_2)$; then $\bar{\jmath}_*$ is the homomorphism induced by

$$j = (o, 1_{B_2}) : \tilde{\omega}B_2 \to \beta.$$

(iii) For the pair $\omega B_1 : B_1 \to O$, one identifies $\bar{\Pi}_{n-1}(A, B_1)$ with

$\bar{\mathrm{P}}_n(A, \omega B_1)$; then $\bar{\partial}_*$ is the homomorphism induced by

$$\partial = (1_{B_1}, o) : \beta \to \omega B_1 .$$

The homomorphisms in $S^*(\beta, A)$ are given dually.

An *i-fibration relative to* A is a pair $\beta : B_1 \to B_2$ in $\mathcal{K}$ with the following properties: (i)[3] the map β has a kernel B_0, called the *fibre* of β; (ii) there is an injective resolution X of A such that, for every map $\varphi_2 : X_n \to B_2$ where X_n is an injective occurring in X, there exists a map $\varphi_1 : X_n \to B_1$ with $\varphi_2 = \beta\varphi_1$. For any object A and an i-fibration $\beta : B_1 \to B_2$ relative to A, we obtain an exact "absolute" sequence

$$T_*(A, \beta) : \cdots \to \bar{\Pi}_n(A, B_1) \xrightarrow{\bar{\beta}_*} \bar{\Pi}_n(A, B_2) \xrightarrow{\bar{\Delta}} \bar{\Pi}_{n-1}(A, B_0)$$
$$\xrightarrow{\bar{\varkappa}_*} \bar{\Pi}_{n-1}(A, B_1) \to \cdots \to \bar{\Pi}(A, B_2),$$

called injective homotopy sequence of the i-fibration β *relative to* A. Here, $\bar{\beta}_*$ is again the homomorphism induced by the map β, $\bar{\varkappa}_*$ is the homomorphism induced by the canonical monomorphism $\varkappa$ from B_0 to B_1, and $\bar{\Delta}$ is a natural homomorphism that we shall not describe in detail here. This result—that for an i-fibration the groups $\bar{\mathrm{P}}_n(A, \beta)$ and the homomorphisms $\bar{\jmath}_*$ and $\bar{\partial}_*$ in $S_*(A, \beta)$ can be replaced by the groups $\bar{\Pi}_{n-1}(A, B_0)$ and the homomorphisms Δ and $\bar{\varkappa}_*$—is that part of the assertion of the so-called excision theorem for i-fibrations (cf. [6, Theorem 12.3] or [4, Théorème 5.2]) which we shall need later on.

A p-fibration relative to A is a pair $\beta : B_2 \to B_1$ in $\mathcal{K}$ with the dual properties. The cokernel of the map β is called *cofibre* B_0 of β. For any object A and a p-fibration $\beta : B_2 \to B_1$ relative to A, we have an exact "absolute" sequence, dual to $T_*(A, \beta)$:

$$T^*(\beta, A) : \cdots \to \underline{\Pi}_n(B_1, A) \xrightarrow{\underline{\beta}^*} \underline{\Pi}_n(B_2, A) \xrightarrow{\underline{\Delta}} \underline{\Pi}_{n-1}(B_0, A)$$
$$\xrightarrow{\underline{\gamma}^*} \underline{\Pi}_{n-1}(B_1, A) \to \cdots \to \underline{\Pi}(B_2, A),$$

called projective homotopy sequence of the p-fibration β *relative to* A. Here, $\underline{\gamma}^*$ is the homomorphism induced by the canonical epimorphism γ from B_1 to B_0.

Finally, we give some propositions concerning injective and projective objects of $\mathcal{K}$ and $P\mathcal{K}$ with the homotopy structure $\mathfrak{S}$ and $P^*\mathfrak{S}$ respectively.

PROPOSITION 2.1. *An object* A *of* $\mathcal{K}$ *is injective if and only if the identity map* 1_A *of* A *is i-homotopic to* o.

Proof. Since $\mathcal{K}$ has enough injectives relative to $\mathfrak{S}$, there is a proper monomorphism $\mu : A \to \bar{A}$ with $\bar{A}$ injective. Assume that $1_A \sim_i o$ (1_A is i-homo-

[3] In an exact category $\mathcal{K}$ with (additional) abelian structure, condition (i) is redundant.

topic to o); then there exists a map $\gamma : \bar{A} \to A$ such that $\gamma\mu = 1_A$. Thus, A is a direct summand of $\bar{A}$, and is therefore injective. Conversely, assume A to be injective. Then, $1_A \sim_i o$ is an immediate consequence of the definition of i-homotopy.

PROPOSITION 2.2. *If every injective object of $\mathfrak{K}$ is projective, then every injective object of $P\mathfrak{K}$ is projective too.*

Proof. This follows immediately from the characterization of the injective pairs with respect to $P^*\mathfrak{S}$ given above.

Dually, we have

PROPOSITION 2.1*. *An object A of $\mathfrak{K}$ is projective if and only if the identity map 1_A of A is p-homotopic to o.*

PROPOSITION 2.2*. *If every projective object of $\mathfrak{K}$ is injective, then every projective object of $P\mathfrak{K}$ is injective too.*

3. Abelian structures with coinciding homotopy

Let $\mathfrak{K}$ be an exact category, and let $\mathfrak{S}$ *be a homotopy structure in* $\mathfrak{K}$, *such that there are enough injectives and projectives with respect to* $\mathfrak{S}$. Assume that the concepts of i-homotopy and of p-homotopy coincide. Then, if an object A of $\mathfrak{K}$ is injective, by Proposition 2.1, the identity map 1_A is i-homotopic to o, thus also p-homotopic to o; therefore, by Proposition 2.1*, A is projective. The same argument shows that a projective object of $\mathfrak{K}$ is injective. Hence, the following conditions are necessary for the two homotopy concepts to coincide:

(I $\to$ P) *Every injective object with respect to the homotopy structure is also a projective object.*

(P $\to$ I) *Every projective object with respect to the homotopy structure is also an injective object.*

We shall show that they are sufficient, too.

PROPOSITION 3.1. *If a map α is i-homotopic to o, then* (I $\to$ P) *implies that α is p-homotopic to o.*

Proof. Assume $\alpha : A \to B$ to be i-homotopic to o $(\alpha \sim_i o)$. There exists a proper monomorphism $\iota : A \to \bar{A}$ with $\bar{A}$ injective. Since $\alpha \sim_i o$, there is a map $\bar{\alpha} : \bar{A} \to B$ such that $\alpha = \bar{\alpha}\iota$. By (I $\to$ P), $\bar{A}$ is projective. Hence, for every proper epimorphism $\varepsilon : B' \to B$, there is a map $\bar{\alpha}' : \bar{A} \to B'$ such that $\bar{\alpha} = \varepsilon\bar{\alpha}'$, and thus a map $\alpha' = \bar{\alpha}'\iota : A \to B'$ such that $\alpha = \varepsilon\alpha'$; i.e., $\alpha \sim_p o$.

By duality, we have

PROPOSITION 3.1*. *If a map α is p-homotopic to o, then* (P $\to$ I) *implies that α is i-homotopic to o.*

Summing up, we have

THEOREM 3.2. *In an (exact) category with homotopy structure $\mathcal{S}$ having enough injectives and projectives, the concepts of i-homotopy and of p-homotopy coincide if and only if the conditions* $(\mathrm{I} \rightarrow \mathrm{P})$ *and* $(\mathrm{P} \rightarrow \mathrm{I})$ *are satisfied.*

An *example* of an abelian structure satisfying $(\mathrm{I} \rightarrow \mathrm{P})$ and $(\mathrm{P} \rightarrow \mathrm{I})$ which is of particular interest to us, is the φ-relative structure $\mathcal{S}_\varphi$ in the exact category of R-modules, where R is a Frobenius K-algebra, φ being the natural embedding $K \rightarrow R$. More explicitly (cf. [6, §1.4, Example a]), let R be an associative algebra over a commutative ring K. Then, the class $\mathcal{S}_\varphi$ consists of those R-homomorphisms $\alpha : A \rightarrow B$ where the kernel of α is a K-direct summand of A, and the cokernel of α a K-direct summand of B. The injectives with respect to $\mathcal{S}_\varphi$ are R-direct summands of R-modules of the form $\mathrm{Hom}_K(R, A)$, where A is a K-module, and the R-module structure of $\mathrm{Hom}_K(R, A)$ is defined by the right-R-module structure of R. The projectives with respect to $\mathcal{S}_\varphi$ are R-direct summands of R-modules of the form $R \otimes_K A$, where A is a K-module, and the R-module structure of $R \otimes_K A$ is defined by that of R. There are enough injectives and projectives with respect to $\mathcal{S}_\varphi$.

We say that R is a *Frobenius K-algebra* if R is K-projective and finitely K-generated, and if there exists a right-R-isomorphism[4] τ from R into $R^0 = \mathrm{Hom}_K(R, K)$. By this definition the group ring $Z(\mathfrak{G})$ of a finite group $\mathfrak{G}$ over the ring Z of integers is a Frobenius Z-algebra (cf. [3, §§2 and 4]).

For any K-module A, consider the R-module structure of $R \otimes_K A$ and $\mathrm{Hom}_K(R^0, A)$, given by that of R and R^0 respectively. We define an R-homomorphism

$$\sigma : R \otimes_K A \rightarrow \mathrm{Hom}_K(R^0, A)$$

by setting

$$[\sigma(r \otimes a)]\alpha = (\alpha r)a$$

for all $r \in R$, $a \in A$, and $\alpha \in R^0 = \mathrm{Hom}_K(R, K)$. If R is K-projective and finitely K-generated, σ is an R-isomorphism. If R is a Frobenius K-algebra, let τ^* denote the (left-) R-isomorphism: $\mathrm{Hom}_K(R^0, A) \rightarrow \mathrm{Hom}_K(R, A)$ induced by $\tau : R \rightarrow R^0$. Thus, for any K-module A, the composite R-homomorphism $\tau^*\sigma$ is an R-isomorphism : $R \otimes_K A \rightarrow \mathrm{Hom}_K(R, A)$. Hence, if R is a Frobenius K-algebra, the conditions $(\mathrm{I} \rightarrow \mathrm{P})$ and $(\mathrm{P} \rightarrow \mathrm{I})$ hold with respect to the structure $\mathcal{S}_\varphi$.

Other examples of abelian structures satisfying $(\mathrm{I} \rightarrow \mathrm{P})$ and $(\mathrm{P} \rightarrow \mathrm{I})$ are: the exact structure in the category of R-modules if R is a quasi-Frobenius ring; the abelian structure $\mathcal{S}_1$ in the category of R-module complexes (cf. [6, §1.4, Example b]).

For the remainder of this section, assume that $\mathcal{K}$ *is an exact category with an abelian structure* $\mathcal{S}$ *such that*

[4] Then there exists a left-R-isomorphism $\tau' : R \rightarrow R^0$ too, and vice versa.

(i) *$\mathcal{K}$ has enough injectives and projectives with respect to $\mathfrak{S}$,*

(ii) *the conditions* (I $\rightarrow$ P) *and* (P $\rightarrow$ I) *hold.*

Then the two homotopy concepts, and thus the functors $\bar{\Pi}$ and $\underline{\Pi}$ coincide.[5] We shall denote the single functor $\bar{\Pi} = \underline{\Pi} : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{G}$ by Π; in particular, we shall put $\bar{\Pi}(A, B) = \underline{\Pi}(A, B) = \Pi(A, B)$. When we consider the higher homotopy groups $\bar{\Pi}_n(A, B)$ and $\underline{\Pi}_n(A, B)$, at first these do not seem to coincide. However, we shall prove that there exist natural isomorphisms $s_n : \bar{\Pi}_n \rightarrow \underline{\Pi}_n$ for all $n > 0$; i.e., that the groups $\bar{\Pi}_n(A, B)$ and $\underline{\Pi}_n(A, B)$ are naturally isomorphic.

LEMMA 3.3. *Let $P\mathcal{K}$ be the category of pairs in $\mathcal{K}$ with homotopy structure $P^*\mathfrak{S}$, let $\iota(A)$ be a pair $\iota : A \rightarrow \bar{A}$ where ι is a proper monomorphism and $\bar{A}$ injective, and let $\varepsilon(\underline{B})$ be a pair $\varepsilon : \underline{B} \rightarrow B$ where ε is a proper epimorphism and $\underline{B}$ projective. Then, for all A, B of $\mathcal{K}$, we have*

$$\bar{\Pi}(\iota(A), \varepsilon(B)) = \underline{\Pi}(\iota(A), \varepsilon(B)).$$

Proof. By Propositions 2.2 and 2.2* the conditions (I $\rightarrow$ P) and (P $\rightarrow$ I) are satisfied in $P\mathcal{K}$ with respect to $P^*\mathfrak{S}$. As already mentioned, $P\mathcal{K}$ generally has not enough injectives and projectives with respect to $P^*\mathfrak{S}$. However, the pairs $\iota(A)$ and $\varepsilon(B)$ are proper objects of $P\mathcal{K}$ (cf. [6, §1.3]); i.e., for these pairs, there exists a proper monomorphism to an injective pair, and a proper epimorphism from a projective pair. Thus, the same argument as in Proposition 3.1 shows that, if a map $a : \iota(A) \rightarrow \varepsilon(B)$ is i-homotopic to o, it is also p-homotopic to o, and vice versa. Hence, the groups $\bar{\Pi}(\iota(A), \varepsilon(B))$ and $\underline{\Pi}(\iota(A), \varepsilon(B))$ coincide.

It should be noted that the family of identities

$$\mathrm{Id}(\iota(A), \varepsilon(B)) : \bar{\Pi}(\iota(A), \varepsilon(B)) \rightarrow \underline{\Pi}(\iota(A), \varepsilon(B))$$

commutes with the homomorphisms induced in these homotopy groups by all maps $a : \iota(A') \rightarrow \iota(A)$ and $b : \varepsilon(B) \rightarrow \varepsilon(B')$.

LEMMA 3.4. *There exists a family of homomorphisms*

$$s_1(A, B) : \bar{\Pi}_1(A, B) \rightarrow \underline{\Pi}_1(A, B)$$

which defines a natural isomorphism $s_1 : \bar{\Pi}_1 \rightarrow \underline{\Pi}_1$.

Proof. For the pairs $\iota(A)$ and $\varepsilon(B)$ of Lemma 3.3, let us consider the last terms of the injective and projective homotopy sequences

$$S_*(A, \varepsilon(B)) : \cdots \rightarrow \bar{\Pi}_1(A, \underline{B}) \rightarrow \bar{\Pi}_1(A, B) \xrightarrow{\bar{J}_*(A, \varepsilon(B))} \bar{P}_1(A, \varepsilon(B)) \rightarrow \bar{\Pi}(A, \underline{B}),$$

$$S^*(\iota(A), B) : \cdots \rightarrow \underline{\Pi}_1(\bar{A}, B) \rightarrow \underline{\Pi}_1(A, B) \xrightarrow{\underline{J}^*(\iota(A), B)} \underline{P}_1(\iota(A), B) \rightarrow \underline{\Pi}(\bar{A}, B).$$

[5] I.e., the family of identities $\mathrm{Id}(A, B) : \bar{\Pi}(A, B) \rightarrow \underline{\Pi}(A, B)$ defines a natural isomorphism $\bar{\Pi} \rightarrow \underline{\Pi}$.

(P $\to$ I) implies $\underline{B}$ injective; thus, $\bar{\Pi}_1(A, \underline{B}) = \bar{\Pi}(A, \underline{B}) = O$. (I $\to$ P) implies $\bar{A}$ projective; thus, $\underline{\Pi}_1(\bar{A}, B) = \underline{\Pi}(\bar{A}, B) = O$. Since the homotopy sequences are exact, $\bar{\jmath}_*(A, \varepsilon(B))$ and $\underline{j}^*(\iota(A), B)$ are isomorphisms. Now, by definition,

$$\bar{\mathrm{P}}_1(A, \varepsilon(B)) = \bar{\Pi}(\iota(A), \varepsilon(B)) \quad \text{and} \quad \underline{\mathrm{P}}_1(\iota(A), B) = \underline{\Pi}(\iota(A), \varepsilon(B)).$$

Therefore,

$$s_1(A, B) = \underline{j}^*(\iota(A), B)^{-1} \circ \mathrm{Id}(\iota(A), \varepsilon(B)) \circ \bar{\jmath}_*(A, \varepsilon(B))$$

gives a family of isomorphisms $\bar{\Pi}_1(A, B) \to \underline{\Pi}_1(A, B)$ for all A, B of $\mathfrak{K}$. There remains to show that the diagrams

$$\begin{tikzcd} \bar{\Pi}_1(A, B) \arrow[r, "s_1(A, B)"] \arrow[d, "\bar{\alpha}^*"'] & \underline{\Pi}_1(A, B) \arrow[d, "\underline{\alpha}^*"] \\ \bar{\Pi}_1(A', B) \arrow[r, "s_1(A', B)"'] & \underline{\Pi}_1(A', B) \end{tikzcd} \qquad \begin{tikzcd} \bar{\Pi}_1(A, B) \arrow[r, "s_1(A, B)"] \arrow[d, "\bar{\beta}_*"'] & \underline{\Pi}_1(A, B) \arrow[d, "\underline{\beta}_*"] \\ \bar{\Pi}_1(A, B') \arrow[r, "s_1(A, B')"'] & \underline{\Pi}_1(A, B') \end{tikzcd}$$

are commutative for all maps $\alpha : A' \to A$ and $\beta : B \to B'$. This can be done by a direct verification, or by just verifying the naturality of the homomorphisms $\bar{\jmath}_*$ and $\underline{j}^*$ in the injective and projective homotopy sequences respectively.

THEOREM 3.5. *The functors $\bar{\Pi}_n$ and $\underline{\Pi}_n$ coincide up to natural isomorphisms* $(n = 0, 1, 2 \cdots)$.

Proof by induction with respect to n. $n = 0$: $s_0(A, B) = \mathrm{Id}$ for all A, B of $\mathfrak{K}$. $n > 0$: Assume there exists a family

$$s_{n-1}(A, B) : \bar{\Pi}_{n-1}(A, B) \to \underline{\Pi}_{n-1}(A, B)$$

which defines a natural isomorphism $s_{n-1} : \bar{\Pi}_{n-1} \to \underline{\Pi}_{n-1}$. Consider the family $s_n(A, B)$ of composite homomorphisms:

$$\begin{aligned} \bar{\Pi}_n(A, B) = \bar{\Pi}_1(\Sigma^{n-1}A, B) &\xrightarrow{s_1(\Sigma^{n-1}A, B)} \underline{\Pi}_1(\Sigma^{n-1}A, B) = \underline{\Pi}(\Sigma^{n-1}A, \Omega B) \\ &\xrightarrow{\mathrm{Id}(\Sigma^{n-1}A, \Omega B)} \bar{\Pi}(\Sigma^{n-1}A, \Omega B) = \bar{\Pi}_{n-1}(A, \Omega B) \\ &\xrightarrow{s_{n-1}(A, \Omega B)} \underline{\Pi}_{n-1}(A, \Omega B) = \underline{\Pi}_n(A, B). \end{aligned}$$

From the transitivity of natural transformations of functors, it then follows that the family $s_n(A, B)$ defines a natural isomorphism $s_n : \bar{\Pi}_n \to \underline{\Pi}_n$.

The functors $\bar{\Pi}_n$ and $\underline{\Pi}_n$ are defined only up to natural isomorphisms (see Section 2). Thus, we denote by Π_n the functor $\bar{\Pi}_n = \underline{\Pi}_n$; in particular, we put $\Pi_n(A, B) = \bar{\Pi}_n(A, B) = \underline{\Pi}_n(A, B)$; the groups $\Pi_n(A, B)$ are called *n^{th} homotopy groups*. Note that we now have a situation similar to that in topology, namely essentially only one kind of homotopy groups $\Pi_n(A, B)$, but two kinds of "relative" homotopy groups $\mathrm{P}_n(A, \beta)$ and $\mathrm{P}_n(\alpha, B)$ occurring in the two dual homotopy sequences $S_*(A, \beta)$ and $S^*(\alpha, B)$ respectively.

4. Extension of the homotopy sequences

Our object in this section is to show that in an exact category $\mathcal{K}$ with a homotopy structure $\mathcal{S}$ having the properties (i) and (ii) (see Section 3), the injective and projective homotopy sequences can be extended to complete exact sequences by introducing in a natural way "negative-dimensional" homotopy groups $\Pi_{-n}(A, B)$, $\mathrm{P}_{-n}(A, \beta)$ and $\mathrm{P}_{-n}(\alpha, B)$. For $n \geqq 1$, the groups $\Pi_{-n}(A, B)$ will appear to be the groups $\mathrm{Ext}^n(A, B)$ with respect to the abelian structure $\mathcal{S}$, more precisely, the satellites of $\mathrm{Hom}(A, B)$, considered as a functor of A, with respect to $\mathcal{S}$. Moreover, there are special pairs (fibrations) for which the extended sequences become "absolute" sequences.

Let $\mathcal{K}$ be an exact category with homotopy structure $\mathcal{S}$ having enough injectives and projectives with respect to $\mathcal{S}$. If condition (P $\to$ I) is satisfied, we shall show that the injective homotopy sequence of a pair can be extended. If the dual condition (I $\to$ P) is satisfied, we obtain a dual extension of the projective homotopy sequence.

LEMMA 4.1. *If condition* (P $\to$ I) *holds, the following properties are valid*:

(a) *if $A \sim_p A'$, then $A \sim_i A'$*;

(b) $\Omega^n A \sim_i \Sigma\Omega^{n+1} A$ *for all* $n \geqq 0$;

(c) *every pair $\beta : B_1 \to B_2$, where β is a proper monomorphism, is a p-fibration* (*relative to each* A).

Proof. (a) Recall that $A \sim_p A'$ ($A \sim_i A'$), if there are maps $\varphi : A \to A'$ and $\psi : A' \to A$ such that $\psi\varphi \sim_p 1_A$ and $\varphi\psi \sim_p 1_{A'}$ ($\psi\varphi \sim_i 1_A$ and $\varphi\psi \sim_i 1_{A'}$). Thus, (a) is a direct consequence of Proposition 3.1*. (b) It is sufficient to prove $A \sim_i \Sigma\Omega A$. In order to do this, consider a dual suspension ΩA of A. By definition, there is a proper s.e.s.

$$O \to \Omega A \to \underline{A} \to A \to O$$

with $\underline{A}$ projective. (P $\to$ I) implies $\underline{A}$ injective, such that A is a suspension of ΩA. Since the i-type of a suspension ΣX is determined by X, we have $A \sim_i \Sigma\Omega A$ for any suspension $\Sigma\Omega A$ of ΩA. (c) Since (P $\to$ I) says that every projective object is injective, (c) is a trivial consequence of the definition of an injective object.

LEMMA 4.1*. *If condition* (I $\to$ P) *holds, the dual properties are valid.*

Assume now condition (P $\to$ I) to be satisfied. Then, by Lemma 4.1 (a), for all $m \geqq 0$, the functors

$$\bar{\Pi}_{-m} = \bar{\Pi}(\Omega^m \times \mathrm{Id}) : \mathcal{K} \times \mathcal{K} \to \mathcal{G}, \quad \text{and}$$

$$\bar{\mathrm{P}}_{-m} = \bar{\mathrm{P}}_1(\Omega^{m+1} \times \mathrm{Id}) : \mathcal{K} \times P\mathcal{K} \to \mathcal{G},$$

where Ω^m is the m-fold dual suspension functor $\mathcal{K} \to \mathcal{K}$, are well-determined up to natural isomorphisms. The groups

$$\bar{\Pi}_{-m}(A, B) = \bar{\Pi}(\Omega^m A, B) \quad \text{and} \quad \bar{P}_{-m}(A, \beta) = \bar{P}_1(\Omega^{m+1} A, \beta)$$

are called *"negative-dimensional" injective homotopy groups.*

Let us consider the exact injective homotopy sequence of the pair $\beta : B_1 \to B_2$ and of a dual suspension $\Omega^m A$ of A:

$$\begin{aligned} S_*(\Omega^m A, \beta): \cdots \to \bar{\Pi}_n(\Omega^m A, B_1) &\xrightarrow{\bar{\beta}_*} \bar{\Pi}_n(\Omega^m A, B_2) \xrightarrow{\bar{J}_*} \bar{P}_n(\Omega^m A, \beta) \\ &\xrightarrow{\bar{\partial}_*} \bar{\Pi}_{n-1}(\Omega^m A, B_1) \to \cdots \to \bar{\Pi}_m(\Omega^m A, B_1) \xrightarrow{\bar{\beta}_*} \bar{\Pi}_m(\Omega^m A, B_2) \\ &\xrightarrow{\bar{J}_*} \bar{P}_m(\Omega^m A, \beta) \xrightarrow{\bar{\partial}_*} \bar{\Pi}_{m-1}(\Omega^m A, B_1) \to \cdots \to \bar{\Pi}(\Omega^m A, B_2). \end{aligned}$$

By Lemma 4.1 (b), we have up to natural isomorphisms,

for $n \geqq m$: $\bar{\Pi}_n(\Omega^m A, B) = \bar{\Pi}(\Sigma^n \Omega^m A, B) = \bar{\Pi}(\Sigma^{n-m} A, B) = \bar{\Pi}_{n-m}(A, B)$,

for $n < m$: $\bar{\Pi}_n(\Omega^m A, B) = \bar{\Pi}(\Sigma^n \Omega^m A, B) = \bar{\Pi}(\Omega^{m-n} A, B) = \bar{\Pi}_{-(m-n)}(A, B)$,

for $n > m$: $\bar{P}_n(\Omega^m A, \beta) = \bar{P}_1(\Sigma^{n-1} \Omega^m A, \beta) = \bar{P}_1(\Sigma^{(n-m)-1} A, \beta) = \bar{P}_{n-m}(A, \beta)$,

for $n = m$: $\bar{P}_m(\Omega^m A, \beta) = \bar{P}_1(\Sigma^{m-1} \Omega^m A, \beta) = \bar{P}_1(\Omega A, \beta) = \bar{P}_0(A, \beta)$,

for $n < m$: $\bar{P}_n(\Omega^m A, \beta) = \bar{P}_1(\Sigma^{n-1} \Omega^m A, \beta) = \bar{P}_1(\Omega^{m-n+1} A, \beta) = \bar{P}_{-(m-n)}(A, \beta)$.

Thus, the exact sequence $S_*(\Omega^m A, \beta)$ can be written as follows:

$$\begin{aligned} S'_*(A, \beta): \cdots \to \bar{\Pi}_p(A, B_1) &\xrightarrow{\bar{\beta}_*} \bar{\Pi}_p(A, B_2) \xrightarrow{\bar{J}_*} \bar{P}_p(A, \beta) \xrightarrow{\bar{\partial}_*} \bar{\Pi}_{p-1}(A, B_1) \to \cdots \\ &\to \bar{\Pi}(A, B_1) \xrightarrow{\bar{\beta}_*} \bar{\Pi}(A, B_2) \xrightarrow{\bar{J}_*} \bar{P}_0(A, \beta) \xrightarrow{\bar{\partial}_*} \bar{\Pi}_{-1}(A, B_1) \\ &\to \cdots \to \bar{\Pi}_{-m}(A, B_2); \end{aligned}$$

and this can be done for arbitrary integers m. In other words, we have an extension of $S_*(A, \beta)$ to a complete exact sequence of positive-, 0- and negative-dimensional homotopy groups.

If the pair $\beta : B_1 \to B_2$ is an i-fibration relative to $\Omega^m A$, we can replace $S_*(\Omega^m A, \beta)$ by an "absolute" exact sequence. Thus, we obtain the following extended exact "absolute" sequence:

$$\begin{aligned} T'_*(A, \beta): \cdots \to \bar{\Pi}_p(A, B_1) &\xrightarrow{\bar{\beta}_*} \bar{\Pi}_p(A, B_2) \xrightarrow{\bar{\Delta}} \bar{\Pi}_{p-1}(A, B_0) \\ &\xrightarrow{\bar{\varkappa}_*} \bar{\Pi}_{p-1}(A, B_1) \to \cdots \to \bar{\Pi}(A, B_1) \xrightarrow{\bar{\beta}_*} \bar{\Pi}(A, B_2) \xrightarrow{\bar{\Delta}} \bar{\Pi}_{-1}(A, B_0) \\ &\xrightarrow{\bar{\varkappa}_*} \bar{\Pi}_{-1}(A, B_1) \to \cdots \to \bar{\Pi}_{-m}(A, B_2), \end{aligned}$$

where B_0 is the fibre of β, and $\varkappa$ the canonical monomorphism from B_0 to B_1.

For $m \geqq 1$, the functor $\bar{\Pi}_{-m} : \mathcal{K} \times \mathcal{K} \to \mathcal{G}$ will now be shown to coincide with the *functor* $\mathrm{Ext}^m : \mathcal{K} \times \mathcal{K} \to \mathcal{G}$, *defined as* n^{th} *satellite of the functor* Hom, Hom *being considered as functor of the contravariant variable, and the satellites being defined with respect to the abelian structure* $\mathfrak{s}$ (i.e., instead of arbitrary s.e.s., only proper s.e.s. are considered in the definition of the satellites).

$\mathrm{Ext}^1 : \mathcal{K} \times \mathcal{K} \to \mathcal{G}$ is defined by means of the groups

$$\mathrm{Ext}^1(A, B) = S^1 \operatorname{Hom}(A, B) = \operatorname{Hom}(\Omega A, B)/\mu^* \operatorname{Hom}(\underline{A}, B)$$

for a proper s.e.s. $O \to \Omega A \to \underline{A} \xrightarrow{\mu} A \to O$, where $\underline{A}$ is projective. Since by $(\mathrm{P} \to \mathrm{I})$, $\underline{A}$ is injective,

$$\bar{\Pi}_{-1}(A, B) = \bar{\Pi}(\Omega A, B) = \operatorname{Hom}(\Omega A, B)/\mu^* \operatorname{Hom}(\underline{A}, B) = \mathrm{Ext}^1(A, B).$$

Moreover, it is easy to see that

$$\bar{\Pi}_{-1}(\alpha, B) = \mathrm{Ext}^1(\alpha, B), \quad \text{and} \quad \bar{\Pi}_{-1}(A, \beta) = \mathrm{Ext}^1(A, \beta)$$

for all maps α and β of $\mathcal{K}$. Hence, $\bar{\Pi}_{-1} = \mathrm{Ext}^1$ up to natural monomorphisms. For $m > 1$,

$$\mathrm{Ext}^m = S^1 \, \mathrm{Ext}^{m-1} = \mathrm{Ext}^{m-1}(\Omega \times \mathrm{Id}) = \mathrm{Ext}^1(\Omega^{m-1} \times \mathrm{Id}).$$

Therefore,

$$\bar{\Pi}_{-m} = \bar{\Pi}(\Omega^m \times \mathrm{Id}) = \bar{\Pi}_1(\Omega^{m-1} \times \mathrm{Id}) = \mathrm{Ext}^1(\Omega^{m-1} \times \mathrm{Id}) = \mathrm{Ext}^m.$$

We do not intend to show here that the functor Ext^m, defined as satellite of Hom with respect to the contravariant variable, coincides with the functor Ext^m, defined as satellite of Hom with respect to the covariant variable; also, we will not compare it to Ext^m defined as derived functor of Hom. However, we shall use from now on the notation Ext^m for $\bar{\Pi}_{-m}$, always keeping in mind its definition as satellite.[6]

THEOREM 4.2. *Let* $O \to B_0 \xrightarrow{\varkappa} B_1 \xrightarrow{\beta} B_2$ *be an exact sequence, and* β *an i-fibration relative to* $\Omega^m A$. *If condition* $(\mathrm{P} \to \mathrm{I})$ *holds, the injective homotopy sequence of* β *can be extended to an exact sequence*

$$\begin{aligned} E_*(A, \beta)\colon \cdots \to \bar{\Pi}_n(A, B_0) &\xrightarrow{\bar{\varkappa}_*} \bar{\Pi}_n(A, B_1) \xrightarrow{\bar{\beta}_*} \bar{\Pi}_n(A, B_2) \\ &\xrightarrow{\bar{\Delta}} \bar{\Pi}_{n-1}(A, B_0) \to \cdots \to \bar{\Pi}(A, B_1) \xrightarrow{\bar{\beta}_*} \bar{\Pi}(A, B_2) \\ &\xrightarrow{\bar{\Delta}} \mathrm{Ext}^1(A, B_0) \xrightarrow{\varkappa_*} \mathrm{Ext}^1(A, B_1) \to \cdots \to \mathrm{Ext}^{m-1}(A, B_2) \\ &\qquad \xrightarrow{\Delta} \mathrm{Ext}^m(A, B_0) \xrightarrow{\varkappa_*} \mathrm{Ext}^m(A, B_1) \xrightarrow{\beta_*} \mathrm{Ext}^m(A, B_2), \end{aligned}$$

[6] Note that the proper monomorphisms with respect to $\mathfrak{s}$ define an h.f. class in the sense of Buchsbaum, and our functors Ext^m could be compared with the functors $\mathcal{J}\text{-}\mathrm{Ext}^m$ introduced in [1].

where Ext^p *is defined as satellite of* Hom, *considered as functor of the contravariant variable, with respect to the abelian structure* $\mathfrak{E}$.

By duality, we have

THEOREM 4.2*. *Let* $B_2 \xrightarrow{\beta} B_1 \xrightarrow{\gamma} B_0 \to O$ *be an exact sequence, and* β *a p-fibration relative to* $\Sigma^m A$. *If condition* $(\mathrm{I} \to \mathrm{P})$ *holds, then the projective homotopy sequence of* β *can be extended to an exact sequence*

$$E^*(\beta, A)\colon \cdots \to \underline{\Pi}_n(B_0, A) \xrightarrow{\gamma^*} \underline{\Pi}_n(B_1, A) \xrightarrow{\beta^*} \underline{\Pi}_n(B_2, A)$$

$$\xrightarrow{\Delta} \underline{\Pi}_{n-1}(B_0, A) \to \cdots \to \underline{\Pi}(B_1, A) \xrightarrow{\beta^*} \underline{\Pi}(B_2, A)$$

$$\xrightarrow{\Delta} \mathrm{Ext}^1(B_0, A) \xrightarrow{\gamma^*} \mathrm{Ext}^1(B_1, A) \to \cdots \to \mathrm{Ext}^{m-1}(B_2, A)$$

$$\xrightarrow{\Delta} \mathrm{Ext}^m(B_0, A) \xrightarrow{\gamma^*} \mathrm{Ext}^m(B_1, A) \xrightarrow{\beta^*} \mathrm{Ext}^m(B_2, A),$$

where Ext^p *is defined as satellite of* Hom, *considered as functor of the covariant variable, with respect to the abelian structure* $\mathfrak{E}$.

Assume now both conditions $(\mathrm{P} \to \mathrm{I})$ and $(\mathrm{I} \to \mathrm{P})$ to be satisfied. Then, we can extend each of the two dual homotopy sequences. Let us consider the extended sequence $E_*(A, \beta)$ of an i-fibration β, which can be obtained if $(\mathrm{P} \to \mathrm{I})$ holds. Since $(\mathrm{I} \to \mathrm{P})$ is satisfied too, by Lemma 4.1* (c), every proper epimorphism ε is an i-fibration relative to $\Omega^m A$ for every A and m. Thus, we have

PROPOSITION 4.3. *Let* $O \to B_0 \xrightarrow{\varkappa} B_1 \xrightarrow{\varepsilon} B_2 \to O$ *be a proper s.e.s. If* $(\mathrm{P} \to \mathrm{I})$ *and* $(\mathrm{I} \to \mathrm{P})$ *hold, then we have the complete exact sequence*

$$E_*(A, \varepsilon)\colon \cdots \to \Pi_n(A, B_0) \xrightarrow{\varkappa_*} \Pi_n(A, B_1) \xrightarrow{\varepsilon_*} \Pi_n(A, B_2)$$

$$\xrightarrow{\Delta} \Pi_{n-1}(A, B_0) \to \cdots \to \Pi(A, B_1) \xrightarrow{\varepsilon_*} \Pi(A, B_2) \xrightarrow{\Delta} \mathrm{Ext}^1(A, B_0)$$

$$\xrightarrow{\varkappa_*} \mathrm{Ext}^1(A, B_1) \to \cdots \to \mathrm{Ext}^{m-1}(A, B_2) \xrightarrow{\Delta} \mathrm{Ext}^m(A, B_0)$$

$$\xrightarrow{\varkappa_*} \mathrm{Ext}^m(A, B_1) \xrightarrow{\varepsilon_*} \mathrm{Ext}^m(A, B_2) \to \cdots.$$

We omit the proposition dual to 4.3.

Remark. In the category of R-modules with $\mathfrak{E}$ being the usual exact structure, $E_*(A, \varepsilon)$ is precisely the derived sequence of the functor Hom. For, by using the notation of [2], it can be shown that the assumptions $(\mathrm{I} \to \mathrm{P})$ and $(\mathrm{P} \to \mathrm{I})$ imply $\bar{\Pi}_{n+1} = L_n$ Hom for $n > 0$, $\Pi_1 = \tilde{L}_0$ Hom, $\Pi = \tilde{R}^0$ Hom (cf. [2, Chapter V, §10]). This indicates another possible approach to the results of this paper.

5. The exact homotopy-cohomology sequence

Let K be a commutative ring, *R an associative algebra over K, which is K-projective.* From now on, we shall consider the exact category $\mathfrak{M}_R$ of R-modules with the φ-relative structure $\mathcal{S}_\varphi$ as homotopy structure, φ being the natural embedding $K \to R$ (cf. Section 3). In contrast to the exact structure of $\mathfrak{M}_R$, the abelian structure $\mathcal{S}_\varphi$ will henceforth be referred to as *weak structure,* and the corresponding homotopy, as *weak homotopy.* Moreover, the injective and projective objects, the suspensions, and the fibrations with respect to $\mathcal{S}_\varphi$ will be called *weakly injective* and *weakly projective* objects, *weak suspensions* and *weak fibrations.* The weak homotopy functors will be denoted by $\bar{\Pi}_n^{(R,K)}$ and $\underline{\Pi}_n^{(R,K)}$, and the extension functors with respect to the weak structure, by $\mathrm{Ext}^n_{(R,K)}$ as opposed to the "usual" functors $\bar{\Pi}_n$, $\underline{\Pi}_n$ and Ext^n_R in $\mathfrak{M}_R$ (cf. [2] and [6]).

We shall now apply the results of the preceding sections to the weak homotopy structure in $\mathfrak{M}_R$. Preliminarily, we shall compare the groups $\mathrm{Ext}^n_{(R,K)}(Q, B)$ with the groups $\mathrm{Ext}^n_R(Q, B)$ for a special choice of the R-module Q.

LEMMA 5.1. *If Q is a K-projective R-module, then there exists an s.e.s. of R-modules $O \to \Omega Q \to Y \to Q \to O$ which splits over K, where Y is R-projective, and ΩQ, K-projective.*

Proof. Set $Y = R \otimes_K Q$, take as R-module structure the one induced by that of R, and define an R-epimorphism $\varepsilon : Y \to Q$ by setting $\varepsilon(r \otimes q) = rq$ for all $r \in R$, $q \in Q$. Since Q is K-projective, Y is R-projective. Put $\Omega Q =$ Ker ε. Then, $O \to \Omega Q \to Y \to Q \to O$ is an s.e.s. which splits over K; in particular, ΩQ is a K-direct summand of Y. Since R and Q are K-projective, $Y = R \otimes_K Q$ is K-projective; therefore, ΩQ is also K-projective.

PROPOSITION 5.2. *If Q is a K-projective R-module, then there are isomorphisms*

$$t(B) : \mathrm{Ext}^n_{(R,K)}(Q, B) \to \mathrm{Ext}^n_R(Q, B),$$

natural with respect to the covariant variable.

Proof. By Lemma 5.1, we have an s.e.s. $O \to \Omega Q \to Y \to Q \to O$ which can be used to define $\mathrm{Ext}^1_{(R,K)}$ as well as Ext^1_R. Hence, the proposition is valid for $n = 1$. To prove it for $n > 1$, recall that

$$\mathrm{Ext}^n_{(R,K)} = \mathrm{Ext}^1_{(R,K)}(\Omega_w^{n-1} \times \mathrm{Id})$$

for a weak dual suspension Ω_w^{n-1}, and that $\mathrm{Ext}^n_R = \mathrm{Ext}^1_R(\Omega^{n-1} \times \mathrm{Id})$ for a dual suspension Ω^{n-1} with respect to the exact structure. Now Lemma 5.1 implies that for a K-projective object Q, there is an object which is at the same time a weak dual suspension $\Omega_w^{n-1}Q$ and an "ordinary" dual suspension $\Omega^{n-1}Q$ of Q. Hence, the proposition follows for $n > 1$.

Assume now that (P $\to$ I) holds with respect to the weak structure of $\mathfrak{M}_R$.

By Theorem 4.2, for every exact sequence $O \to B_0 \to B_1 \xrightarrow{\beta} B_2$, where β is a weak i-fibration relative to a weak dual suspension $\Omega^m Q$ of Q, we then have an exact sequence $E_*(Q, \beta)$, consisting of weak homotopy groups $\bar{\Pi}_n^{(R,K)}(Q, B_i)$ and of groups $\mathrm{Ext}^n_{(R,K)}(Q, B_i)$ $(i = 0, 1, 2)$. Proposition 5.2 says now that for a K-projective R-module Q, the groups $\mathrm{Ext}^n_{(R,K)}(Q, B_i)$ can be replaced by the "ordinary" groups $\mathrm{Ext}^n_R(Q, B_i)$. If we assume that also condition $(\mathrm{I} \to \mathrm{P})$ holds, we know a special class of weak i-fibrations relative to any weak suspension $\Omega^m Q$, namely the proper epimorphisms with respect to the weak structure (cf. Proposition 4.3). We shall show that, under the assumption that Q is K-projective, every R-epimorphism (not necessarily being proper) is a weak i-fibration relative to a properly-chosen weak suspension $\Omega^m Q$ for every m.

LEMMA 5.3. *If Q is a K-projective R-module, and if condition* $(\mathrm{I} \to \mathrm{P})$ *holds with respect to the weak structure, then there exists an s.e.s.*

$$O \to Q \to X \to \Sigma Q \to O$$

which splits over K, where X is at the same time weakly injective and R-projective, and where ΣQ is K-projective.

Proof. Set $X = \mathrm{Hom}_K(R, Q)$, take as R-module structure the one induced by the right-R-module structure of R, and define an R-monomorphism $\mu : Q \to X$ by setting $[\mu(q)](r) = rq$ for all $r \in R$, $q \in Q$. Clearly, $\mathrm{Hom}_K(R, Q)$ is weakly injective.[7] Put $\Sigma X = \mathrm{Coker}\, \mu$. Then,

$$O \to Q \to X \to \Sigma Q \to O$$

is an s.e.s. which splits over K. Since Q and R are K-projective, $X = \mathrm{Hom}_K(R, Q)$, and therefore ΣQ, are K-projective. By $(\mathrm{I} \to \mathrm{P})$, X is weakly projective, and thus an R-direct summand of $R \otimes_K X$. Since X is K-projective, $R \otimes_K X$ is R-projective, and thus X is R-projective too.

PROPOSITION 5.4. *If Q is a K-projective R-module, and if condition* $(\mathrm{I} \to \mathrm{P})$ *holds with respect to the weak structure, then for any $m > 0$, there exists a weak dual suspension $\Omega^m Q$ of Q such that every R-epimorphism is a weak i-fibration relative to $\Omega^m Q$.*

Proof. By Lemma 5.1, there is a weak dual suspension $\Omega^m Q$ of Q which is K-projective. By Lemma 5.3, there exists a proper injective resolution of $\Omega^m Q$ with respect to the weak structure (i.e., an exact sequence

$$O \to \Omega^m Q \to X_1 \to X_2 \to \cdots,$$

splitting over K, where the X_i are weakly injective), such that the X_i are R-projective modules. If we recall the definition of an R-projective module, the proposition follows immediately.

[7] Even R-injective.

The functor Ext^n_R in the category $\mathfrak{M}_R$ of R-modules can be used to introduce the cohomology groups of various algebraic structures (e.g., associative algebras, groups). There is a unifying concept of "algebraic" cohomology theory, namely the cohomology theory of augmented rings, of which the usual theories are special instances. Let R be an augmented ring, i.e., a ring R with augmentation module Q (cf. [2, Chapter VIII]). For an arbitrary R-module B, the groups $\mathrm{Ext}^n_R(Q, B)$ are called n^{th} *cohomology groups of* R *with coefficients in* B. For these groups, we shall use the notation $H^n(R; B)$.

Let us consider the category $\mathfrak{M}_R$ *of* R*-modules, where* R *is an augmented ring with augmentation module* Q. Assume the following conditions to be satisfied:

(i) *R is an associative algebra over a commutative ring K,*
(ii) *R and Q are K-projective.*

Then, by Proposition 5.2, we have, up to natural isomorphisms,

$$H^n(R; B) = \mathrm{Ext}_R(Q, B) = \mathrm{Ext}^n_{(R,K)}(Q, B), \qquad n > 0.$$

Assume that condition $(\mathrm{P} \to \mathrm{I})$ holds with respect to the weak structure. If we apply Theorem 4.2 to this situation, for every exact sequence

$$O \to B_0 \to B_1 \xrightarrow{\beta} B_2,$$

where β is a weak i-fibration relative to a weak dual suspension $\Omega^m Q$ of Q, we obtain an exact sequence $E_*(Q, \beta)$ consisting of weak injective homotopy groups and cohomology groups. If we assume that also $(\mathrm{I} \to \mathrm{P})$ holds, by the results of Section 3, we have only one kind of weak homotopy groups, and by Proposition 5.4, all R-epimorphisms are weak i-fibrations relative to a properly-chosen weak dual suspension $\Omega^m Q$ of Q. Summing up, we have

THEOREM 5.5. *Let R be an augmented ring with augmentation module Q, satisfying the conditions* (i) *and* (ii), *and let* $O \to B_0 \xrightarrow{\varkappa} B_1 \xrightarrow{\beta} B_2$ *be an exact sequence of R-modules. If the conditions* $(\mathrm{P} \to \mathrm{I})$ *and* $(\mathrm{I} \to \mathrm{P})$ *hold with respect to the weak structure of* $\mathfrak{M}_R$, *and if*

(1) *β is a weak i-fibration relative to a weak dual suspension $\Omega^m Q$ of Q for every $m > 0$, or*

(2) *β is an R-epimorphism,*

then we have a complete exact sequence

$$\begin{aligned} E_*(Q, \beta)\colon \cdots &\to \Pi_n^{(R,K)}(Q, B_0) \xrightarrow{\varkappa_*} \Pi_n^{(R,K)}(Q, B_1) \xrightarrow{\beta_*} \Pi_n^{(R,K)}(Q, B_2) \\ &\xrightarrow{\Delta} \Pi_{n-1}^{(R,K)}(Q, B_0) \to \cdots \to \Pi^{(R,K)}(Q, B_1) \xrightarrow{\beta_*} \Pi^{(R,K)}(Q, B_2) \\ &\xrightarrow{\Delta} H^1(R; B_0) \xrightarrow{\varkappa_*} H^1(R; B_1) \to \cdots \to H^{m-1}(R; B_2) \xrightarrow{\Delta} H^m(R; B_0) \\ &\xrightarrow{\varkappa_*} H^m(R; B_1) \xrightarrow{\beta_*} H^m(R; B_2) \to \cdots. \end{aligned}$$

The complete exact sequence $E_*(Q, \beta)$ will be referred to as *homotopy-cohomology sequence of R* associated with the fibration β, or with the s.e.s.

$$O \to B_0 \xrightarrow{\varkappa} B_1 \xrightarrow{\beta} B_2 \to O.$$

Remarks. 1. Under the assumption that only condition (P $\to$ I) holds, but by demanding in case (2) that β be proper, we obtain an analogous complete exact sequence now containing injective homotopy groups.

2. In case (2), the "cohomology part" is precisely the cohomology sequence of the augmented ring R; Δ can be shown to coincide with the "usual" connecting homomorphism.

6. Applications

(a) *F is a Frobenius K-algebra* (*cf. Section* 3)

Let us take as augmented ring R the enveloping algebra $F \otimes_K F^*$ of F, and as augmentation module F, being considered as $F \otimes_K F^*$-module, or, which is the same, as two-sided F-module. For any two-sided F-module B, the n^{th} cohomology groups $H^n(R; B)$ of R are then the n^{th} *Hochschild cohomology groups $H^n(F, B)$ of the algebra F* (cf. [2, Chapter IX]). Since F is a Frobenius K-algebra, so is the inverse ring F^*. Moreover, the enveloping algebra $F \otimes_K F^*$ is still Frobenius (Proposition 2 of [3]). Thus, conditions (P $\to$ I) and (I $\to$ P) hold with respect to the weak structure of the category $\mathfrak{M}_R$ of R-modules or two-sided F-modules. In analogy with the definition of the Hochschild cohomology groups, we define the weak homotopy groups of a Frobenius algebra as follows.

Definition. *The n^{th} weak homotopy groups $\pi_n(F, B)$ of the Frobenius K-algebra F* with coefficients in a two-sided F-module B are the homotopy groups $\Pi_n^{(F\otimes F^*, K)}(F, B)$ in the category of two-sided F-modules with respect to the weak homotopy structure.

By definition, $F \otimes_K F^*$ and F are K-projective modules. Thus Theorem 5.5 of the preceding section can be formulated as follows:

Let $O \to B_0 \xrightarrow{\varkappa} B_1 \xrightarrow{\beta} B_2$ be an exact sequence of two-sided F-modules. If F is a Frobenius K-algebra, and if

(1) *β is a weak i-fibration relative to a weak dual suspension $\Omega^m F$ of F for every $m > 0$, or*

(2) *β is an epimorphism,*

then we have an exact homotopy-cohomology sequence

$$\begin{aligned}
&\cdots \to \pi_n(F, B_0) \xrightarrow{\varkappa_*} \pi_n(F, B_1) \xrightarrow{\beta_*} \pi_n(F, B_2) \\
&\xrightarrow{\Delta} \pi_{n-1}(F, B_0) \to \cdots \to \pi(F, B_1) \xrightarrow{\beta_*} \pi(F, B_2) \xrightarrow{\Delta} H^1(F, B_0) \\
&\xrightarrow{\varkappa_*} H^1(F, B_1) \to \cdots \to H^{m-1}(F, B_2) \xrightarrow{\Delta} H^m(F, B_0) \\
&\qquad\qquad \xrightarrow{\varkappa_*} H^m(F, B_1) \xrightarrow{\beta_*} H^m(F, B_2) \to \cdots .
\end{aligned} \tag{6.1}$$

(b) *$\mathfrak{G}$ is a finite group (multiplicatively written)*

Let us take as augmented ring R the group algebra $Z(\mathfrak{G})$ of a group $\mathfrak{G}$ over the ring Z of integers, and as augmentation module the additive group I of Z on which $\mathfrak{G}$ operates trivially. For any $\mathfrak{G}$-module B, the n^{th} cohomology groups $H^n(R; B)$ of R are then the *n^{th} cohomology groups $H^n(\mathfrak{G}; B)$ of the group* $\mathfrak{G}$ (cf. [2, Chapter X]). If the group $\mathfrak{G}$ is finite, $Z(\mathfrak{G})$ is a Frobenius Z-algebra (in the sense of Section 3). Thus, conditions (P $\rightarrow$ I) and (I $\rightarrow$ P) hold with respect to the weak structure of the category $\mathfrak{M}_R$ of $Z(\mathfrak{G})$-modules or $\mathfrak{G}$-modules.

DEFINITION. *The n^{th} weak homotopy groups $\pi_n(\mathfrak{G}; B)$ of the finite group $\mathfrak{G}$* with coefficients in a $\mathfrak{G}$-module B are the homotopy groups $\Pi_n^{(Z(\mathfrak{G}),Z)}(I, B)$ in the category of $\mathfrak{G}$-modules with respect to the weak structure, I being the additive group of integers with trivial $\mathfrak{G}$-module structure.

The group algebra $Z(\mathfrak{G})$ and I are free modules over Z. Thus, Theorem 5.5 reads:

Let $O \rightarrow B_0 \xrightarrow{\varkappa} B_1 \xrightarrow{\beta} B_2$ be an exact sequence of $\mathfrak{G}$-modules. If $\mathfrak{G}$ is a finite group, and if

(1) *β is a weak i-fibration relative to a weak dual suspension $\Omega^m I$ of I for every $m > 0$, or*

(2) *β is an epimorphism,*

then we have an exact homotopy-cohomology sequence

$$
(6.2)\qquad
\begin{aligned}
\cdots \rightarrow \pi_n(\mathfrak{G}; B_0) &\xrightarrow{\varkappa_*} \pi_n(\mathfrak{G}; B_1) \xrightarrow{\beta_*} \pi_n(\mathfrak{G}; B_2) \\
&\xrightarrow{\Delta} \pi_{n-1}(\mathfrak{G}; B_0) \rightarrow \cdots \rightarrow \pi(\mathfrak{G}; B_1) \xrightarrow{\beta_*} \pi(\mathfrak{G}; B_2) \xrightarrow{\Delta} H^1(\mathfrak{G}; B_0) \\
&\xrightarrow{\varkappa_*} H^1(\mathfrak{G}; B_1) \rightarrow \cdots \rightarrow H^{m-1}(\mathfrak{G}; B_2) \xrightarrow{\Delta} H^m(\mathfrak{G}; B_0) \\
&\qquad\qquad \xrightarrow{\varkappa_*} H^m(\mathfrak{G}; B_1) \xrightarrow{\beta_*} H^m(\mathfrak{G}; B_2) \rightarrow \cdots .
\end{aligned}
$$

(c) *An interpretation of the weak homotopy groups $\pi_n(\mathfrak{G}, B)$ of a finite group $\mathfrak{G}$*

For an integral index n, let $\hat{H}^n(\mathfrak{G}; B)$ denote the cohomology (homology) groups of a finite group $\mathfrak{G}$ introduced by J. Tate. For a $\mathfrak{G}$-module B, these are defined as follows:

$$
\begin{aligned}
&\hat{H}^n(\mathfrak{G}; B) = H^n(\mathfrak{G}; B), \qquad n > 0 \qquad \text{(usual cohomology groups)},\\
&\hat{H}^n(\mathfrak{G}; B) = H_{-n-1}(\mathfrak{G}; B), \quad n < -1 \quad \text{(usual homology group)},\\
&\hat{H}^{-1}(\mathfrak{G}; B) = \text{kernel of } N^*, \qquad H^0(\mathfrak{G}; B) = \text{cokernel of } N^*,
\end{aligned}
$$

where N^* is the homomorphism $B_{\mathfrak{G}} \rightarrow B^{\mathfrak{G}}$ induced by the norm homomorphism $N : B \rightarrow B$ (cf. [2, Chapter XII]).

Let $O \rightarrow B_0 \rightarrow B_1 \rightarrow B_2 \rightarrow O$ be an exact sequence of $\mathfrak{G}$-modules. Then, one obtains an *exact* sequence

$$(6.3)\qquad \cdots \to \hat{H}^{n-1}(\mathfrak{G}; B_2) \to \hat{H}^n(\mathfrak{G}; B_0) \to \hat{H}^n(\mathfrak{G}; B_1) \to \hat{H}^n(\mathfrak{G}; B_2) \to \hat{H}^{n+1}(\mathfrak{G}; B_0) \to \cdots,$$

called the complete derived sequence of the group $\mathfrak{G}$. The positive-dimensional groups of (6.3) coincide by definition with those of (6.2). We shall show that the negative-dimensional groups of (6.3) are essentially the weak homotopy groups of (6.1).

LEMMA 6.4. *For a weak dual suspension* ΩB *of a* $\mathfrak{G}$*-module* B, *we have*

$$\hat{H}^{n-1}(\mathfrak{G}; B) = \hat{H}^n(\mathfrak{G}; \Omega B),$$

up to natural isomorphisms.

Proof. This is a direct consequence of the exactness of (6.3) and of the following known facts: $H^n(\mathfrak{G}; B) = O$ if B is weakly injective, $n > 0$; $H_n(\mathfrak{G}; B) = O$ if B is weakly projective, $n > 0$, (see [2, Chapter X, Corollary 8.3]); $N^* : B_{\mathfrak{G}} \to B^{\mathfrak{G}}$ is an isomorphism if B is weakly projective (see [2, Chapter XII, Proposition 1.3]).

By Lemma 6.4, it follows immediately that, for a weak dual suspension $\Omega^m B$ of B, $\hat{H}^{1-n}(\mathfrak{G}; B) \cong \hat{H}^1(\mathfrak{G}; \Omega^n B) \cong H^1(\mathfrak{G}; \Omega^n B)$ $(n \geqq 0)$. Besides, we have by definition $\pi_n(\mathfrak{G}; B) \cong \pi(\mathfrak{G}; \Omega^n B)$ $(n \geqq 0)$, and, by Proposition 5.2, $\pi(\mathfrak{G}; B) = H^1(\mathfrak{G}; \Omega B)$. Therefore, $\hat{H}^{-(n-1)}(\mathfrak{G}; B) \cong \pi_n(\mathfrak{G}; B)\,(n \geqq 0)$. Thus, we obtain the following interpretation of the n^{th} weak homotopy groups $\pi_n(\mathfrak{G}; B)$:

$$\pi_n(\mathfrak{G}; B) \cong H_{n-1}(\mathfrak{G}; B), \quad n > 1,$$

$$\pi_1(\mathfrak{G}; B) = \text{kernel of } N^*,$$

$$\pi(\mathfrak{G}; B) = \text{cokernel of } N^*.$$

Moreover, the above isomorphisms are all natural. Hence, the homotopy-cohomology sequence (6.2) and the complete derived sequence (6.3) of the group $\mathfrak{G}$ are essentially the same sequences.

An analogous interpretation can be obtained for a Frobenius K-algebra which is free over K. The n^{th} weak homotopy groups $\pi_n(F, B)$ of a Frobenius K-algebra F are then the modified homology groups $H^*_{n-1}(F, B)$ of F in the sense of Nakayama, and the sequence (6.1) coincides with the corresponding complete homology-cohomology sequence (cf. [7]). One obtains this result by exactly the same procedure as above, which we will not repeat here. Instead of the propositions used here in that context and taken from [2], one has of course to use the corresponding ones for Frobenius algebras, which can be found in [7].

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