COHOMOLOGY OF LIE GROUPS

Dedicated to Reinhold Baer on the occasion of his sixtieth birthday

BY

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1. Introduction

Given a Lie group G and a finite-dimensional continuous G-module V, there present themselves several kinds of "cohomology groups" for G in V that have representation-theoretical, topological, or group-structural interest. In a concrete, but conceptually unsatisfactory fashion, they may be described as the cohomology groups based on continuous cochains, differentiable cochains, or representative cochains. If G is a real linear algebraic group, there is a further specialization of cochains to rational representative cochains.

In order to bring these various cohomology theories under control, one must define and analyze the underlying categories of G-modules, as well as the appropriate notions of "resolution" of a G-module, capable of yielding technically efficient definitions of the cohomology groups. The key for obtaining satisfactory functorial definitions of cohomology groups of the type considered here is the notion of "injectivity" of a module, which (in the context of the general theory of modules) was considered and analyzed first by Reinhold Baer (Bull. Amer. Math. Soc., vol. 46 (1940), pp. 800–806) who showed, in particular, that every module can be imbedded in an injective module, thus ensuring the existence of injective resolutions. It turns out that, contrary to what is the case for projective resolutions, the mechanism of injective resolutions can be adapted to take account of additional structure (topological, differentiable, or rational).

For algebraic linear groups over arbitrary fields of characteristic 0, such a theory has been presented in [3], where it has also been shown how the rational cohomology of such a group can be expressed in terms of the usual cohomology of Lie algebras. The exactly analogous development for the representative cohomology of a Lie group is included below. The continuous cohomology theory has been presented in [11], which also contains the main results on the passage to Lie algebra cohomology. Examination of the technicalities involved in this passage has revealed difficulties which were not fully appreciated at the time when [11] was written, and which stand in the way of a truly categorical treatment.

These difficulties reside in the requirements of "differentiability" and "integrability" of a G-module, the first of which is the essential link to the Lie algebra cohomology, while the second is an indispensable technical aid. While it is immediate from classical results that requirements of this type

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are always met by any finite-dimensional continuous G-module, direct verification of such analytical requirements tends to become extremely awkward or even unfeasible for the vastly more complicated modules occurring in a category that permits a general construction of the appropriate injective resolutions for every member of the category. The way out consists in balancing the definitions of differentiability and integrability with respect to a standard construction of injective resolutions and with respect to each other so that these properties are inherited in a formal fashion by the modules occurring in the various resolutions. This accounts for the seemingly disproportionate length of Sections 3–8 below. Most of the results of these sections are amendments of results appearing in Sections 2, 3, and 4 of [11].

Sections 9 and 10 deal with the representative cohomology. Sections 11 and 12 answer the questions that originally motivated this investigation in a surprisingly simple way (Theorems 11.1 and 12.1): if G is a real linear Lie group with finite group of components, and V is a finite-dimensional continuous G-module, then the continuous cohomology group for G in V is naturally isomorphic with the tensor product of the representative cohomology group for G in V by the continuous cohomology group of a certain factor group of G in the trivial 1-dimensional G-module. If G is an algebraic linear group there is an exactly analogous result, with "representative" in the place of "continuous", and "rational" in the place of "representative".

2. Continuous cohomology

Let G be a locally compact topological group. By a *continuous G-module* we mean a Hausdorff topological vector space A over the field R of the real numbers that is equipped with a G-module structure such that the corresponding map $G \times A \to A$, written $(x, a) \to x \cdot a$, is continuous, and the map $a \to x \cdot a$ is a *linear* automorphism of A, for every $x \in G$.

An exact sequence $\cdots \to A_i \xrightarrow{\alpha_i} A_{i+1} \to \cdots$ of continuous G-module homomorphisms is said to be *strongly exact* if there is a sequence of continuous linear maps $\gamma_i : A_i \to A_{i-1}$ such that, for each i, $\gamma_{i+1} \circ \alpha_i + \alpha_{i-1} \circ \gamma_i$ is the identity map of A_i onto itself. Such a sequence (γ_i) is called a continuous contracting homotopy.

We say that a continuous G-module A is *continuously injective* if, for every strongly exact sequence $0 \to U \xrightarrow{\rho} V \to W \to 0$ of continuous G-module homomorphisms and every continuous G-module homomorphism $\alpha : U \to A$, there is a continuous G-module homomorphism $\beta : V \to A$ such that $\beta \circ \rho = \alpha$.

If A and B are continuous G-modules, a *strong imbedding* of A in B is a continuous G-module monomorphism $\alpha : A \to B$ such that there is a continuous linear map $\beta : B \to A$ whose composite $\beta \circ \alpha$ with α is the identity map of A onto itself. The continuous cohomology theory is based on the fact that *every continuous G-module has a strong imbedding in a continuously injective G-module.* We shall exhibit such an imbedding presently.

Let $F(G, A)$ denote the space of all continuous maps of G into a topological vector space A, and topologize $F(G, A)$ by the compact-open topology, i.e., the topology in which a fundamental system of neighborhoods of 0 consists of the sets $N(C, U)$, where C ranges over the compact subsets of G, U ranges over the neighborhoods of 0 in A, and $f \in N(C, U)$ means that $f(C) \subset U$. We make $F(G, A)$ into a G-module via the left translations $f \to x \cdot f$, where $(x \cdot f)(y) = f(yx)$. It is not difficult to verify that this makes $F(G, A)$ into a continuous G-module.

Next we show that $F(G, A)$ *is continuously injective*. Let

$$0 \to U \xrightarrow{\rho} V \to W \to 0$$

be a strongly exact sequence of continuous G-module homomorphisms, and let α be a continuous G-module homomorphism of U into $F(G, A)$. The strong exactness of the sequence means that there is a continuous linear map $\sigma : V \to U$ such that $\sigma \circ \rho$ is the identity map on U. For $v \in V$, define the map $\beta(v)$ of G into A by $\beta(v)(x) = \alpha(\sigma(x \cdot v))(1)$, where 1 stands for the identity element of G. Clearly, $\beta(v) \in F(G, A)$. Moreover, one verifies immediately that β is a continuous linear map of V into $F(G, A)$. Evidently, $\beta(y \cdot v) = y \cdot \beta(v)$, for every $y \in G$, so that β is a continuous G-module homomorphism of V into $F(G, A)$. Finally, it is seen directly that $\beta \circ \rho = \alpha$. Thus we have shown that $F(G, A)$ is continuously injective.

Now suppose that A is a continuous G-module. With each element $a \in A$, we associate the element $f_a \in F(G, A)$ that is given by $f_a(x) = x \cdot a$. Then the map $a \to f_a$ is evidently a continuous G-module monomorphism of A into $F(G, A)$. The map $f \to f(1)$ is a continuous linear map of $F(G, A)$ into A whose composite with the map $a \to f_a$ is the identity map on A. Thus the map $a \to f_a$ is a strong imbedding of A in $F(G, A)$.

It follows immediately from this that, for every continuous G-module A, there is a strongly exact sequence of continuous G-module homomorphisms $0 \to A \to X_0 \to X_1 \to \cdots$, where each X_i is continuously injective. Such a sequence is called a *continuously injective resolution* of A. From such a resolution, we obtain a complex of topological vector spaces $0 \to X_0^G \to X_1^G \to \cdots$, where X_i^G denotes the G-fixed part of X_i. The homology space $H(X^G)$ of this complex is independent, up to natural isomorphisms, of the choice of the continuously injective resolution X of A. We denote it by $H_c(G, A)$ and call it the continuous cohomology space for G in A.

More generally, let $0 \to A \to Y_0 \to Y_1 \to \cdots$ be a strongly exact sequence of continuous G-module homomorphisms, and let $0 \to B \to Z_0 \to Z_1 \to \cdots$ be a sequence of continuous G-module homomorphisms such that the composite of any two successive homomorphisms is 0 and each Z_i is continuously injective. Let α be any continuous G-module homomorphism of A into B. Then α can be extended to a continuous G-module homomorphism of the first sequence into the second so that the resulting diagram

$$\begin{array}{ccccccc}
0 \to & A & \to & Y_0 & \to & Y_1 & \to \cdots \\
& \alpha \downarrow & & \downarrow & & \downarrow & \\
0 \to & B & \to & Z_0 & \to & Z_1 & \to \cdots
\end{array}$$

is commutative. Moreover, this extension is unique up to a continuous G-module homotopy, i.e., if α_1 and α_2 are any two such extensions, there exists a continuous G-module homomorphism γ sending Y_0 into B and Y_i into Z_{i-1}, for $i > 0$, such that $\alpha_1 - \alpha_2 = d_Y \circ \gamma + \gamma \circ d_Z$, where d_Y and d_Z stand for the sequences of maps in the complexes Y and Z, respectively. The proofs of all these statements follow the usual pattern of homological algebra. In particular, one sees from these facts that α induces a homomorphism $H_c(G, A) \to H_c(G, B)$ in the canonical way. Thus $H_c(G, *)$ becomes an additive functor. More precisely, we obtain a connected sequence of functors $H_c^n(G, *)$, $n = 0, 1, \cdots$, with connecting homomorphisms $H_c^n(G, C) \to H_c^{n+1}(G, A)$ corresponding to every strongly exact sequence $0 \to A \to B \to C \to 0$ of continuous G-module homomorphisms, such that the sequence $\cdots \to H_c^n(G, A) \to H_c^n(G, B) \to H_c^n(G, C) \to H_c^{n+1}(G, A) \to \cdots$ is exact (cf. [1, Ch. V]).

An explicit definition of $H_c(G, A)$ and the associated homomorphisms may be based on any functorial construction of continuously injective resolutions. We proceed to give such a construction. For each $f \in F(G, A)$, let f' be the map of G into A given by $f'(x) = x \cdot f(x^{-1})$. Clearly, $f' \in F(G, A)$, $(f')' = f$, and the map $f \to f'$ is continuous. Thus the map $f \to f'$ is a continuous linear involution of the topological vector space $F(G, A)$. We have $(x \cdot f)'(y) = x \cdot f'(x^{-1}y)$. Thus the involution $f \to f'$ transports the G-module structure on $F(G, A)$ into a new G-module structure in which $(x \cdot f)(y) = x \cdot f(x^{-1}y)$. We shall denote by $F^0(G, A)$ the continuous G-module whose underlying topological vector space is $F(G, A)$ and whose G-module structure is given by this last formula. The map $f \to f'$ is then a topological G-module isomorphism of $F(G, A)$ onto $F^0(G, A)$. For $i \geqq 0$, we define $F^i(G, A)$ inductively, setting $F^{i+1}(G, A) = F^0(G, F^i(G, A))$. Clearly, $F^i(G, A)$ may then be identified with the continuous G-module whose underlying topological vector space is the space of all continuous maps of the $(i + 1)$-fold direct product of copies of G into A, topologized by the compact-open topology, and whose G-module structure is given by the formula

$$(x \cdot f)(y_0, \cdots, y_i) = x \cdot f(x^{-1}y_0, \cdots, x^{-1}y_i).$$

Each $F^i(G, A)$ is continuously injective, as is clear from the above.

Now we define a continuously injective resolution

$$0 \to A \to F^0(G, A) \to F^1(G, A) \to \cdots$$

as follows. The first map $A \to F^0(G, A)$ is the map associating with every $a \in A$ the constant map $G \to A$ with value a. For $i \geqq 0$, the map $d : F^i(G, A) \to F^{i+1}(G, A)$ is given by the formula

$$(df)(x_0, \cdots, x_{i+1}) = \sum_{j=0}^{i+1} (-1)^j f(x_0, \cdots, \hat{x}_j, \cdots, x_{i+1}),$$

where $\hat{}$ indicates that the argument below it is to be omitted. It is easily verified that this is indeed a continuously injective resolution of A; a contracting homotopy γ is given by $\gamma_0 : F^0(G, A) \to A; f \to f(1)$, and, for $i > 0$, $\gamma_i : F^i(G, A) \to F^{i-1}(G, A)$; $(\gamma_i f)(x_0, \cdots, x_{i-1}) = f(1, x_0, \cdots, x_{i-1})$.

We shall call this resolution the *homogeneous* resolution. Now we observe that $F^i(G, A)^G$ is topologically isomorphic with $F^{i-1}(G, A)$, by the map $f \to f^*$, where

$$f^*(x_1, \cdots, x_i) = f(1, x_1, x_1 x_2, \cdots, x_1 \cdots x_i).$$

The complex of the G-fixed part of the homogeneous resolution thus becomes the complex of the *nonhomogeneous continuous cochains*

$$0 \to A \to F^0(G, A) \to F^1(G, A) \to \cdots,$$

the coboundary map, δ, being given by

$$\delta_0 : A \to F^0(G, A); \qquad (\delta_0 a)(x) = x \cdot a - a,$$

$$\delta_i : F^{i-1}(G, A) \to F^i(G, A);$$

$$(\delta_i f)(x_0, \cdots, x_i) = x_0 \cdot f(x_1, \cdots, x_i)$$
$$+ \textstyle\sum_{j=0}^{i-1} (-1)^{j+1} f(x_0, \cdots, x_j x_{j+1}, \cdots, x_i) + (-1)^i f(x_0, \cdots, x_{i-1}).$$

3. Auxiliary results on continuous modules

Let A be a topological vector space, and let G be a locally compact topological group. Let $F^*(G, A)$ denote the space of all continuous maps of G into A with compact support, topologized by the compact-open topology. We say that A is *G-integrable* if there is a continuous map J_G of $F^*(G, A)$ into A and a separating family A' of continuous linear functionals on A such that, for every $\gamma \in A'$ and every $f \in F^*(G, A)$, we have $\gamma(J_G(f)) = I_G(\gamma \circ f)$, where I_G is a Haar integral on G, invariant in the sense that $I_G(x \cdot g) = I_G(g)$, for every $x \in G$ and every $g \in F^*(G, A)$.

LEMMA. 3.1. *Let A be a G-integrable topological vector space, and let S be a locally compact topological space. Let $F(S, A)$ be the space of all continuous maps of S into A, topologized by the compact-open topology. Then $F(S, A)$ is G-integrable, with respect to the family $F(S, A)'$ consisting of the functionals $t_{\gamma,s}$, where γ ranges over A', s ranges over S, and $t_{\gamma,s}(f) = \gamma(f(s))$.*

Proof. Let $\varphi \in F^*(G, F(S, A))$. For each $s \in S$, define the map $\varphi_s : G \to A$ by setting $\varphi_s(x) = \varphi(x)(s)$. Clearly, $\varphi_s \in F^*(G, A)$, so that $J_G(\varphi_s)$ is defined as an element of A. Now define the map $J_G(\varphi) : S \to A$ by $J_G(\varphi)(s) = J_G(\varphi_s)$. First we show that $J_G(\varphi) \in F(S, A)$. Let $s_1 \in S$, and let U be a neighborhood of 0 in A. Since J_G is continuous from $F^*(G, A)$ to A, we can find a compact subset C of G and a neighborhood V of 0 in A such that $J_G(N(C, V)) \subset U$. Now choose a neighborhood W of 0 in A such that $W + W - W \subset V$. For each $x \in G$, we can find a compact neighborhood T_x of s_1 in S such that $\varphi(x)(t) - \varphi(x)(s_1) \in W$, for all $t \in T_x$. Since φ is continuous, there is a neighborhood P_x of x in G such that

$$(\varphi(y) - \varphi(x))(T_x) \subset W,$$

for all $y \in P_x$. The compact set C is contained in the union of a finite family $P_{x_1}, \cdots, P_{x_n}$. Let T be intersection of the corresponding neighborhoods T_{x_i} of s_1 in S. Now let $x \in C$ and $t \in T$. Then $x \in P_{x_i}$, for some i, and we have

$$\varphi(x)(t) - \varphi(x)(s_1) = [\varphi(x)(t) - \varphi(x_i)(t)] + [\varphi(x_i)(t) - \varphi(x_i)(s_1)] - [\varphi(x)(s_1) - \varphi(x_i)(s_1)],$$

which lies in V. Thus we have $\varphi_t - \varphi_{s_1} \in N(C, V)$, whence $J_G(\varphi_t - \varphi_{s_1}) \in U$, i.e., $J_G(\varphi)(t) - J_G(\varphi)(s_1) \in U$. We have shown that $J_G(\varphi)$ is continuous, i.e., that $J_G(\varphi) \in F(S, A)$.

Next we show that J_G is continuous from $F^*(G, F(S, A))$ to $F(S, A)$. Let T be a compact subset of S, and let U be a neighborhood of 0 in A. The condition $J_G(\varphi) \subset N(T, U)$ means that $J_G(\varphi_t) \in U$, for every $t \in T$. This will be the case if $\varphi_t \in N(C, V)$, for every $t \in T$, i.e., if $\varphi \in N(C, N(T, V))$. Hence it is clear that J_G is continuous.

Finally, we have $t_{\gamma,s}(J_G(\varphi)) = \gamma(J_G(\varphi_s)) = I_G(\gamma \circ \varphi_s) = I_G(t_{\gamma,s} \circ \varphi)$, which completes the proof of Lemma 3.1.

If K is any other locally compact group, and A is a continuous K-module, we say that A is a G-integrable continuous K-module if A is G-integrable as a topological vector space with respect to a separating family A' of continuous linear functionals such that, for every $\gamma \in A'$ and every $x \in K$, the map $a \to \gamma(x \cdot a)$ belongs to A'. Then Lemma 3.1 shows immediately that *if A is a G-integrable continuous K-module, so is $F(K, A)$*.

LEMMA 3.2. *Let G be a locally compact topological group, P a compact normal subgroup of G, S a closed subgroup of G such that $G = SP$. Let A be a P-integrable continuous G-module that is continuously injective as an S-module. Then A is continuously injective also as a G-module.*

Proof. Let V be a continuous G-module having a continuous linear projection onto a submodule U, and let α be a continuous G-module homomorphism of U into A. Since A is continuously injective as an S-module, there exists a continuous S-module homomorphism β of V into A such that β coincides with α on U. For each $v \in V$, define a map φ_v of P into A by $\varphi_v(p) = p^{-1} \cdot \beta(p \cdot v)$. Clearly, $\varphi_v \in F(P, A)$, and the map $v \to \varphi_v$ is continuous. Now define the continuous map ρ of V into A by $\rho(v) = J_P(\varphi_v)$. Here, making use of the fact that P is compact, we normalize J_P so that it maps constant functions onto their values. For $s \in S$, we have

$$\varphi_{s \cdot v}(p) = p^{-1} \cdot \beta(ps \cdot v) = s \cdot [s^{-1} p^{-1} s \cdot \beta(s^{-1} ps \cdot v)] = s \cdot \varphi_v(s^{-1} ps).$$

For any $f \in F(P, A)$, define f^s and $s(f)$ as elements of $F(P, A)$ by

$$f^s(p) = f(s^{-1} ps) \quad \text{and} \quad s(f)(p) = s \cdot f(p).$$

Then we have $\varphi_{s\cdot v} = s(\varphi_v{}^s)$. We have $J_P(s(f)) = s\cdot J_P(f)$. Furthermore, the uniqueness of the normalized Haar integral on the compact group P implies that $J_P(f^s) = J_P(f)$. Hence we obtain

$$\rho(s\cdot v) = J_P(\varphi_{s\cdot v}) = s\cdot J_P(\varphi_v) = s\cdot \rho(v).$$

On the other hand, with $p \in P$, we have

$$\rho(p\cdot v) = J_P(\varphi_{p\cdot v}) = J_P(p(p\cdot\varphi_v)) = p\cdot J_P(p\cdot\varphi_v) = p\cdot J_P(\varphi_v) = p\cdot\rho(v).$$

Thus we conclude that ρ is a continuous G-module homomorphism of V into A.

Finally, if $v \in U$, then φ_v is the constant function with value $\alpha(v)$, whence $\rho(v) = \alpha(v)$, which completes the proof of Lemma 3.2.

LEMMA 3.3. *Let G be a locally compact topological group, and let K be a closed subgroup of G such that the space G/K of the cosets xK is compact. Let A be a continuously injective G-module, and suppose that, as a topological vector space, A is K-integrable. Then A is continuously injective as a K-module.*

Proof. We need an auxiliary weighting function on G, and we use the following construction due to P. Cartier [13, p. 22-02]. We shall construct a continuous nonnegative real-valued function f on G with compact support and such that $I_K((x\cdot f)_K) = 1$, for every $x \in G$, where the subscript K denotes restriction of the function to K. Let C be a compact neighborhood of the identity in G. Since a finite set of translates of the canonical image of C in G/K covers the compact space G/K, there is a compact subset D of G such that $G = DK$. We can find a nonnegative real-valued continuous function g with compact support on G such that $g(x) = 1$, for every $x \in D^{-1}$. Then, for every $x \in G$, $(x\cdot g)_K$ is a continuous function with compact support on K. Hence we may define a function h on G by setting $h(x) = I_K((x\cdot g)_K)$. Since $G = KD^{-1}$ and g takes the value 1 at each point of D^{-1}, the function $(x\cdot g)_K$ is not identically zero. Since it is nonnegative, it follows that $h(x) > 0$, for every $x \in G$. Since g has compact support, it is uniformly continuous, whence h is continuous. Now define $f(x) = g(x)/h(x)$. Then f is nonnegative and continuous and has compact support. We have $(x\cdot f)_K = (x\cdot g)_K/(x\cdot h)_K$. For $y \in K$, we have $(x\cdot h)(y) = h(yx) = I_K((yx\cdot g)_K) = I_K(y\cdot(x\cdot g)_K) = I_K((x\cdot g)_K) = h(x)$. Thus $(x\cdot h)_K$ is the constant function with value $h(x)$, and it follows that $I_K((x\cdot f)_K) = 1$, for every $x \in G$.

Since A is continuously injective as a G-module, we may identify it with a direct topological G-module summand of $F(G, A)$. Hence it suffices to show that $F(G, A)$ is continuously injective as a K-module. Let V be a continuous K-module having a continuous linear projection γ onto a submodule U, and let α be a continuous K-module homomorphism of U into $F(G, A)$. For $v \in V$ and $x \in G$, let $\rho_{v,x}$ be the map of K into A defined by $\rho_{v,x}(y) = \alpha(\gamma(y\cdot v))(xy^{-1})$. Clearly, $\rho_{v,x} \in F(K, A)$, whence $(x^{-1}\cdot f)_K\rho_{v,x} \in F^*(K, A)$. Define $\beta(v)$ as a map of G into A by $\beta(v)(x) = J_K((x^{-1}\cdot f)_K\rho_{v,x})$. We see immediately that $\beta(v) \in F(G, A)$ and that β is a continuous linear map of V into $F(G, A)$.

Now let $x \in G$ and $y \in K$. Then we have

$$(y\cdot\beta(v))(x) = \beta(v)(xy) = J_K((y^{-1}x^{-1}\cdot f)_K\,\rho_{v,xy}) = J_K((x^{-1}\cdot f)_K\, y\cdot\rho_{v,xy}).$$

But

$$(y\cdot\rho_{v,xy})(z) = \rho_{v,xy}(zy) = \alpha(\gamma(zy\cdot v))(xyy^{-1}z^{-1}) = \rho_{y\cdot v,x}(z),$$

so that $y\cdot\rho_{v,xy} = \rho_{y\cdot v,x}$. Hence we obtain $(y\cdot\beta(v))(x) = \beta(y\cdot v)(x)$, showing that β is a continuous K-module homomorphism of V into $F(G, A)$.

Finally, if $v \in U$ we have $\rho_{v,x}(y) = \alpha(y\cdot v)(xy^{-1}) = \alpha(v)(x)$, i.e., $\rho_{v,x}$ is the constant function with value $\alpha(v)(x)$. It follows that

$$\beta(v)(x) = I_K((x^{-1}\cdot f)_K)\alpha(v)(x) = \alpha(v)(x),$$

so that β coincides with α on U. This completes the proof of Lemma 3.3.

The next result was already obtained in [11, Section 2.8] and is included here for the convenience of the reader.

LEMMA 3.4. *Let G be a locally compact topological group, and let K be a closed subgroup of G. Suppose that the space G/K is paracompact and that there is a continuous local cross-section $G/K \to G$. Then every continuously injective continuous G-module is continuously injective also as a K-module.*

Proof. It suffices to show that, for every topological vector space A, the continuous G-module $F(G, A)$ is continuously injective as a K-module. The assumptions of the lemma imply that there is a locally finite open covering $(C_i)_{i\in I}$ of G/K satisfying the following conditions. If p is the canonical map $G \to G/K$, then $p^{-1}(C_i)$ is homeomorphic with $C_i \times K$, by a homeomorphism commuting with the right K-action. There is a partition of the constant function on G/K with value 1 into continuous nonnegative functions γ_i with support contained in C_i.

Now let U, V, α be as in the proof of Lemma 3.3. Let α_i be the continuous K-module homomorphism of U into $F(p^{-1}(C_i), A)$ defined by making $\alpha_i(u)$ the restriction to $p^{-1}(C_i)$ of $\alpha(u) \in F(G, A)$, for every $u \in U$. Now $F(p^{-1}(C_i), A)$ may be identified, as a continuous K-module, with $F(K, F(C_i, A))$ by means of the homeomorphism between $p^{-1}(C_i)$ and $C_i \times K$. Hence the continuous K-module $F(p^{-1}(C_i), A)$ is continuously injective. Hence there is a continuous K-module homomorphism β_i of V into $F(p^{-1}(C_i), A)$ such that β_i coincides with α_i on U. Now consider the value-wise product $(\gamma_i \circ p)\beta_i(v)$, where $v \in V$. This is an element of $F(p^{-1}(C_i), A)$ which may evidently be regarded as an element $\psi_i(v)$ of $F(G, A)$ vanishing outside $p^{-1}(C_i)$. Clearly, ψ_i is a continuous K-module homomorphism of V into $F(G, A)$, and ψ_i coincides with $(\gamma_i \circ p)\alpha$ on U. Since the covering by the C_i is locally finite, it is clear that, for each $v \in V$, the sum $\sum_{i\in I}\psi_i(v)$ is defined as a map of G into A, in the sense that, for every $x \in G$, only a finite number of the $\psi_i(v)(x)$ are different from 0. Moreover, it is evident that this sum represents an element of $F(G, A)$. If we write $\beta(v) = \sum_{i \in I} \psi_i(v)$, we see immediately that β coincides with α on U, and it is not difficult to verify that β is a continuous K-module homomorphism of V into $F(G, A)$, so that Lemma 3.4 is proved.

4. Differentiable modules

Let G be a real analytic group, $\mathfrak{G}$ the Lie algebra of G. Let A be a real topological vector space, and assume that the points of A are separated by the continuous linear functionals on A. We say that a continuous map φ of G into A is differentiable if it satisfies the following conditions.

(1) For $x \in G$, $\zeta \in \mathfrak{G}$, and $t \in R$, we have

$$\varphi(x \exp(t\zeta)) = \varphi(x) + t\varphi'(x, \zeta, t),$$

where φ' is a continuous map of $G \times \mathfrak{G} \times R$ into A.

(2) If, with $\tau \in \mathfrak{G}$, $\tau(\varphi) : G \to A$ is defined by $\tau(\varphi)(x) = \varphi'(x, \tau, 0)$, then $\tau(\varphi)$ satisfies (1); if $\tau_1, \tau_2 \in \mathfrak{G}$, then $\tau_1(\tau_2(\varphi))$ satisfies (1), etc.

Let D denote the space of all real-valued differentiable functions on G. By a differential operator on D we mean a real linear endomorphism of D that can be written as a polynomial in multiplications by elements of D and real linear derivations. As is well known, one may identify the algebra of all those differential operators which commute with the right translations ($f \to f \cdot x$, where $(f \cdot x)(y) = f(xy)$) with the universal enveloping algebra $U(\mathfrak{G})$ of $\mathfrak{G}$, and the D-module of all differential operators is canonically isomorphic with $D \otimes U(\mathfrak{G})$.

Clearly, if γ is any continuous linear functional on A, and if φ is a differentiable map of G into A, then $\gamma \circ \varphi \in D$, and we have $\gamma \circ \zeta(\varphi) = \zeta(\gamma \circ \varphi)$, for every $\zeta \in \mathfrak{G}$. Using the above facts concerning the differential operators, and that the elements of A are separated by the continuous linear functionals, we see that the action of $\mathfrak{G}$ on differentiable maps of G into A can be extended uniquely to an action of the algebra of all differential operators on differentiable maps such that the formula $\gamma \circ \delta(\varphi) = \delta(\gamma \circ \varphi)$ holds for all differential operators δ.

Let $F_d(G, A)$ denote the space of all differentiable maps of G into A. It is clear that every differential operator sends $F_d(G, A)$ into itself. We claim that, furthermore, $F_d(G, A)$ is stable under the left translations with the elements of G. Let $f \in F_d(G, A)$, x, $y \in G$, $\zeta \in \mathfrak{G}$, $t \in R$. Then we have

$$(y \cdot f)(x \exp(t\zeta)) = f(xy \exp(ty^*(\zeta))),$$

where y^* is the element of the adjoint group of G that corresponds to y^{-1}. Hence

$$(y \cdot f)(x \exp(t\zeta)) = (y \cdot f)(x) + tf'(xy, y^*(\zeta), t).$$

Define $(y\cdot f)' : G \times \mathfrak{G} \times R \to A$ by $(y\cdot f)'(x, \zeta, t) = f'(xy, y^*(\zeta), t)$. Then $(y\cdot f)'$ is evidently continuous, and we have shown that $y\cdot f$ satisfies (1). Moreover, we have

$$\zeta(y\cdot f)(x) = (y\cdot f)'(x, \zeta, 0) = f'(xy, y^*(\zeta), 0) = y^*(\zeta)(f)(xy),$$

so that $\zeta(y\cdot f) = y\cdot y^*(\zeta)(f)$. Since $y^*(\zeta)(f) \in F_d(G, A)$, what we have already proved shows that $\zeta(y\cdot f)$ satisfies (1), and that, with $\tau \in \mathfrak{G}$, $\tau(\zeta(y\cdot f)) = y\cdot y^*(\tau)(y^*(\zeta)(f))$. Clearly, this argument can be repeated, and we conclude that $y\cdot f \in F_d(G, A)$. Similarly, but more easily, one sees that $F_d(G, A)$ is stable under the right translations.

We topologize $F_d(G, A)$ by taking for a fundamental system of neighborhoods of 0 the sets $N(C, E, U)$, where C ranges over the compact subsets of G, E over the finite sets of differential operators on D, U over the neighborhoods of 0 in A, and $f \in N(C, E, U)$ means that $\delta(f)(C) \subset U$, for every $\delta \in E$.

We claim that $F_d(G, A)$ thus becomes a continuous G-module. Let δ be a differential operator on D that commutes with the left translations; let $x \in G$ and $f \in F_d(G, A)$. Then we have $\delta(x\cdot f) = x\cdot\delta(f)$, whence it is clear that the map $(x, f) \to \delta(x\cdot f)$ is continuous from $G \times F_d(G, A)$ to $F(G, A)$. Since every differential operator on D is a D-linear combination of such δ's, it follows that the map $(x, f) \to \delta(x\cdot f)$ is continuous from $G \times F_d(G, A)$ to $F(G, A)$, for every differential operator δ. But this is equivalent to the statement that the map $(x, f) \to x\cdot f$ is continuous from $G \times F_d(G, A)$ to $F_d(G, A)$, so that our claim is proved.

We shall say that a continuous G-module B is a *differentiable G-module* if, for every $b \in B$, the map $f_b : G \to B$, defined by $f_b(x) = x\cdot b$, is a differentiable map and the map $b \to f_b$ is continuous from B to $F_d(G, B)$. We shall show that $F_d(G, A)$ *is a differentiable G-module.*

Let $\varphi \in F_d(G, A)$. Then we have

$$((x \exp(t\zeta))\cdot\varphi)(y) = \varphi(yx \exp(t\zeta)) = \varphi(yx) + t\varphi'(yx, \zeta, t).$$

Define the map $\varphi'_{\zeta,t} : G \to A$ by $\varphi'_{\zeta,t}(x) = \varphi'(x, \zeta, t)$. Then the above gives

$$(x \exp(t\zeta))\cdot\varphi = x\cdot\varphi + tx\cdot\varphi'_{\zeta,t}.$$

Hence we have

$$f_\varphi(x \exp(t\zeta)) = f_\varphi(x) + t(f_\varphi)'(x, \zeta, t),$$

where $(f_\varphi)'(x, \zeta, t) = x\cdot\varphi'_{\zeta,t}$. If $t \neq 0$, we have $\varphi'_{\zeta,t} = t^{-1}(\exp(t\zeta)\cdot\varphi - \varphi)$, which belongs to $F_d(G, A)$. On the other hand, $\varphi'_{\zeta,0} = \zeta(\varphi)$, which also belongs to $F_d(G, A)$. It follows that $(f_\varphi)'$ is a map of $G \times \mathfrak{G} \times R$ into $F_d(G, A)$.

In order to show that $(f_\varphi)'$ is continuous, it suffices to prove that, for every differential operator δ that commutes with the left translations, the map $(x, \zeta, t) \to \delta(x\cdot\varphi'_{\zeta,t}) = x\cdot\delta(\varphi'_{\zeta,t})$ is continuous from $G \times \mathfrak{G} \times R$ to $F(G, A)$. For $t \neq 0$, we have $\delta(\varphi'_{\zeta,t}) = \delta(t^{-1}(\exp(t\zeta)\cdot\varphi - \varphi)) =$

$t^{-1}(\exp(t\zeta)\cdot\delta(\varphi) - \delta(\varphi)) = \delta(\varphi)'_{\zeta,t}$. On the other hand, $\varphi'_{\zeta,0} = \zeta(\varphi)$. Since δ commutes with the left translations, it commutes also with $\zeta \in \mathfrak{G}$, so that $\delta(\varphi'_{\zeta,0}) = \zeta(\delta(\varphi)) = \delta(\varphi)'_{\zeta,0}$. Thus we have, for all $t \in R$, $\delta(\varphi'_{\zeta,t}) = \delta(\varphi)'_{\zeta,t}$. Hence it suffices to show that, if ψ $(= \delta(\varphi))$ is any element of $F_d(G, A)$, the map $(x, \zeta, t) \to x\cdot\psi'_{\zeta,t}$ is continuous from $G \times \mathfrak{G} \times R$ to $F(G, A)$. Since ψ' is continuous, this is evidently the case.

We have now shown that the map f_φ of G into $F_d(G, A)$ satisfies condition (1) of the definition of a differentiable map. Moreover, we have $\zeta(f_\varphi)(x) = (f_\varphi)'(x, \zeta, 0) = x\cdot\varphi'_{\zeta,0} = x\cdot\zeta(\varphi)$, so that $\zeta(f_\varphi) = f_{\zeta(\varphi)}$. Hence our above argument applies again to show that $\zeta(f_\varphi)$ satisfies condition (1), etc., and we may conclude that f_φ is a differentiable map of G into $F_d(G, A)$.

Now observe that, if δ is any element of the universal enveloping algebra of $\mathfrak{G}$, we have $\delta(f_\varphi) = f_{\delta(\varphi)}$, whence we see immediately that the map $\varphi \to \delta(f_\varphi)$ is continuous from $F_d(G, A)$ to $F(G, F_d(G, A))$. But this implies that the map $\varphi \to f_\varphi$ is continuous from $F_d(G, A)$ to $F_d(G, F_d(G, A))$. Thus $F_d(G, A)$ has been shown to be a differentiable G-module.

LEMMA 4.1. *Let U be a convex neighborhood of 0 in a topological vector space A, and let ψ be a map of R into A such that $\psi(0) = 0$ and ψ has a derivative $\psi'(t) \in U$, for all t in the closed interval $[-e, e]$, where e is some positive real number. Then, for every positive real number c, we have $\psi(t) \in (1 + c)tU$, for all $t \in [-e, e]$.*

Proof. By symmetry, it suffices to show that the final assertion holds for all $t \in [0, e]$. Let S be the subset of this interval consisting of all $x \in [0, e]$ such that $\psi(t) \in (1 + c)tU$ whenever $0 \leqq t \leqq x$. Let s be the least upper bound of S. We have $\psi(s + h) = \psi(s) + h\psi'(s) + h\varphi(h)$, where $\varphi(h)$ approaches the 0-element of U as h approaches 0. Writing this with $h = -z$ and with $h = y$, where z and y are nonnegative real numbers, and subtracting, we obtain $\psi(s + y) = \psi(s - z) + (y + z)\psi'(s) + y\varphi(y) + z\varphi(-z)$. There is a positive real number c_1 such that $\varphi(h) \in cU$ whenever $|h| \leqq c_1$. By the definition of s, there is a real number z such that $0 \leqq z \leqq \min(c_1, s)$ and $\psi(s - z) \in (1 + c)(s - z)U$. Then we have, for all y such that $0 \leqq y \leqq c_1$, $\psi(s + y) \in (1 + c)(s - z)U + (y + z)U + ycU + zcU$. Since U is convex, this means that $\psi(s + y) \in (1 + c)(s + y)U$. We may evidently conclude from this that $s = e$ and $S = [0, e]$, Q.E.D.

PROPOSITION 4.1. *Let G be a real analytic group, K an arbitrary locally compact group. Let A be a K-integrable locally convex real topological vector space. Then $F_d(G, A)$ is locally convex, and differentiable and K-integrable as a G-module.*

Proof. $F_d(G, A)$ is evidently locally convex. We have already shown that it is a differentiable G-module. What remains to be shown is that $F_d(G, A)$ is a K-integrable G-module.

Let $\varphi \in F^*(K, F_d(G, A))$. For each $x \in G$, let φ_x denote the map of K into

A that is given by $\varphi_x(y) = \varphi(y)(x)$. Then $\varphi_x \in F^*(K, A)$, and $J_K(\varphi_x)$ is defined as an element of A. We know from the proof of Lemma 3.1 that if $J_K(\varphi)(x) = J_K(\varphi_x)$, then $J_K(\varphi) \in F(G, A)$. We must show that, actually, $J_K(\varphi) \in F_d(G, A)$. We have

$$\varphi_{x \exp(t\zeta)}(y) = \varphi(y)(x \exp(t\zeta)) = \varphi(y)(x) + t\varphi(y)'(x, \zeta, t).$$

Thus $\varphi_{x \exp(t\zeta)} = \varphi_x + t\varphi'_{x,\zeta,t}$, where $\varphi'_{x,\zeta,t}(y) = \varphi(y)'(x, \zeta, t)$. For $t \neq 0$, it is clear from this that $\varphi'_{x,\zeta,t} \in F^*(K, A)$. On the other hand, $\varphi'_{x,\zeta,0}(y) = \varphi(y)'(x, \zeta, 0) = \zeta(\varphi(y))(x)$, whence it is clear that $\varphi'_{x,\zeta,0} \in F^*(K, A)$. Hence, for every $t \in R$, $J_K(\varphi'_{x,\zeta,t})$ is defined as an element of A, and we have

$$J_K(\varphi)(x \exp(t\zeta)) = J_K(\varphi)(x) + tJ_K(\varphi'_{x,\zeta,t}).$$

We must show, therefore, that the map $(x, \zeta, t) \to J_K(\varphi'_{x,\zeta,t})$ is continuous from $G \times \mathfrak{G} \times R$ to A. Since J_K is continuous from $F^*(K, A)$ to A, this will follow as soon as we have shown that the map $(x, \zeta, t) \to \varphi'_{x,\zeta,t}$ is continuous from $G \times \mathfrak{G} \times R$ to $F^*(K, A)$. In order to do this, it is evidently sufficient to show that the map $(y, x, \zeta, t) \to \varphi(y)'(x, \zeta, t)$ is continuous from $K \times G \times \mathfrak{G} \times R$ to A. Now the continuity of this map is clear except at $t = 0$.

Hence it suffices to show that, given y_0, x_0, ζ_0 and a neighborhood U of 0 in A, there is a neighborhood V of $(y_0, x_0, \zeta_0, 0)$ in $K \times G \times \mathfrak{G} \times R$ such that $\varphi(y)'(x, \zeta, t) - \varphi(y_0)'(x_0, \zeta_0, 0) \in U$, for all $(y, x, \zeta, t) \in V$, i.e., such that $\varphi(y)'(x, \zeta, t) - \zeta_0(\varphi(y_0))(x_0) \in U$, for all $(y, x, \zeta, t) \in V$. Now define a map ψ of R into A by

$$\begin{aligned} \psi(t) &= t[\varphi(y)'(x, \zeta, t) - \zeta(\varphi(y))(x)] \\ &= \varphi(y)(x \exp(t\zeta)) - \varphi(y)(x) - t\zeta(\varphi(y))(x). \end{aligned}$$

Evidently, ψ has a derivative ψ' at every $t \in R$, and

$$\psi'(t) = \zeta(\varphi(y))(x \exp(t\zeta)) - \zeta(\varphi(y))(x).$$

Since A is locally convex, we can find a convex neighborhood U_1 of 0 in A such that $3U_1 \subset U$. We can find a neighborhood W of (y_0, x_0, ζ_0) in $K \times G \times \mathfrak{G}$ and a positive real number e such that

$$\zeta(\varphi(y))(x) - \zeta_0(\varphi(y_0))(x_0) \in U_1 \quad \text{and}$$

$$\zeta(\varphi(y))(x \exp(t\zeta)) - \zeta(\varphi(y))(x) \in U_1,$$

whenever $(y, x, \zeta) \in W$ and $|t| \leqq e$. In particular, we have then $\psi'(t) \in U_1$. By Lemma 4.1, this implies that $\psi(t) \in 2tU_1$, whenever $(y, x, \zeta) \in W$ and $|t| \leqq e$. For these points (y, x, ζ, t), we have therefore

$$\zeta(\varphi(y))(x) - \zeta_0(\varphi(y_0))(x_0) \in U_1 \quad \text{and} \quad \varphi(y)'(x, \zeta, t) - \zeta(\varphi(y))(x) \in 2U_1,$$

whence

$$\varphi(y)'(x, \zeta, t) - \zeta_0(\varphi(y_0))(x_0) \in 3U_1 \subset U.$$

Hence, if $J_K(\varphi)'$ is defined by $J_K(\varphi)'(x, \zeta, t) = J_K(\varphi'_{x,\zeta,t})$, we have shown that $J_K(\varphi)'$ is continuous from $G \times \mathfrak{G} \times R$ to A, and

$$J_K(\varphi)(x \exp(t\zeta)) = J_K(\varphi)(x) + tJ_K(\varphi)'(x, \zeta, t).$$

This gives $\zeta(J_K(\varphi))(x) = J_K(\varphi)'(x, \zeta, 0) = J_K(\varphi'_{x,\zeta,0}) = J_K((\zeta \circ \varphi)_x)$, so that $\zeta(J_K(\varphi)) = J_K(\zeta \circ \varphi)$. Since $\zeta \circ \varphi$ is still in $F^*(K, F_d(G, A))$, we may repeat the above arguments and conclude that $J_K(\varphi) \in F_d(G, A)$. Moreover, it is clear that, for every element δ of the universal enveloping algebra of $\mathfrak{G}$, we have $\delta(J_K(\varphi)) = J_K(\delta \circ \varphi)$.

Taking into account what is known from the proof of Lemma 3.1 concerning the composition of J_K with continuous linear functionals, we see that there remains only to show that the map J_K is continuous from $F^*(K, F_d(G, A))$ to $F_d(G, A)$. For this, it suffices to show that, for every element δ of the universal enveloping algebra of $\mathfrak{G}$, the map $\varphi \to \delta(J_K(\varphi)) = J_K(\delta \circ \varphi)$ is continuous from $F^*(K, F_d(G, A))$ to $F(G, A)$. Since the map $\varphi \to \delta \circ \varphi$ is continuous from $F^*(K, F_d(G, A))$ to itself, and since J_K is continuous from $F^*(K, F_d(G, A))$ to $F(G, A)$ (as we know from the proof of Lemma 3.1), this is indeed the case, so that Proposition 4.1 is proved.

5. Differentiable cohomology

The differentiable cohomology for a real analytic group G is defined in exactly the same way as the continuous cohomology, the sole change being that only differentiable G-modules are admitted. In particular, we say that a differentiable G-module A is *differentiably injective* if, for every strongly exact sequence $0 \to U \xrightarrow{\rho} V \to W \to 0$ of continuous G-module homomorphisms between differentiable G-modules U, V, W, and every continuous G-module homomorphism $\alpha : U \to A$, there is a continuous G-module homomorphism $\beta : V \to A$ such that $\beta \circ \rho = \alpha$.

LEMMA 5.1. *Let A be a real topological vector space whose points are separated by the continuous linear functionals. Then the differentiable G-module $F_d(G, A)$ is differentiably injective.*

Proof. Using the notation of the above definition, let σ be a continuous linear map of V into U such that $\sigma \circ \rho$ is the identity map on U. For $v \in V$, define the map $\beta(v)$ of G into A by $\beta(v)(x) = \alpha(\sigma(x \cdot v))(1)$. Since V is a differentiable G-module and since σ and α are continuous, it is clear that $\beta(v) \in F_d(G, A)$. One verifies immediately that β is a G-module homomorphism and that $\beta \circ \rho = \alpha$. There remains only to show that β is a continuous map of V into $F_d(G, A)$, or that, for every differential operator δ, the map $v \to \delta(\beta(v))$ is continuous from V to $F(G, A)$. As usual, let us define $f_v : G \to V$ by $f_v(x) = x \cdot v$. Define $\alpha_1 : U \to A$ by $\alpha_1(u) = \alpha(u)(1)$. Then we have $\beta(v) = \alpha_1 \circ \sigma \circ f_v$, whence $\delta(\beta(v)) = \alpha_1 \circ \sigma \circ \delta(f_v)$. Since the map $v \to \delta(f_v)$ is continuous from V to $F(G, V)$, it follows that the map $v \to \delta(\beta(v))$ is continuous from V to $F(G, A)$, Q.E.D.

Lemma 5.1 ensures the existence of a differentiably injective resolution for every differentiable G-module A, so that the spaces $H_d^n(G, A)$ of the differentiable cohomology for G in A can indeed be defined. The construction of the homogeneous continuously injective resolution that we made at the end of Section 2 can be carried out with $F_d(G, A)$ in the place of $F(G, A)$ and, as in the continuous case, this leads to a description of $H_d(G, A)$ as the cohomology space of the *complex of nonhomogeneous differentiable cochains*. The differentiable n-cochains for G in A are the differentiable maps of the direct product of n copies of G into A, and the coboundary map is given by the formula at the end of Section 2.

Clearly, if A is any differentiable G-module, a differentiably injective resolution of A can be mapped into any continuously injective resolution of A. These maps are unique up to continuous homotopies and induce a natural homomorphism $H_d(G, A) \rightarrow H_c(G, A)$, or, more precisely, a morphism of the functor $H_d(G, *)$ to the functor obtained by restricting the functor $H_c(G, *)$ to the category of differentiable G-modules. Moreover, this morphism evidently commutes with the connecting homomorphisms.

LEMMA 5.2. *If A is a locally convex G-integrable topological vector space, then $F_d(G, A)$ is continuously injective.*

Proof. Choose an element $w \in D$ with compact support and such that $I_G(w) = 1$. Let $\varphi \in F(G, F_d(G, A))$, and define $\varphi_1 \in F(G, F_d(G, A))$ by $\varphi_1(x)(y) = w(xy^{-1})\varphi(x)(yx^{-1})$. Let S be the support of w, and let $F^S(G, F_d(G, A))$ denote the subspace of $F(G, F_d(G, A))$ consisting of all functions ψ such that the support of ψ_y is contained in Sy, where $\psi_y(x) = \psi(x)(y)$. In particular, $\psi_y \in F^*(G, A)$, so that we may define $J_G(\psi)$ as a map from G to A by $J_G(\psi)(y) = J_G(\psi_y)$. Evidently, the proof of Proposition 4.1 applies to the elements of $F^S(G, F_d(G, A))$ and shows that the map $\psi \rightarrow J_G(\psi)$ is a continuous map of $F^S(G, F_d(G, A))$ into $F_d(G, A)$. We have $\varphi_1 \in F^S(G, F_d(G, A))$, and we write $\varphi^* = J_G(\varphi_1) \in F_d(G, A)$.

Clearly, the map $\varphi \rightarrow \varphi_1$ is continuous, so that the map $\varphi \rightarrow \varphi^*$ is also continuous. We have $(z\cdot\varphi)_1(x)(y) = \varphi_1(xz)(yz)$, so that $(z\cdot\varphi)_1(x) = z\cdot[(z\cdot\varphi_1)(x)]$. Thus $(z\cdot\varphi)_1 = z(z\cdot\varphi_1)$, where, generally $z(\psi)$ is defined by $z(\psi)(x) = z\cdot\psi(x)$. This gives

$$(z\cdot\varphi)^* = J_G((z\cdot\varphi)_1) = J_G(z(z\cdot\varphi_1)) = z\cdot J_G(z\cdot\varphi_1) = z\cdot J_G(\varphi_1) = z\cdot\varphi^*.$$

Thus the map $\varphi \rightarrow \varphi^*$ is a continuous G-module homomorphism of $F(G, F_d(G, A))$ into $F_d(G, A)$.

Now suppose that $\varphi = f_u$, where $u \in F_d(G, A)$. Then we have $\varphi_1(x)(y) = w(xy^{-1})u(y)$, which gives $\varphi^*(y) = I_G(y^{-1}w)u(y) = u(y)$. Thus $(f_u)^* = u$. Hence we conclude that the map $u \rightarrow f_u$ is a topological G-module isomorphism of $F_d(G, A)$ onto a direct topological G-module summand of $F(G, F_d(G, A))$, whence $F_d(G, A)$ is continuously injective.

THEOREM 5.1. *If A is a locally convex, differentiable and G-integrable G-module, then the canonical homomorphism $H_d(G, A) \to H_c(G, A)$ is an isomorphism.*

Proof. By Proposition 4.1, $F_d(G, A)$ is a locally convex, differentiable, and G-integrable G-module. By Lemma 5.2, it is continuously injective. Hence we see inductively that the differentiably injective resolution of A by the modules $F_d^i(G, A)$ is actually also a continuously injective resolution and therefore effects an identification of $H_d(G, A)$ with $H_c(G, A)$.

6. Differential forms

Let G be a real analytic group, D the algebra of all differentiable functions on G, T the D-module of all real linear derivations of D. Let V be a differentiable G-module. For every positive integer q, let $A^q(T, V)$ denote the D-module of all (D, q)-linear alternating maps with arguments in T and values in $F_d(G, V)$. We agree that $A^0(T, V) = F_d(G, V)$ and $A^q(T, V) = (0)$ if q is a negative integer. We let $A(T, V)$ denote the weak direct sum of the D-modules $A^q(T, V)$. An element of $A^q(T, V)$ is called a V-valued differentiable differential form of degree q.

To every $\tau \in T$, there is attached a D-linear homogeneous endomorphism c_τ of degree -1, the contraction with respect to τ, which is given by

$$c_\tau(\alpha)(\tau_1, \cdots, \tau_q) = \alpha(\tau, \tau_1, \cdots, \tau_q).$$

The natural action of T on $F_d(G, V)$ (in which the G-module structure of V does not enter) is extended to an action of T on $A(T, V)$ by homogeneous real linear endomorphisms t_τ of degree 0, where

$$t_\tau(\alpha)(\tau_1, \cdots, \tau_q) = \tau(\alpha(\tau_1, \cdots, \tau_q)) + \textstyle\sum_{i=1}^q \alpha(\tau_1, \cdots, [\tau_i, \tau], \cdots, \tau_q)$$

One has the identities

$$c_\tau^2 = 0, \qquad [t_{\tau_1}, c_{\tau_2}] = c_{[\tau_1, \tau_2]}, \qquad [t_{\tau_1}, t_{\tau_2}] = t_{[\tau_1, \tau_2]}.$$

The differential operator, δ, on $A(T, V)$ is a homogeneous real linear endomorphism of degree 1, which, as such, is characterized by the identity $\delta c_\tau + c_\tau \delta = t_\tau$. One has $\delta^2 = 0$, and δ commutes with each t_τ. The explicit formula for δ is

$$\delta(\alpha)(\tau_0, \cdots, \tau_q) = \textstyle\sum_{i=0}^q (-1)^i \tau_i(\alpha(\tau_0, \cdots, \hat{\tau}_i, \cdots, \tau_q))$$
$$+ \textstyle\sum_{r<s} (-1)^{r+s} \alpha([\tau_r, \tau_s], \tau_0, \cdots, \hat{\tau}_r, \cdots, \hat{\tau}_s, \cdots, \tau_q).$$

Now let K be a closed subgroup of G, and consider the homogeneous space G/K of the cosets xK. This is a differentiable manifold, with D^K as the algebra of all differentiable functions, where D^K is the subalgebra of D consisting of all $f \in D$ such that $x \cdot f = f$, for every $x \in K$. The complex of the V-valued differential forms on G/K is defined exactly as above, with D^K

in the place of D and $F_d(G/K, V) = F_d(G, V)^K$ in the place of $F_d(G, V)$; again, $F_d(G, V)^K$ is defined as the D^K-submodule of $F_d(G, V)$ consisting of all $f \in F_d(G, V)$ such that $x \cdot f = f$, for all $x \in K$. We denote the D^K-module of all real linear derivations of D^K by T_K, and the complex of the differential forms on G/K by $A(T_K, V)$.

We regard T and T_K as right G-modules, with

$$(\tau \cdot x)(f) = \tau(f \cdot x^{-1}) \cdot x,$$

where $x \in G, f \in D$ or D^K, and $\tau \in T$ or T_K.

In addition to the G-module structure of $F_d(G, V)$ by left translations, we define a right G-module structure on $F_d(G, V)$ by

$$(f \cdot x)(y) = x^{-1} \cdot f(xy).$$

Now we make $A(T, V)$ and $A(T_K, V)$ into G-modules by

$$(x \cdot \alpha)(\tau_1, \cdots, \tau_q) = \alpha(\tau_1 \cdot x, \cdots, \tau_q \cdot x) \cdot x^{-1}.$$

It is verified directly that δ is a G-module endomorphism of $A(T, V)$ and $A(T_K, V)$.

The Lie algebra $\mathfrak{G}$ of G is identified with the real linear subspace of T consisting of all $\tau \in T$ such that $\tau \cdot x = \tau$, for every $x \in G$. In addition to the right G-module structure used here, T has a left G-module structure given by

$$(x \cdot \tau)(f) = x \cdot \tau(x^{-1} \cdot f).$$

We denote by T^G the subspace of T consisting of all $\tau \in T$ such that $x \cdot \tau = \tau$, for all $x \in G$. The natural maps of $D \otimes \mathfrak{G}$ into T and of $D \otimes T^G$ into T are both isomorphisms.

We define a real linear projection $T \to T^G$, denoted $\tau \to \tau^*$, by setting

$$\tau^*(f)(x) = \tau(x \cdot f)(1),$$

where $f \in D$ and $x \in G$. Observe that we have $(f\tau)^* = f(1)\tau^*$, and hence $(x \cdot (f\tau))^* = f(x)(x \cdot \tau)^*$. Evidently, the elements of T^G map D^K into itself, so that we have a restriction map $\sigma \to \sigma_K$ of T^G into T_K.

Now let $\alpha \in A^q(T_K, V)$. We define a function $\rho(\alpha)$ on q-tuples of elements of T and with values that are maps of G into V by

$$\rho(\alpha)(\tau_1, \cdots, \tau_q)(x) = \alpha((x \cdot \tau_1)^*_K, \cdots, (x \cdot \tau_q)^*_K)(x).$$

We see immediately that $\rho(\alpha)$ is (D, q)-linear and alternating. Moreover, if the τ_i's are in T^G, we have $\rho(\alpha)(\tau_1, \cdots, \tau_q) = \alpha((\tau_1)_K, \cdots, (\tau_q)_K)$, which lies in $F_d(G, V)^K$. Since the elements of T^G span T over D, it follows that the values of $\rho(\alpha)$ lie in $F_d(G, V)$, so that we have $\rho(\alpha) \in A^q(T, V)$.

Next we show that ρ commutes with δ. Since the elements ζ^* with $\zeta \in \mathfrak{G}$ span T over D, it suffices to show that

$$\delta(\rho(\alpha))(\zeta_0^*, \cdots, \zeta_q^*) = \rho(\delta(\alpha))(\zeta_0^*, \cdots, \zeta_q^*),$$

whenever each ζ_i lies in $\mathfrak{G}$. Since

$$(x\cdot\zeta_i^*)^* = (\zeta_i^*)^* = \zeta_i^* \quad \text{and} \quad [\zeta_r^*, \zeta_s^*] = [\zeta_s, \zeta_r]^*,$$

the desired equality becomes immediately apparent from the explicit formula for δ.

Furthermore, ρ is a G-module homomorphism. Indeed, we have

$$\begin{aligned}(x\cdot\rho(\alpha))(\tau_1, \cdots, \tau_q)(y) &= x\cdot\rho(\alpha)(\tau_1\cdot x, \cdots, \tau_q\cdot x)(x^{-1}y)\\ &= x\cdot\alpha((x^{-1}y\cdot\tau_1\cdot x)^*_K, \cdots, (x^{-1}y\cdot\tau_q\cdot x)^*_K)(x^{-1}y).\end{aligned}$$

Now one verifies in a straightforward fashion that $(x^{-1}\cdot\tau\cdot x)^* = \tau^*\cdot x$. Hence we obtain

$$\begin{aligned}(x\cdot\rho(\alpha))(\tau_1, \cdots, \tau_q)(y) &= x\cdot\alpha(((y\cdot\tau_1)^*\cdot x)_K, \cdots, ((y\cdot\tau_q)^*\cdot x)_K)(x^{-1}y)\\ &= (x\cdot\alpha)((y\cdot\tau_1)^*_K, \cdots, (y\cdot\tau_q)^*_K)(y)\\ &= \rho(x\cdot\alpha)(\tau_1, \cdots, \tau_q)(y).\end{aligned}$$

Thus $x\cdot\rho(\alpha) = \rho(x\cdot\alpha)$, and we conclude that ρ is a homomorphism of the G-module complex $A(T_K, V)$ into the G-module complex $A(T, V)$.

Now we define a right G-module structure on $A(T, V)$ by

$$(\alpha\cdot x)(\tau_1, \cdots, \tau_q) = x^{-1}\cdot\alpha(x\cdot\tau_1, \cdots, x\cdot\tau_q).$$

The explicit formula for δ shows immediately that δ is also a right G-module endomorphism of $A(T, V)$. Let $A_K(T, V)$ be the set of all $\alpha \in A(T, V)$ such that $\alpha\cdot x = \alpha$, for all $x \in K$ and $c_\zeta(\alpha) = 0$, for all $\zeta \in \mathfrak{K}$, the Lie algebra of K. We claim that $A_K(T, V)$ is a subcomplex of $A(T, V)$, i.e., that it is stable under δ. Clearly, if $\alpha \in A_K(T, V)$, we have $\delta(\alpha)\cdot x = \delta(\alpha\cdot x) = \delta(\alpha)$, for all $x \in K$. Moreover, if $\zeta \in \mathfrak{K}$, $c_\zeta(\delta(\alpha)) = t_\zeta(\alpha)$. If we identify T with $D \otimes \mathfrak{G}$, we see immediately that $A(T, V)$ may be identified with $E(\mathfrak{G}') \otimes F_d(G, V)$, where $E(\mathfrak{G}')$ is the exterior R-algebra constructed over the dual $\mathfrak{G}' = \mathrm{Hom}_R(\mathfrak{G}, R)$ of $\mathfrak{G}$. The representation of G on $A(T, V)$ given by $\alpha \to \alpha\cdot x^{-1}$ thereby becomes the tensor product of the representation of G on $E(\mathfrak{G}')$ that is induced by the (dual of the) adjoint representation of G on $\mathfrak{G}$ and the representation of G on $F_d(G, V)$ by left translations. Hence it is clear that $A(T, V)$ is a differentiable G-module with the G-action $\alpha \to \alpha\cdot x^{-1}$. Moreover, it is easily verified that the differential of this representation of G on $A(T, V)$ is the representation $\zeta \to t_\zeta$ of $\mathfrak{G}$ on $A(T, V)$. It follows that t_ζ annihilates $A_K(T, V)$ whenever $\zeta \in \mathfrak{K}$, so that the above shows that $c_\zeta(\delta(\alpha)) = 0$ whenever $\alpha \in A_K(T, V)$ and $\zeta \in \mathfrak{K}$. Thus $A_K(T, V)$ is indeed a subcomplex of $A(T, V)$. Moreover, it is seen immediately from the definitions that $A_K(T, V)$ is also a left G-submodule of $A(T, V)$. We shall show that ρ maps $A(T_K, V)$ isomorphically onto $A_K(T, V)$.

It follows at once from the definitions that $\rho(\alpha)\cdot x = \rho(\alpha)$, for every

$\alpha \in A(T_K, V)$ and every $x \in K$. Now let $\zeta \in \mathfrak{K}$. We have

$$c_\zeta(\rho(\alpha))(\tau_1, \cdots, \tau_{q-1}) = \rho(\alpha)(\zeta, \tau_1, \cdots, \tau_{q-1}), \quad \text{and}$$

$$\rho(\alpha)(\zeta, \tau_1, \cdots, \tau_{q-1})(x) = \alpha((x\cdot\zeta)^*_K, (x\cdot\tau_1)^*_K, \cdots, (x\cdot\tau_{q-1})^*_K)(x).$$

Now this is the element of V obtained by evaluating the local differential form α_{xK} at the q-tuple of elements of the tangent space at xK to G/K defined by the infinitesimal transformations $(x\cdot\zeta)^*_K$, $(x\cdot\tau_i)^*_K$. But if $f \in F_d(G, V)^K$, we have $(x\cdot\zeta)^*_K(f)(xK) = (x\cdot\zeta)(x\cdot f)(1) = \zeta(f)(x)$, which is equal to 0 because ζ annihilates $F_d(G, V)^K$. Hence we have $c_\zeta(\rho(\alpha)) = 0$, and we conclude that $\rho(A(T_K, V)) \subset A_K(T, V)$.

Now we recall that $\rho(\alpha)(\tau_1, \cdots, \tau_q) = \alpha((\tau_1)_K, \cdots, (\tau_q)_K)$ whenever the τ_i's lie in T^G. Since the differential of the canonical map $G \to G/K$ sends the tangent space to G at x *onto* the tangent space to G/K at xK, the elements $(\tau_K)_{xK}$, with $\tau \in T^G$, make up the whole tangent space to G/K at xK. Hence we see, on localizing as above, that ρ is a monomorphism.

Let $\mathfrak{L}$ be a linear complement of $\mathfrak{K}$ in $\mathfrak{G}$. The tangent space to G at a point x is the space of the tangents τ_x as τ ranges over $\mathfrak{G}$, where $\tau_x(f) = \tau(f)(x)$. The differential at x of the canonical map $G \to G/K$ is an epimorphism of $\mathfrak{G}_x$ onto the tangent space to G/K at xK, and its kernel is precisely $\mathfrak{K}_x$. Hence it maps $\mathfrak{L}_x$ isomorphically onto the tangent space to G/K at xK. Now let σ be any element of T_K. We define a tangent vector field τ on G by making τ_x the unique element of $\mathfrak{L}_x$ whose image under the differential of the canonical map $G \to G/K$ is the tangent σ_{xK} at xK that is defined by σ. It is easily seen that τ is a differentiable vector field, so that it may be identified with an element $\tau \in T$ such that $\tau(f)(x) = \tau_x(f)$, for every $x \in G$. Thus we conclude that every element of T_K is the restriction to D^K of an element of T.

Now let $\gamma \in A^q_K(T, V)$. If we identify T with $D \otimes \mathfrak{G}$ we see that the kernel of the restriction map of T into the D-module of all real linear derivations of D^K into D becomes identified with $D \otimes \mathfrak{K}$. Hence we see that $\gamma(\tau_1, \cdots, \tau_q)$ depends only on the restrictions of the τ_i's to D^K. Next we note that an element τ of T restricts to an element of T_K if and only if $x\cdot\tau = \tau$ on D^K, for every $x \in K$. Since $\gamma\cdot x = \gamma$ whenever $x \in K$, it follows that $\gamma(\tau_1, \cdots, \tau_q) \in F_d(G, V)^K$ whenever the τ_i's send D^K into itself. Hence the restriction of γ to such q-tuples defines an element $\alpha \in A^q(T_K, V)$. Then we have $\rho(\alpha)(\tau_1, \cdots, \tau_q) = \gamma(\tau_1, \cdots, \tau_q)$ whenever the elements τ_i belong to T^G and hence whenever they belong to $DT^G = T$. Thus $\rho(\alpha) = \gamma$, and we have shown that ρ maps $A(T_K, V)$ isomorphically onto $A_K(T, V)$.

As before, let us write $A(T, V) = E(\mathfrak{G}') \otimes F_d(G, V)$. Then the left G-module structure of $A(T, V)$ takes the form $x\cdot(e \otimes f) = e \otimes (f\cdot x^{-1})$, where $(f\cdot x^{-1})(y) = x\cdot f(x^{-1}y)$. We topologize $A(T, V)$ by giving it the coarsest topology making all the maps $\alpha \to \alpha(\tau_1, \cdots, \tau_q)$ continuous from $A^q(T, V)$ to $F_d(G, V)$. Clearly, every R-basis of $E(\mathfrak{G}')$ then yields a homeomorphism of $A(T, V)$ onto a direct sum of copies of the topological vector space $F_d(G, V)$. Since V is a differentiable G-module, the map $f \to f'$, where $f'(x) = x \cdot f(x^{-1})$, is a continuous involution of $F_d(G, V)$ transporting the G-module structure $f \to f \cdot x^{-1}$ into the G-module structure of $F_d(G, V)$ based on the left translations. Hence it is clear that $F_d(G, V)$ is a differentiable G-module for the structure $f \to f \cdot x^{-1}$, whence $A(T, V)$ is a differentiable G-module for the structure $\alpha \to x \cdot \alpha$.

On the other hand, we regard $A(T, V)$ as a K-module with the action $\alpha \to \alpha \cdot x^{-1}$. We have already remarked above that this is the structure of a differentiable K-module. Clearly, the subspace $E((\mathfrak{G}/\mathfrak{K})') \otimes F_d(G, V)$ is both a G-submodule and a K-submodule of $A(T, V)$. Combining these structures, we evidently obtain the structure of a continuous (even differentiable) $G \times K$-module.

Now suppose that V is locally convex and G- and K-integrable as a G-module. Then it follows from Proposition 4.1 that $F_d(G, V)$ is G- and K-integrable as a G-module, and also as a $G \times K$-module. By Lemma 5.2, $F_d(G, V)$ is continuously injective as a G-module.

Quite generally, if P and Q are locally compact groups, M a Q-integrable continuously injective P-module, and U a finite-dimensional continuous P-module, then $U \otimes M$ is a Q-integrable continuously injective P-module; the Q-integrability is seen quite directly by using the linear functionals on U; the injectivity follows from the fact that $U \otimes F^0(P, M)$ may be identified with $F^0(P, U \otimes M)$.

In particular, we conclude that $E((\mathfrak{G}/\mathfrak{K})') \otimes F_d(G, V)$ is a K-integrable $G \times K$-module and continuously injective as a G-module. If K is compact, it follows therefore from Lemma 3.2 that $E((\mathfrak{G}/\mathfrak{K})') \otimes F_d(G, V)$ is continuously injective as a $G \times K$-module. It follows trivially from this that its K-fixed part, i.e., $A_K(T, V)$, is continuously injective as a G-module.

In order to proceed, we must impose an additional integrability condition on V. We assume that there is a separating family V' of continuous linear functionals on V, stable under composition with the G-operators on V, and a continuous linear map $J_1 : F(R, V) \to V$, where R stands for the space of the real numbers, such that, for every $\gamma \in V'$ and every $f \in F(R, V)$, $\gamma(J_1(f))$ is the ordinary integral from 0 to 1 of the continuous real-valued function $\gamma \circ f$. Let us refer to this property of V by saying that V *is* [0, 1]*-integrable*. The proof of Proposition 4.1 carries over without change to show that *if V is locally convex and* [0, 1]*-integrable, then $F_d(G, V)$ is* [0, 1]*-integrable.*

Now let us suppose that K is a maximal compact subgroup of G and that V is a differentiable G-module that, as such, is G-, K- and [0, 1]-integrable. Since $F_d(G, V)$ is [0, 1]-integrable, so is evidently $F_d(G, V)^K$, and we know from the proof of Proposition 4.1 that $\tau(J_1(f)) = J_1(\tau \circ f)$, for every $\tau \in T_K$ and every $f \in F(R, F_d(G, V)^K)$. Since the differentiable manifold G/K is now diffeomorphic with Euclidean space, and since $F_d(G, V)^K$ is [0, 1]-integrable, one sees that the usual constructive proof of the Poincaré Lemma can be applied to produce a continuous real linear contracting homotopy of the V-augmented complex $A(T_K, V)$. Identifying $A(T_K, V)$ with $A_K(T, V)$ by means of the isomorphism ρ, which is easily seen to be a topological isomorphism, we conclude that the sequence of continuous G-module homomorphisms $(0) \to V \to A_K^0(T, V) \to A_K^1(T, V) \to \cdots$ is strongly exact, and hence is a continuously injective resolution of V.

Now $(A_K^q(T, V))^G$ is the K-fixed part of $E((\mathfrak{G}/\mathfrak{K})') \otimes V$, for the representation of K obtained as the tensor product of the given representation of K on V and the representation of K on $E((\mathfrak{G}/\mathfrak{K})')$ that is induced from the adjoint representation of K on $\mathfrak{G}$. Since K is connected, this means that $(A_K^q(T, V))^G$ is the space $C^q(\mathfrak{G}, \mathfrak{K}, V)$ of the relative Lie algebra cochains for $(\mathfrak{G}, \mathfrak{K})$ in V. Moreover, the map $C^q(\mathfrak{G}, \mathfrak{K}, V) \to C^{q+1}(\mathfrak{G}, \mathfrak{K}, V)$ that is induced by δ is the usual coboundary map of the relative Lie algebra cochain complex. In this way, $H_c(G, V)$ is identified with the relative Lie algebra cohomology space $H(\mathfrak{G}, \mathfrak{K}, V)$, the topology being ignored. Note, however, that if V is finite-dimensional, then the spaces $C^q(\mathfrak{G}, \mathfrak{K}, V)$ are finite-dimensional, whence we see that the induced topology of $H_c(G, V)$ is then Hausdorff, so that, with its induced topology, $H_c(G, V)$ becomes identified with the finite-dimensional Euclidean space $H(\mathfrak{G}, \mathfrak{K}, V)$. Thus we have the following result (cf. [11, Theorem 3.6.1]).

THEOREM 6.1. *Let G be a real analytic group, and let K be a maximal compact subgroup of G. Let V be a locally convex, G-, K-, and* [0, 1]*-integrable, differentiable G-module. Then the resolution of V by the V-valued differential forms on G/K yields an isomorphism of $H_c(G, V)$ onto the relative Lie algebra cohomology space $H(\mathfrak{G}, \mathfrak{K}, V)$. If V is finite-dimensional, this isomorphism is a topological isomorphism.*

7. Groups of finite homology type

We say that a locally compact group G is of *finite homology type* if, for every finite-dimensional continuous G-module V, the cohomology space $H_c^q(G, V)$, with its naturally induced topology, is a finite-dimensional Hausdorff space, for every q. By Theorem 6.1, a real analytic group is always of finite homology type.

THEOREM 7.1. *Let G be a locally compact group, K a closed normal subgroup of G, V a finite-dimensional continuous G-module. Assume that there is a continuously injective resolution of V as a G-module that is also a continuously injective resolution of V as a K-module. Suppose that $H_c^q(K, V)$ is a finite-dimensional Hausdorff space, for each q, and that $H_c^p(G/K, H_c^q(K, V))$ is a finite-dimensional Hausdorff space, for all p and q. Then $H_c^n(G, V)$ is a finite-dimensional Hausdorff space, for each n.*

Proof. Let $0 \to V \to X_0 \to X_1 \to \cdots$ be a continuously injective resolution of V as a G-module such that each X_i is continuously injective also as a K-module. Since each X_i is continuously injective as a G-module, each X_i^K is continuously injective as a G/K-module.

Let $C^{p,q} = F^{p-1}(G/K, X_q^K)$, where we agree that $F^{-1}(G/K, X_q^K) = X_q^K$. Let d denote the map $X_q^K \to X_{q+1}^K$ obtained from our resolution, and let δ be the coboundary map $F^p(G/K, X_q^K) \to F^{p+1}(G/K, X_q^K)$, as defined at the end of Section 2. Let D be the map $C^{p,q} \to C^{p+1,q} + C^{p,q+1}$ defined by $D(f) = \delta(f) + (-1)^p \, d \circ f$. Then the weak direct sum C of the $C^{p,q}$, equipped with the map D, is a bigraded complex of topological vector spaces. On the subspace X^G of C, the boundary map D coincides with the boundary map d, so that X^G is a subcomplex of C. Hence the injection $X^G \to C$ induces a continuous map of $H_c(G, V)$ into the homology group of the complex C. Since each X_q^K is continuously injective as a G/K-module, the homology group of the "partial" complex $(\sum_p C^{p,q}, \delta)$ reduces to its component of (partial) degree 0, which is X_q^G. If we filter the complex C with respect to the secondary degree q, we see that this implies that the above continuous map of $H_c(G, V)$ into the homology group of C is an isomorphism of untopologized vector spaces; the argument is carried out, for instance, in [8, Ch. I, Section 4]. Hence it suffices to show that, for each degree n, $H^n(C)$ is a finite-dimensional Hausdorff space.

In order to do this, we consider the spectral sequence associated with the filtration of the complex C with respect to the primary degree p. In the usual notation of spectral sequences, we have $E_0^{p,q} = C^{p,q}$, and the differential operator d_0 on E_0 is the map $f \to (-1)^p d \circ f$ sending $C^{p,q}$ into $C^{p,q+1}$. Let $Z(X_q^K)$ denote the kernel of d in X_q^K. Then the kernel of d_0 in $E_0^{p,q}$ is $F^{p-1}(G/K, Z(X_q^K))$, while $d_0(E_0^{p,q-1}) = F^{p-1}(G/K, d(X_{q-1}^K))$. Hence we have

$$E_1^{p,q} = F^{p-1}(G/K, Z(X_q^K))/F^{p-1}(G/K, d(X_{q-1}^K)).$$

Since $H_c^q(K, V)$ is a finite-dimensional Hausdorff space, the continuous G/K-module sequence $0 \to d(X_{q-1}^K) \to Z(X_q^K) \to H_c^q(K, V) \to 0$ is strongly exact. Hence it is clear that $E_1^{p,q}$ may be identified with $F^{p-1}(G/K, H_c^q(K, V))$. The differential operator $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ is the map

$$\delta : F^{p-1}(G/K, H_c^q(K, V)) \to F^p(G/K, H_c^q(K, V)).$$

Hence we have $E_2^{p,q} = H_c^p(G/K, H_c^q(K, V))$, which, by assumption, is a finite-dimensional Hausdorff space. It follows that each $E_\infty^{p,q}$ is a finite-dimensional Hausdorff space.

Now let $H(C)_p$ be the p^{th} filtration subspace of $H(C)$, for the filtration of $H(C)$ induced by the filtration of C. Then we have the continuous canonical isomorphism $H^{p+q}(C)_p/H^{p+q}(C)_{p+1} \to E_\infty^{p,q}$. Hence we see that, for each n, $H^n(C)_p/H^n(C)_{p+1}$ is a finite-dimensional Hausdorff space. Since $H^n(C)_0 = H^n(C)$, and $H^n(C)_p = 0$ when p exceeds n, it follows that $H^n(C)$ is a finite-dimensional Hausdorff space. This completes the proof of Theorem 7.1.

Let G be any connected locally compact group. Then, as is shown in the solution of Hilbert's fifth problem, G has a compact normal subgroup K such that G/K is a real analytic group [10, Ch. IV, Theorem 4.6]. Thus G/K has finite homology type. Let V be a finite-dimensional continuous G-module. Then V is K-integrable, and it follows from Lemmas 3.1 and 3.2 that the resolution of V by the modules $F^p(G, V)$ is also a continuously injective resolution of V as a K-module. Moreover, by Lemma 3.2, the continuous cohomology of K in any finite-dimensional continuous K-module W reduces to $H_c^0(K, W) = W^K$. Thus we may apply Theorem 7.1 to conclude that G is of finite homology type.

Now let T be a discrete free group. If V is any continuous T-module, we may compute $H_c(T, V)$ from the complex of the nonhomogeneous cochains for T in V, as described at the end of Section 2. Since T is discrete, every cochain for T in V is automatically continuous. The 2-cocycles for T in V are factor sets for group extensions of V by T. Since T is a free group, these group extensions are split, whence it follows that every 2-cocycle for T in V is a coboundary, i.e., that $H_c^2(T, V) = (0)$. The exact sequences resulting from strong imbeddings of continuous T-modules in continuously injective continuous T-modules now show that $H_c^n(T, V) = (0)$, for all $n > 1$. Let S be a free system of generators for T. Then it is easy to see from the nonhomogeneous cochain description that $H_c^1(T, V)$ is isomorphic, as a topological vector space, with the factor space of the space of all maps of S into V (topologized by the finite-open topology) modulo the subspace consisting of the maps of the form $x \longrightarrow x \cdot v - v$, where $v \in V$. Finally, $H_c^0(T, V)$ may be identified with V^T, of course. Hence it is clear that if T is finitely generated, then it is of finite homology type.

Let G be a locally compact group, K a normal closed subgroup of G such that K and G/K are of finite homology type and G/K is either compact or discrete. Then, in virtue of Lemma 3.3 or Lemma 3.4, Theorem 7.1 applies to show that G is of finite homology type. Let us say that a group G is *quasi-solvable* if there is a finite series

$$G_0 = G \supset G_1 \supset \cdots \supset G_n = (1)$$

of closed subgroups of G such that each G_i is normal in G_{i-1} and G_{i-1}/G_i is either compact or discrete, finitely generated and free. Then we have the following result.

THEOREM 7.2. *Let G be a locally compact group, G_1 the connected component of the identity in G. Then, if G/G_1 is quasi-solvable, G is of finite homology type.*

An important special case is that of a closed normal subgroup G of a Lie group M such that M/M_1 is finite. In order to deal with this case, we need not appeal to the difficult theorem that a connected locally compact group has a compact normal subgroup such that the factor group is a Lie group. We argue as follows. Since $G/(G \cap M_1)$ is finite, it will follow that G is of finite homology type as soon as we have shown that $G \cap M_1$ is of finite homology type. Thus we may assume that M is connected. Now G/G_1 is a discrete subgroup of the center of M/G_1, and hence is a finitely generated abelian group. Hence it is clear from the above that G is of finite homology type.

Note. Theorem 7.1 fills a gap in the proof of [11, Theorem 4.1]. The definition of *finite homology type* of [11] should be amended to include the requirement that $H^k(G, V)$ be Hausdorff, as well as finite-dimensional. This additional requirement is needed to make legitimate line 8 of p. 37 of [11].

8. The restriction homomorphism

We shall need the following strengthening of the integrability condition. Let G be a locally compact topological group, and let A be a topological vector space over R. We say that A is *G-summable* if it is G-integrable and, given any neighborhood U of 0 in A, there is a neighborhood V of 0 in A such that, for every nonnegative real-valued continuous function g on G with compact support S_g and $I_G(g) = 1$, we have $J_G(gN(S_g, V)) \subset U$.

Let us verify that R is G-summable. Let U and g be as above. There is a positive real number c such that every real number of absolute value no greater than c lies in U. We claim that if V is the set of real numbers of absolute value less than c, then V satisfies the above requirement. Indeed, let $f \in N(S_g, V)$. Since G is locally compact and S_g is compact, we can extend the restriction of f to S_g to a continuous real-valued function f^* on G such that the absolute value of $f^*(x)$ is less than c, for every $x \in G$. Now we have $gf = gf^*$, whence $-cg(x) \leqq g(x)f(x) \leqq cg(x)$, for every $x \in G$. Hence $-c \leqq I_G(gf) \leqq c$, Q.E.D.

Now suppose that A is G-summable and that S is a locally compact topological space. We claim that then $F(S, A)$ is also G-summable. Let X be a neighborhood of 0 in $F(S, A)$. Then X contains a neighborhood of the form $N(T, U)$, where T is a compact subset of S and U is a neighborhood of 0 in A. Let V be a neighborhood of 0 in A that is adapted to U as described in the summability condition. Now put $Y = N(T, V)$. We claim that Y is adapted to X. Indeed, let $\varphi \in N(S_g, Y)$. Then, in the notation of the proof of Lemma 3.1, $(g\varphi)_s = g\varphi_s$. If $s \in T$, then $\varphi_s \in N(S_g, V)$, whence $J_G(g\varphi_s) \in U$. Thus we have $J_G(g\varphi)(s) \in U$ whenever $s \in T$, i.e.,

$$J_G(g\varphi) \in N(T, U) \subset X, \qquad \text{Q.E.D.}$$

If A is any G-module, we denote by A^* the maximum locally finite G-submodule of A, i.e., the sum of all finite-dimensional G-stable subspaces of A.

LEMMA 8.1. *Let A be a G-summable continuous G-module, where G is a compact topological group. Then A^* is dense in A.*

Proof. Let $a \in A$, and let U be a neighborhood of 0 in A. Let f_a be the element of $F(G, A)$ defined by $f_a(x) = x \cdot a$, for every $x \in G$. Let V be a neighborhood of 0 in A that is adapted to U as in the above definition of summability. Now choose a neighborhood X of 1 in G such that $a - x \cdot a \in V$, for all $x \in X$. Choose a real-valued nonnegative continuous function g on G such that the support S_g of g lies in X and $I_G(g) = 1$. Then we have $J_G(g(a - f_a)) \in U$, where we identify $a \in A$ with the constant map on G with value $a \in A$. On the other hand, since J_G is continuous, we see from the Peter-Weyl Theorem that there is a representative function h on G such that $J_G((g - h)f_a) \in U$. Now we have

$$a - J_G(hf_a) = J_G(g(a - f_a)) + J_G((g - h)f_a) \in U + U.$$

With $x \in G$, we have

$$x \cdot J_G(hf_a) = J_G(hx(f_a)) = J_G(h(f_a \cdot x)) = J_G((h \cdot x^{-1})f_a).$$

Since h is a representative function, this shows that $J_G(hf_a) \in A^*$, so that Lemma 8.1 is proved.

THEOREM 8.1. *Let G be a locally compact topological group, K a closed normal subgroup of G such that G/K is compact. Let V be a K-integrable and G/K-summable continuous G-module. Then the restriction homomorphism $H_c^q(G, V) \to H_c^q(K, V)^{G/K}$ is a topological isomorphism onto a dense subspace of $H_c^q(K, V)^{G/K}$, for each q.*

Proof. We consider the homogeneous resolution of V by the continuously injective G-modules $X_i = F^i(G, V)$. By what we have seen above, each X_i is K-integrable and G/K-summable with respect to a separating family X_i' of continuous linear functionals that is stable under composition with the G-operators and such that $X_{i+1}' \circ d \subset X_i'$, where d is the boundary map of the resolution. By Lemma 3.3, each X_i is continuously injective also as a K-module. Hence $H_c(K, V)$ is the homology group of the complex X^K, and the restriction homomorphism $H_c(G, V) \to H_c(K, V)$ is the homomorphism induced by the injection $X^G \to X^K$. Clearly, it is continuous, and the image of $H_c(G, V)$ lies in the subspace $H_c(K, V)^{G/K}$ of $H_c(K, V)$.

Our assumptions imply that each X_i^K is a G/K-summable continuous G/K-module. Hence we have a continuous linear projection $t_{G/K} : X_i^K \to X_i^G$, defined by $t_{G/K}(u) = J_{G/K}(f_u)$, where $f_u(z) = z \cdot u$, for every $z \in G/K$. Moreover, since $X_{i+1}' \circ d \subset X_i'$, we see that $t_{G/K}$ commutes with d. Hence $t_{G/K}$ induces a continuous homomorphism $H_c(K, V) \to H_c(G, V)$ whose composite with the restriction homomorphism is the identity map on $H_c(G, V)$. By Lemma 8.1, each $(X_i^K)^*$ is dense in X_i^K. Hence the image of the cohomology space of $(X^K)^*$ in $H_c^i(K, V)$ is a dense subspace of $H_c^i(K, V)$. Since $(X^K)^*$ is semisimple as a G/K-module, the G/K-fixed part of its cohomology space coincides with the cohomology space of X^G, i.e., with the restriction image of $H_c(G, V)$. This completes the proof of Theorem 8.1.

It is an immediate corollary that if $H_c^q(K, V)^{G/K}$ is a finite-dimensional Hausdorff space the restriction map is a topological isomorphism of $H_c^q(G, V)$ onto $H_c^q(K, V)^{G/K}$, because a finite-dimensional Hausdorff topological vector space has no proper dense vector subspaces.

If V is finite-dimensional, we can show, without making any assumptions on the cohomology groups, that the restriction map $H_c^1(G, V) \to H_c^1(K, V)^{G/K}$ is a topological isomorphism. A cocycle version of this result is due to P. Cartier and was used by him to give a new simple proof that the fundamental group of a compact semisimple analytic group is finite; cf. [13, p. 22-04, Théorème 2]. The result on the restriction map can be proved as follows.

Let $Y_i = F^i(K, V)$. Then $d(Y_0^K)$ may be identified with the space of maps $x \to x \cdot v - v$ of K into V, with v ranging over V. Thus $d(Y_0^K)$ is a finite-dimensional subspace of the kernel $Z(Y_1^K)$ of d in Y_1^K. Now let φ be the restriction map $X_1^K \to Y_1^K$, where, as before, $X_i = F^i(G, V)$. Let P be the kernel of φ. Then P is the intersection of the kernels of a subfamily of X_1', so that the family S of all elements of X_1' that vanish on P separates the elements of X_1^K/P. Let Q be the inverse image, under φ, of $d(Y_0^K)$ in X_1^K. Then Q/P is evidently finite-dimensional. Hence it follows that Q is the intersection of the kernels of elements of S. Thus the family of all elements of X_1' that vanish on Q separates the elements of X_1^K/Q. It follows that $J_{G/K}(f) \in Q$, for all $f \in F(G/K, Q)$.

Now let u be any element of $H_c^1(K, V)^{G/K}$, and let t be a representative of u in X_1^K. Then we have $z \cdot t - t \in Q$, for every $z \in G/K$. Hence we obtain $t_{G/K}(t) - t = J_{G/K}(f_t - t) \in Q$, because $f_t - t \in F(G/K, Q)$. Hence u coincides with the element of $H_c^1(K, V)^{G/K}$ that is represented by the element $t_{G/K}(t) \in X_1^G$, and thus belongs to the restriction image of $H_c^1(G, V)$, Q.E.D.

9. Representative cohomology

Let G be a locally compact topological group. By a *representative G-module* we shall mean a real vector space V on which G acts by linear automorphisms in such a way that each element of V lies in a finite-dimensional G-stable subspace of V (which we express by saying that V is locally finite) and that, for each finite-dimensional G-submodule W of V, the induced representation of G on W is continuous. We say that the representative G-module V is *representatively injective* if, for every exact sequence

$$0 \to A \xrightarrow{\rho} B \to C \to 0$$

of G-module homomorphisms between representative G-modules A, B, C, and for every G-module homomorphism α of A into V, there is a G-module homomorphism β of B into V such that $\beta \circ \rho = \alpha$. A *representatively injective resolution* of the representative G-module V is an exact sequence of G-module homomorphisms $0 \to V \to X_0 \to \cdots$ where each X_i is a representatively injective representative G-module.

Let $R(G)$ denote the algebra of all real-valued representative functions on G, i.e., let $R(G) = F(G, R)^*$. Then $R(G)$ is a representative G-module, with G operating by left translations. It is easy to verify that, for every real vector space A, the G-module $R(G) \otimes A$, with G operating by left translations on the factor $R(G)$, is a representatively injective representative G-module.

If A is a representative G-module, the map $f_a : G \to A$ may be regarded as an element of $R(G) \otimes A$, so that A is imbedded in the representatively injective G-module $R(G) \otimes A$. We note also that this G-module $R(G) \otimes A$ is isomorphic, as a G-module, with the G-module $R(G) \otimes A$ in which the action of G is given by $x \cdot (f \otimes a) = (f \cdot x^{-1}) \otimes (x \cdot a)$; cf. Section 2.

In particular, it is now clear that every representative G-module V has a representatively injective resolution X. We define the *representative cohomology group* $H_r(G, V)$ for G in V as the homology group of the complex X^G.

If V is any representative G-module, we topologize V by the coarsest topology for which all linear functionals are continuous. Then V becomes a continuous G-module, and every exact sequence of G-module homomorphisms between representative G-modules, topologized in this manner, is strongly exact. Hence, if X is a representatively injective resolution of V, and Y is a continuously injective resolution of V, there is a continuous G-module homomorphism of the G-module complex X into the G-module complex Y that extends the identity map of V onto itself. Such maps induce a unique canonical homomorphism of $H_r(G, V)$ into $H_c(G, V)$, for all representative G-modules V. Note that the induced topology on $H_r(G, V)$ is again the coarsest topology for which all linear functionals are continuous, and that the canonical map $H_r(G, V) \to H_c(G, V)$ is continuous.

Now let G be a real Lie group with only a finite number of connected components. It is immediately clear from the definition that the representative cohomology for G is the same as the representative cohomology for G/P, where P is the intersection of all kernels of finite-dimensional continuous representations of G. It is known, due to M. Goto, that G/P has a faithful continuous representation as a group of automorphisms of a finite-dimensional vector space (cf. [5, Theorem 7.1]). Hence we may assume without loss of generality that G is a linear group. In that case, there is a simply connected solvable closed normal analytic subgroup K of G such that G/K is *reductive*, in the sense that every continuous finite-dimensional representation of G/K is semisimple and that G/K has a faithful such representation. Moreover, G is a semidirect product $S \cdot K$. Such a subgroup K is called a *nucleus* of G; cf. [5, Theorem 9.1], [7, Section 2].

LEMMA 9.1. *Let G be a real linear Lie group with finite component group, and let K be a nucleus of G. Then every representatively injective representative G-module is representatively injective also as a K-module.*

Proof. Let V be a representatively injective representative G-module. Then V may be identified with a direct G-module summand of $R(G) \otimes V$, where G operates by left translations on the factor $R(G)$ only. On the other hand, if B is any representatively injective representative K-module, then, for any real vector space W, $B \otimes W$ is still representatively injective, because B may be identified with a direct K-module summand of $R(K) \otimes B$ (with K operating on the factor $R(K)$ by left translations), so that $B \otimes W$ is isomorphic with a direct K-module summand of $R(K) \otimes (B \otimes W)$. Hence it suffices to show that $R(G)$ is representatively injective as a K-module.

Since G is a semidirect product $S \cdot K$, $R(G)$ is a tensor product $R(G)^K \otimes T$, where T is a subalgebra of $R(G)$ that is stable under the left translations with the elements of K, and that is mapped isomorphically onto the restriction image $R(G)_K$ of $R(G)$ in $R(K)$, by the restriction homomorphism $R(G) \to R(K)$ [5, Proposition 2.4]. Hence it is clear from what we said above that it will suffice to prove that $R(G)_K$ is representatively injective as a K-module. It is easily seen that, in doing this, we may assume without loss of generality that G is connected.

Actually, we shall show that then $R(G)_K$ is a direct K-module summand of $R(K)$. In order to do this, it will be convenient to consider the algebra $C(G)$ of the complex-valued representative functions on G. We have

$$C(G) = R(G) \otimes C, \quad C(G)_K = R(G)_K \otimes C, \quad \text{and} \quad C(K) = R(K) \otimes C.$$

Hence it suffices to show that $C(G)_K$ is a direct K-module summand of $C(K)$.

This can be read off easily from the known results [6, Section 4] on the structure of $C(G)$ and $C(K)$, as follows. There are algebraically independent elements $u_1, \cdots, u_n$ of $C(K)$ and indices $0 \leqq p \leqq q \leqq n$ satisfying the following conditions. The elements $u_{p+1}, \cdots, u_n$ span $\mathrm{Hom}(K, C)$ over C, and $u_{q+1}, \cdots, u_n$ span $\mathrm{Hom}(G, C)_K$ over C, where C denotes the field of the complex numbers. Let Q_p be the multiplicative group consisting of the exponentials of the C-linear combinations of the u_i's with $i > p$, and define Q_q similarly. Then $C(K)$ is the group algebra of Q_p over $C[u_1, \cdots, u_n]$, while $C(G)_K$ is the group algebra of Q_q over $C[u_1, \cdots, u_n]$. Thus $C(K)$ is the direct K-module sum of $C(G)_K$ and the submodule consisting of the $C(G)_K$-linear combinations of the exponentials of the nonzero C-linear combinations of $u_{p+1}, \cdots, u_q$. This completes the proof of Lemma 9.1.

THEOREM 9.1. *Let G be a real linear Lie group with finite component group, and let K be a nucleus of G. Let V be a representative G-module. Then the restriction map is an isomorphism of $H_r(G, V)$ onto $H_r(K, V)^{G/K}$.*

Proof. By Lemma 9.1, a representatively injective resolution X of V is also a representatively injective resolution of V as a K-module. Hence $H_r(K, V)$ may be identified with the homology group of the complex X^K, and the restriction map is induced by the injection $X^G \to X^K$. Now X^K, as a G/K-module, is the sum of finite-dimensional continuous representation spaces for G/K. Since G/K is reductive, this implies that X^K is semisimple as a G/K-module. Hence the G/K-fixed part of the homology group of X^K, i.e., $H_r(K, V)^{G/K}$ is the homology group of the G/K-fixed part of X^K, i.e., of X^G, and this homology group is $H_r(G, V)$. This completes the proof.

10. Representative differential forms

Let G be a real analytic group, and let V be a representative G-module. We consider the complex $A(T, V)$ of the differentiable V-valued differential forms on G. We say that a differential form $\alpha \in A(T, V)$ is a *representative differential form* if its transforms $x \cdot \alpha$, with x ranging over G, all lie in a finite-dimensional subspace of $A(T, V)$. It is easily seen that the representative differential forms of degree 0 are precisely the canonical images in $F_d(G, V)$ of the elements of $R(G) \otimes V$. We have seen that $A(T, V)$ may be identified with $E(\mathfrak{G}') \otimes F_d(G, V)$, the G-module structure being such that $x \cdot (e \otimes f) = e \otimes (f \cdot x^{-1})$. Hence we see that the subspace of the representative differential forms is identified with $E(\mathfrak{G}') \otimes R(G) \otimes V$, the G-module structure being such that $x \cdot (e \otimes f \otimes v) = e \otimes (f \cdot x^{-1}) \otimes (x \cdot v)$. One sees immediately that this is a subcomplex of $A(T, V)$ and a G-submodule of $A(T, V)$ for both the left and the right G-module structure. Moreover, it is clear from the definition of the differential operator δ on $A(T, V)$ that the induced differential operator on $E(\mathfrak{G}') \otimes R(G) \otimes V$ acts only on the factor $E(\mathfrak{G}') \otimes R(G)$. Thus the cohomology space of the V-valued representative differential forms is the tensor product of the cohomology space of the R-valued differential forms by V.

We recall that δ is a G-module endomorphism of $A(T, V)$ for both the left and the right G-module structure. We have noted in Section 6 that the differential of the representation sending $x \in G$ onto the transformation $\alpha \rightarrow \alpha \cdot x^{-1}$ of $A(T, V)$ is the representation $\zeta \rightarrow t_\zeta$ of $\mathfrak{G}$ on $A(T, V)$. The formula $\delta c_\zeta + c_\zeta \delta = t_\zeta$ shows that the induced representation of $\mathfrak{G}$ on the cohomology space of $A(T, V)$ is trivial. Since G is connected, this implies that the representation of G on the cohomology space of $A(T, V)$ that is induced by the right G-module structure of $A(T, V)$ is trivial. We wish to obtain the same result for the left G-module structure.

For this purpose, we define an involution $\alpha \rightarrow \alpha'$ of $A(T, V)$, as follows. For $f \in F_d(G, V)$, set $f'(x) = x \cdot f(x^{-1})$. For $\tau \in T$, define $\tau' \in T$ by setting $\tau'(g) = (\tau(g'))'$. For $\alpha \in A(T, V)$, define $\alpha' \in A(T, V)$ by $\alpha'(\tau_1, \cdots, \tau_q) = \alpha(\tau_1', \cdots, \tau_q')'$. Evidently, $\alpha'' = \alpha$, for every $\alpha \in A(T, V)$. With $x \in G$, we have

$$\begin{aligned}(\alpha \cdot x)'(\tau_1, \cdots, \tau_q) &= (\alpha \cdot x)(\tau_1', \cdots, \tau_q')' = (x^{-1} \cdot \alpha(x \cdot \tau_1', \cdots, x \cdot \tau_q'))' \\ &= \alpha((\tau_1 \cdot x^{-1})', \cdots, (\tau_q \cdot x^{-1})')' \cdot x \\ &= \alpha'(\tau_1 \cdot x^{-1}, \cdots, \tau_q \cdot x^{-1}) \cdot x \\ &= (x^{-1} \cdot \alpha')(\tau_1, \cdots, \tau_q).\end{aligned}$$

Thus $(\alpha \cdot x)' = x^{-1} \cdot \alpha'$. Moreover, we have $c_\tau(\alpha)' = c_{\tau'}(\alpha')$ and $t_\tau(\alpha)' = t_{\tau'}(\alpha')$, whence $\delta(\alpha') = \delta(\alpha)'$. Thus the map $\alpha \rightarrow \alpha'$ is an involution of the complex $A(T, V)$ transporting the right G-module structure onto the left

G-module structure. Hence we conclude that the representation of G on the cohomology space of $A(T, V)$ that is induced by the left G-module structure of $A(T, V)$ is trivial also.

Moreover, it is easy to verify that the involution $\alpha \to \alpha'$ maps the subcomplex of the representative differential forms onto itself. Hence the above argument also applies to this subcomplex, and we conclude that the representation of G on the cohomology space of the complex of the representative differential forms that is induced by the left G-module structure is trivial.

THEOREM 10.1. *Let G be a real analytic group, and let K be a nucleus of G. Then the canonical injection $E((\mathfrak{G}/\mathfrak{K})') \to E(\mathfrak{G}') \otimes R(G)$ induces an isomorphism of the Lie algebra cohomology space $H(\mathfrak{G}/\mathfrak{K}, R)$ onto the cohomology space of the complex of the representative differential forms on G.*

Proof. Let us write G as a semidirect product $S \cdot K$. Then S is a reductive analytic group, whence the representative G-module $E(\mathfrak{G}') \otimes R(G)$ is semisimple as an S-module. Since the induced representation of G, and thus of S, on the cohomology space of the complex $E(\mathfrak{G}') \otimes R(G)$ is trivial, it follows that this cohomology space may be identified with the cohomology space of the S-fixed part of $E(\mathfrak{G}') \otimes R(G)$. This S-fixed part is $E(\mathfrak{G}') \otimes R(G)^S$, where $R(G)^S$ is the S-fixed part of $R(G)$ for the representation of S on $R(G)$ by *right* translations.

The canonical injection identifies $E((\mathfrak{G}/\mathfrak{K})')$ with a subcomplex of $E(\mathfrak{G}') \otimes R(G)^S$. Our theorem will be proved as soon as we have shown that the cohomology space of the factor complex, P say, is (0). Let J denote the ideal of $E(\mathfrak{G}') \otimes R(G)^S$ that is generated by $(\mathfrak{G}/\mathfrak{K})'$. We define a decreasing sequence of subcomplexes of P by setting

$$P_q = (E((\mathfrak{G}/\mathfrak{K})') + J^q)/E((\mathfrak{G}/K)'),$$

agreeing that $P_q = P$ for $q \leqq 0$. We have $P_q = (0)$ when q exceeds the dimension of $\mathfrak{G}/\mathfrak{K}$. Hence, in order to show that the cohomology space $H(P)$ of P is (0), it suffices to show that $H(P_q/P_{q+1}) = (0)$, for all q.

We may write

$$P_q/P_{q+1} = (E^q((\mathfrak{G}/\mathfrak{K})')E(\mathfrak{G}') \otimes R(G)^S)/(E^q((\mathfrak{G}/\mathfrak{K})') + J^{q+1}).$$

The restriction map $R(G) \to R(G)_K \subset R(K)$ induces an isomorphism of $R(G)^S$ onto $R(G)_K$. Combining this with the restriction epimorphism $\mathfrak{G}' \to \mathfrak{K}'$, we obtain an epimorphism $E(\mathfrak{G}') \otimes R(G)^S \to E(\mathfrak{K}') \otimes R(G)_K$ whose kernel is evidently our ideal J. Moreover, we see immediately that $E(\mathfrak{K}') \otimes R(G)_K$ is a subcomplex of $E(\mathfrak{K}') \otimes R(K)$, and that our epimorphism is an epimorphism of complexes. The above form of P_q/P_{q+1} shows that this epimorphism induces an isomorphism of $H(P_q/P_{q+1})$ onto

$$E^q((\mathfrak{G}/\mathfrak{K})') \otimes H((E(\mathfrak{K}') \otimes R(G)_K)/R).$$

Hence it suffices to show that the cohomology space of the complex $(E(\mathfrak{K}') \otimes R(G)_K)/R$ is trivial. In order to do this, we appeal (as in the proof of Lemma 9.1) to the following known results on the structure of $R(G)$ [6, Section 4]. There are algebraically independent elements $u_1, \cdots, u_n$ in $R(K)$ and an index q, $0 \leqq q \leqq n$, such that the elements $u_{q+1}, \cdots, u_n$ span $\mathrm{Hom}(G, R)_K$ over R and $R(G)_K$ is the tensor product of $R[u_1, \cdots, u_n]$ with the R-algebra consisting of the elements of $R(G)_K$ that can be written as complex linear combinations of exponentials of complex linear combinations of $u_{q+1}, \cdots, u_n$. Furthermore, $R(G)_K \mathfrak{K}$ is precisely the $R(G)_K$-module of all those derivations of $R(G)_K$ that are restrictions to $R(G)_K$ of derivations of the algebra $R[[u_1, \cdots, u_n]]$ of all integral power series in the n variables $u_1, \cdots, u_n$.

In order to see this, note first that if $\tau \in \mathfrak{K}$ and $h \in \mathrm{Hom}(K, C)$, then $\tau(h)$ is a constant, and $\tau(\exp(h)) = \tau(h) \exp(h)$. Hence it is clear that every element of $\mathfrak{K}$, and therefore also every element of $R(G)_K \mathfrak{K}$, can be extended to a derivation of the algebra $R[[u_1, \cdots, u_n]]$. The derivations of $R[[u_1, \cdots, u_n]]$ are the $R[[u_1, \cdots, u_n]]$-linear combinations of the derivations τ_i such that $\tau_i(u_j) = \delta_{ij}$. The derivations of $R[[u_1, \cdots, u_n]]$ that send $R(G)_K$ into itself are therefore the $R(G)_K$-linear combinations of the τ_i's. Hence it suffices to show that the restrictions of the τ_i's to $R(G)_K$ belong to $R(G)_K \mathfrak{K}$. It is clear from what we have already said in proving Lemma 9.1 that $R[[u_1, \cdots, u_n]]$ also contains $R(K)$ and that the τ_i's send $R(K)$ into itself. Let τ be the restriction to $R(K)$ of any one of the τ_i's (or of any $R(G)_K$-linear combinations of the τ_i's). We show first that $\tau \in R(K)\mathfrak{K}$.

Considering the tangent space to K at the identity element 1, we see that there is a basis $\zeta_1, \cdots, \zeta_n$ of $\mathfrak{K}$ such that $\zeta_i(u_j)(1) = \delta_{ij}$. Let d denote the determinant formed from the $\zeta_i(u_j)$. Then we can evidently find elements $g_i \in R(K)$ such that $\sum_{i=1}^n g_i \zeta_i(u_j) = d\tau(u_j)$, for all j, which implies that $\sum_{i=1}^n g_i \zeta_i = d\tau$. Applying these derivations to $u_j \cdot x^{-1}$, with $x \in K$, and evaluating at x, we obtain $\sum_{i=1}^n g_i(x)\zeta_i(u_j \cdot x^{-1})(x) = d(x)\tau(u_j \cdot x^{-1})(x)$, which reduces to $g_j(x) = d(x)\tau(u_j \cdot x^{-1})(x)$. Thus we have $g_j = dh_j$, where $h_j(x) = \tau(u_j \cdot x^{-1})(x)$. Clearly, $h_j \in R(K)$ and $\sum_{i=1}^n h_i \zeta_i = \tau$. Now it is clear from what we have said in proving Lemma 9.1 concerning the structure of the algebra of representative functions that there is an $R(G)_K$-linear projection, φ say, of $R(K)$ onto $R(G)_K$. Our last result shows that the restriction of τ to $R(G)_K$ coincides with $\sum_{i=1}^n \varphi(h_i)\zeta_i \in R(G)_K \mathfrak{K}$.

Now note that the subalgebra $R(G)_K$ of $R[[u_1, \cdots, u_n]]$ is stable not only under the derivations τ_i of $R[[u_i, \cdots, u_n]]$, but also under the process of formal integration with respect to any one of the variables u_i. Hence it follows immediately from the proof of the formal Poincaré Lemma for $R[[u_1, \cdots, u_n]]$, as given, for instance, in [2, pp. 65–67], that the complex $(E(\mathfrak{K}') \otimes R(G)_K)/R$ has cohomology space (0). Theorem 10.1 is thus established.

THEOREM 10.2. *Let G be a real linear Lie group with finite component group, and let K be a nucleus of G. Let V be a representative G-module. Then the complex of the V-valued representative differential forms on K is a representatively injective resolution of V as a K-module and yields an isomorphism of $H_r(G, V)$ onto the Lie algebra cohomology space $H(\mathfrak{K}, V)^{G/K}$.*

Proof. By Theorem 9.1, $H_r(G, V)$ is isomorphic, by the restriction homomorphism, with $H_r(K, V)^{G/K}$. By Theorem 10.1, the V-augmented complex $E(\mathfrak{K}') \otimes R(K) \otimes V$ has cohomology space (0), and therefore is a representatively injective resolution of V as a K-module. Hence $H_r(K, V)$ may be identified with the homology space of the complex

$$(E(\mathfrak{K}') \otimes R(K) \otimes V)^K = E(\mathfrak{K}') \otimes V,$$

i.e., with the Lie algebra cohomology space $H(\mathfrak{K}, V)$. Moreover, the resulting isomorphism $H_r(K, V) \to H(\mathfrak{K}, V)$ is evidently a G/K-module isomorphism, so that it maps $H_r(K, V)^{G/K}$ isomorphically onto $H(\mathfrak{K}, V)^{G/K}$. This completes the proof of Theorem 10.2.

11. Relation between representative and continuous cohomology

Let G be a real linear Lie group with finite component group, and let K be a nucleus of G. Let V be a finite-dimensional continuous G-module. Let X be a continuously injective resolution of R as a G/K-module, and let Y be a representatively injective resolution of V as a G-module. We regard each Y_q as a topological vector space, giving it the coarsest topology making all linear functionals continuous. Now consider the tensor product complex $X \otimes Y$. We regard this as a complex of topological vector spaces by giving each $X_p \otimes Y_q$ the coarsest topology making all linear maps $X_p \otimes Y_q \to X_p$ that are induced by linear functionals on Y_q continuous. Clearly, if we augment the complex $X \otimes Y$ by the map $V \to X_0 \otimes Y_0$ defined from the given maps $R \to X_0$ and $V \to Y_0$ we obtain a strongly exact sequence of continuous G-module homomorphisms.

Now let Z be a continuously injective resolution of V. Then there is a continuous homomorphism of continuous G-module complexes $X \otimes Y \to Z$ extending the natural map $R \otimes V \to V$. This map defines a homomorphism $H_c(G/K, R) \otimes H_r(G, V) \to H_c(G, V)$ which, by the general facts of homological algebra, is independent of the choices of X, Y, Z and the map $X \otimes Y \to Z$. We shall call this map the *canonical map*. It may be viewed as the composite of the "inflation" $H_c(G/K, R) \to H_c(G, R)$ and a "cup product" $H_c(G, R) \otimes H_r(G, V) \to H_c(G, V)$.

THEOREM 11.1. *Let G be a real linear Lie group with finite component group, and let K be a nucleus of G. Then, for every finite-dimensional continuous G-module V, the canonical map $H_c(G/K, R) \otimes H_r(G, V) \to H_c(G, V)$ is an isomorphism.*

Proof. Let G_1 be the connected component of the identity in G. Since G/G_1 is finite, the restriction from G to G_1 gives isomorphisms

$$H_c(G/K, R) \to H_c(G_1/K, R)^{G/G_1}, \qquad H_r(G, V) \to H_r(G_1, V)^{G/G_1},$$
$$H_c(G, V) \to H_c(G_1, V)^{G/G_1}.$$

It is easy to see that the canonical map is compatible with restriction and with the action of G/G_1. Hence it is clear that it suffices to prove our theorem in the case where G is connected, which we shall now assume.

Write G as a semidirect product $S \cdot K$, and let P be a maximal compact subgroup of S. Let X be the continuously injective resolution of R as an S-module that is obtained from the complex of the differential forms on the homogeneous space S/P. Since the canonical epimorphism $G \to G/K$ maps S isomorphically onto G/K, we may regard X as a continuously injective resolution of R as a G/K-module. In fact, we shall write X in the form $(E((\mathfrak{G}/(\mathfrak{P} + \mathfrak{K}))') \otimes D)^{PK}$, the superscript referring to the right module structure.

On the other hand, consider the complex $E(\mathfrak{G}') \otimes R(G) \otimes V$ of the representative differential forms on G. This has a left G-module structure and a right S-module structure which combine to form the structure of a representative $G \times S$-module. The subspace $E(\mathfrak{G}/\mathfrak{S})') \otimes R(G) \otimes V$ is evidently a $G \times S$-submodule, and it is representatively injective as a G-module. Since S is reductive, the S-fixed part of this $G \times S$-module is a direct G-module summand and hence is still representatively injective as a G-module. Under the right S-module structure of $E((\mathfrak{G}/\mathfrak{S})') \otimes R(G) \otimes V$, the factor V is left inert, so that the S-fixed part is $(E((\mathfrak{G}/\mathfrak{S})') \otimes R(G))^S \otimes V$. We know from our discussion of differential forms on homogeneous spaces that this is a subcomplex, Y say, of $E(\mathfrak{G}') \otimes R(G) \otimes V$. We claim that the V-augmented complex Y has cohomology space (0). This amounts to saying that the R-augmented complex $(E((\mathfrak{G}/\mathfrak{S})') \otimes R(G))^S$ has cohomology space (0). It is clear from the results of Section 6 that the G-module complex $(E((\mathfrak{G}/\mathfrak{S})') \otimes R(G))^S$ may be identified with the complex of the representative differential forms on the homogeneous space G/S, i.e., with the maximum locally finite G-submodule of the complex of all differential forms on G/S. On the other hand, the homogeneous space G/S may be identified with K (as a differentiable manifold). Then the complex of the differential forms on G/S takes the form $E(\mathfrak{K}') \otimes D^S$, with G operating so that $x \cdot (e \otimes f) = e \otimes (f \cdot x^{-1})$. Hence the complex of the representative differential forms on G/S is isomorphic with $E(\mathfrak{K}') \otimes R(G)^S$. From the proof of Theorem 10.1, we know that the cohomology space of the complex $(E(\mathfrak{K}') \otimes R(G)^S)/R$ is (0). Hence the same is true for the complex $(E((\mathfrak{G}/\mathfrak{S})') \otimes R(G))^S/R$. Hence we conclude that Y is a representatively injective resolution of V.

Now let Z denote the complex of the differential forms on the homogeneous space G/P. Since P is a maximal compact subgroup of S, and since K is solvable and simply connected, it follows that P is also a maximal compact subgroup of G. Hence, as we know from Section 6, Z is a continuously injective resolution of V. There is an evident continuous map of G-module complexes of $X \otimes Y$ into Z, namely the tensor product map induced by the multiplication in $E(\mathfrak{G}')$ and the multiplication $D \otimes F_d(G, V) \to F_d(G, V)$. The canonical map $H_c(G/K, R) \otimes H_r(G, V) \to H_c(G, V)$ is induced by the composite of the maps $X^G \otimes Y^G \to (X \otimes Y)^G \to Z^G$.

Now X^G may be identified with the complex $C(\mathfrak{S}, \mathfrak{P}, R)$ of the relative Lie algebra cochains for $(\mathfrak{S}, \mathfrak{P})$ in R, Y^G may be identified with $C(\mathfrak{G}, \mathfrak{S}, V)$, and Z^G with $C(\mathfrak{G}, \mathfrak{P}, V)$. Our canonical map of the group cohomology spaces is thereby transported into the map of Lie algebra cohomology spaces

$$H(\mathfrak{S}, \mathfrak{P}, R) \otimes H(\mathfrak{G}, \mathfrak{S}, V) \to H(\mathfrak{G}, \mathfrak{P}, V)$$

composed of the cup product

$$H(\mathfrak{G}, \mathfrak{P}, R) \otimes H(\mathfrak{G}, \mathfrak{P}, V) \to H(\mathfrak{G}, \mathfrak{P}, V),$$

the canonical map

$$H(\mathfrak{G}, \mathfrak{S}, V) \to H(\mathfrak{G}, \mathfrak{P}, V),$$

and a map

$$H(\mathfrak{S}, \mathfrak{P}, R) \to H(\mathfrak{G}, \mathfrak{P}, R)$$

inverting the restriction *epimorphism* $H(\mathfrak{G}, \mathfrak{P}, R) \to H(\mathfrak{S}, \mathfrak{P}, R)$. This map is discussed in [4, Section 4], where it is shown to be an isomorphism, provided only that $\mathfrak{S}$ is a reductive subalgebra of $\mathfrak{G}$, and V is semisimple as an $\mathfrak{S}$-module, which is evidently the case here. The fact that the restriction map $H(\mathfrak{G}, \mathfrak{P}, R) \to H(\mathfrak{S}, \mathfrak{P}, R)$ is an epimorphism follows easily from the fact that $\mathfrak{G}$ is the semidirect sum $\mathfrak{S} + \mathfrak{K}$. We remark that the above result on the Lie algebra cohomology is a straightforward generalization of [9, Theorem 12], which is the case where $\mathfrak{P} = (0)$. This completes the proof of Theorem 11.1.

12. Representative and rational cohomology for real linear algebraic groups

Let G be a real linear algebraic group. The *rational cohomology* of G is to be understood in the sense of [3]; the definition is obtained by replacing "continuous representation" by "rational representation" in the definition of the representative cohomology. If V is a rational G-module, we denote the rational cohomology space for G in V by $H_\rho(G, V)$. If V is finite-dimensional, the relation between $H_\rho(G, V)$ and $H_r(G, V)$ is exactly analogous to the relation between $H_r(G, V)$ and $H_c(G, V)$, as given in Section 11. We proceed to discuss this in detail.

Note first that, by [12, Appendix], the component group of G is finite. Let N be the maximum unipotent normal algebraic subgroup of G. Then G is a semidirect product $T \cdot N$, where T is a fully reducible algebraic subgroup of G. Let L be a nucleus of T, and put $K = LN$. Then K is evidently a nucleus of G. We define a canonical homomorphism

$$H_r(G/N, R) \otimes H_\rho(G, V) \to H_r(G, V)$$

in a fashion strictly analogous to the construction underlying Theorem 11.1. Note that, here, there is no need to consider any topology of the resolutions and cohomology spaces. The result is as follows.

THEOREM 12.1. *Let G be a real linear algebraic group, let N be the maximum unipotent normal subgroup of G, and let V be a finite-dimensional rational G-module. Then the canonical homomorphism*

$$H_r(G/N, R) \otimes H_\rho(G, V) \to H_r(G, V)$$

is an isomorphism.

Proof. We know that the restriction from G to K yields isomorphisms $H_r(G/N, R) \to H_r(K/N, R)^G$ and $H_r(G, V) \to H_r(K, V)^G$. Following these up with the isomorphism of Theorem 10.2, we obtain isomorphisms

$$H_r(G/N, R) \to H(\mathfrak{K}/\mathfrak{N}, R)^G \quad \text{and} \quad H_r(G, V) \to H(\mathfrak{K}, V)^G.$$

On the other hand, it is known from [3] that the restriction from G to N yields an isomorphism $H_\rho(G, V) \to H_\rho(N, V)^G$. Moreover, it was shown in [3] that the complex of the rational differential forms on N (obtained from the complex of the representative differential forms by replacing the algebra of the representative functions with the subalgebra of the rational representative functions) gives a rationally injective resolution of V, and hence a natural isomorphism of $H_\rho(N, V)^G$ onto $H(\mathfrak{N}, V)^G$. Composing this with the restriction map, we obtain a natural isomorphism $H_\rho(G, V) \to H(\mathfrak{N}, V)^G$.

By means of these isomorphisms, the canonical homomorphism

$$H_r(G/N, R) \otimes H_\rho(G, V) \to H_r(G, V)$$

is transported into a homomorphism

$$H(\mathfrak{K}/\mathfrak{N}, R)^G \otimes H(\mathfrak{N}, V)^G \to H(\mathfrak{K}, V)^G.$$

It was shown in [3, Section 5] that the map from rational group cohomology to Lie algebra cohomology is obtained from any injective group resolution by regarding it as a Lie algebra complex and mapping it into any injective Lie algebra resolution. This is easily seen to apply in exactly the same way also to representative cohomology. It follows that the above homomorphism $H(\mathfrak{K}/\mathfrak{N}, R)^G \otimes H(\mathfrak{N}, V)^G \to H(\mathfrak{K}, V)^G$ must be the restriction to the G-fixed parts of a homomorphism $H(\mathfrak{K}/\mathfrak{N}, R) \otimes H(\mathfrak{N}, V)^K \to H(\mathfrak{K}, V)$ that is obtained in the canonical way from injective Lie algebra resolutions. Using the standard injective Lie algebra resolutions described in [3, Section 5], one sees that this canonical homomorphism of Lie algebra cohomology spaces is the usual composite of the lift $H(\mathfrak{K}/\mathfrak{N}, R) \to H(\mathfrak{K}, R)$, the lift

$H(\mathfrak{N}, V)^K \to H(\mathfrak{K}, V)$ obtained by identifying $H(\mathfrak{N}, V)^K$ with the relative $H(\mathfrak{K}, \mathfrak{L}, V)$, and the cup product $H(\mathfrak{K}, R) \otimes H(\mathfrak{K}, V) \to H(\mathfrak{K}, V)$. Now, if we note that $H(\mathfrak{K}/\mathfrak{N}, R)$ may be identified with $H(\mathfrak{L}, R)$, [9, Theorem 12] says precisely that this homomorphism is an isomorphism. This completes the proof of Theorem 12.1.

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