FELLER AND MARTIN BOUNDARIES FOR COUNTABLE SETS

BY
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1. Introduction

Let X be a countable set, which will also be considered as a topological space with the discrete topology. For each $x \in X$, let $P(x, \cdot)$ be a measure on X, of total mass $\leqq 1$. P is thus a Markoff transition function with state space X, but with a possibility of "annihilation" at each step. We also use the notation P for the operator on $\mathfrak{L}_\infty(X)$ given by

$$Pf(x) = \int P(x, dy)f(y).$$

$P^{(n)}(x, \cdot)$ is then defined in the usual way by

$$P^{(n+1)}(x, S) = \int P^{(n)}(x, dy)P(y, S),$$

and has as its corresponding operator the n-fold iterate of the operator P. $P^{(0)}(x, \cdot)$ is defined as a unit mass at x.

There have been defined for such operators two sorts of "exit boundaries": the Martin boundary, described in [2] or [4], and the Feller boundary, constructed in [3]. These serve to describe the long-term behavior of the stochastic processes constructed from P, and to classify the functions left fixed by P. In these papers, there is raised the natural question: Precisely what is the relationship between these two boundaries? Here we shall try to give some sort of answer to this question. It should be remarked that the Martin boundary was originally constructed in connection with classical potential theory, but has been made part of a far-reaching extension of the ideas of classical potential theory to kernels arising from Markoff processes. The construction has been carried through by G. A. Hunt for a wide class of Markoff processes.

It will be assumed that all points are transient, i.e., that for all finite sets S the sum $\sum_{n=0}^{\infty} P^{(n)}(x, S)$ converges to a finite limit, which we call $G(x, S)$. This is no real loss, as the additional boundary points which are introduced by the existence of recurrent states play a rather transparent role, for either type of boundary.

Briefly, the Martin boundary B (for a given starting measure) is a separable compact metric space with a certain measure μ on its Borel sets, and attached to X in a certain way.

We shall see that the Feller boundary Γ can be described as the Stone space of the measure algebra of μ. Furthermore, if C is the support of μ (in the

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topological sense, i.e., the minimal closed set whose complement has measure 0), then C can be realized in a natural way as a quotient space of Γ. This simply amounts to doing for C what Kendall does in [5] for the unit interval. Furthermore, the construction of C from Γ can be carried out without explicit reference to C, so that the Martin boundary can to some extent be recaptured from the Feller boundary.

The projection from Γ onto C extends in a natural way to one from $X \cup \Gamma$ onto $X \cup C$, which suggests an alternative way of extending the topology of C to $X \cup C$.

We will have occasion to use stochastic processes constructed in the usual way from P. Let Ω be a set, τ a function from Ω to the nonnegative integers and ∞, and $\xi_0, \xi_1, \cdots$ a sequence of functions from Ω to X, $\xi_n(\omega)$ being defined for all nonnegative integers $n \leqq \tau(\omega)$. For each $x \in X$, let $\Pr_x$ be a measure on the σ-algebra generated by the ξ_n and τ, for which $\xi_0, \xi_1, \cdots$ is a Markoff process starting at x and with transition probabilities

$$\Pr_x\{\xi_{n+1} \in S \mid \xi_n = y\} = P(y, S).$$

Such Ω, τ, ξ_n, $\Pr_x$ exist, as is well known. Terms such as "almost everywhere in Ω" will mean almost everywhere with respect to all the measures $\Pr_x$. Also, for any measure α on X, a measure $\Pr_\alpha$ can be defined by

$$\Pr_\alpha\{\cdot\} = \int \Pr_x\{\cdot\}\, \alpha(dx).$$

This notation will be used throughout.

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2. The Feller boundary

Let $\mathcal{B}$ be the real Banach space of bounded solutions to the equation $Pf = f$, with supremum norm. Solutions to the equation $Pf = f$ will also be called *harmonic* functions, after Feller's and Doob's usage. Hunt calls these "concordant" functions. Let $\mathcal{S}$ be the set of nonnegative extreme elements of the unit sphere in $\mathcal{B}$. Feller shows in [3] that these are precisely the functions of the form

$$s_S(x) = \Pr_x\{\xi_n \in S \text{ for all } n \geqq \text{some } n_0\},$$

where S is any subset of X. In particular, let $e(x) = s_X(x) = \Pr_x\{\tau = \infty\}$. It is further shown that $\mathcal{S}$ is a lattice; indeed,

$$s_S \wedge s_T = s_{S \cap T}, \quad \text{and} \quad s_1 \vee s_2 = e - (s_1 \wedge s_2).$$

$s \to e - s$ provides a complementation operation. Let Γ be the set of maximal lattice-ideals, topologized by using sets of the form

$$\Delta(s) = \{\gamma \in \Gamma \mid s \notin \gamma\}$$

as a subbasis for the open sets.

Some of the results of Feller, and improvements on them by Kendall [4], can be summarized as follows.

(1) $\mathcal{S}$ is a complete Boolean algebra, with e as unit element.

(2) Γ is the Stone space of $\mathcal{S}$ (see [6]), and hence an extremely disconnected compact Hausdorff space. The sets $\Delta(s)$, $s \in \mathcal{S}$, are precisely the both closed and open sets of Γ.

(3) The map $s \to 1_{\Delta(s)}$ extends to an isometry from $\mathcal{B}$ onto $\mathcal{C}(\Gamma)$, which we shall call j (the notation 1_A means the indicator function of the set A).

(4) The space Γ is "hyperstonian" (see [1]). This just comes down to the fact that the map $x \to s(x)$, for each $x \in X$, is countably additive as a function on $\mathcal{S}$.

Feller furthermore topologizes $X \cup \Gamma$ in a manner which can be described as follows: X is open in $X \cup \Gamma$; X and Γ have their original topologies induced on them from $X \cup \Gamma$; and finally, a net x_n of points in X is said to converge to a point $\gamma \in \Gamma$ if, for any $S \subset X$ such that $j(s_S)(\gamma) = 1$, x_n lies in S for all sufficiently great n. This makes $X \cup \Gamma$ into a Hausdorff space, with X as a dense subset.

Remark 2.1. $X \cup \Gamma$ is not necessarily compact, even though Γ is. Indeed, let X be the integers (both positive and negative), and let $P(x, \cdot)$ be a point mass at $x + 1$. The only functions in $\mathcal{B}$ are then the constants, Γ is a single point γ, and a set $S \subset X$ has γ as a limit point if and only if S is unbounded above. Thus, the set of negative integers has no limit point, and $X \cup \Gamma$ is not compact.

Remark 2.2. *$X \cup \Gamma$ is normal.*

Proof. Let Y and Z be disjoint closed subsets of $X \cup \Gamma$. There are disjoint open neighborhoods V, W of $Y \cap \Gamma$ and $Z \cap \Gamma$ in $X \cup \Gamma$, since $Y \cap \Gamma$ and $Z \cap \Gamma$ are compact and $X \cup \Gamma$ is Hausdorff. Then $(V \cap Z^{\perp}) \cup (X \cap Y)$ and $(W \cap Y^{\perp}) \cup (X \cap Z)$ are open, disjoint, and contain Y and Z respectively.

Feller shows further that if $x_n \to \gamma$, then for any $f \in \mathcal{B}$ we have $f(x_n) \to jf(\gamma)$. In fact, this property can be considered the main goal of the construction.

3. The Martin boundary

Recall that $G(x, S)$ was defined as $\sum_{n=0}^{\infty} P^{(n)}(x, S)$. Then for each finite measure α on X there is defined a measure αG, finite on finite sets, by the formula

$$\alpha G(S) = \int G1_S \, d\alpha.$$

We fix once and for all a finite measure ε for which $\varepsilon(S) > 0$ for all nonempty S, and define a function K of two variables in X by

$$K(x, y) = \frac{G(x, dy)}{\varepsilon G(dy)} = \frac{G(x, \{y\})}{\int G(z, \{y\}) \varepsilon(dz)}.$$

The use of the Radon-Nikodym derivative notation, although somewhat high-brow for a discrete space, is natural in view of what happens for more general X.

We complete X with respect to the uniformity induced by calling $y_1, y_2, \cdots$ a Cauchy sequence if either

(1) y_n is constant for some n onward, or

(2) for any finite set S, $y_n \notin S$ if n is sufficiently large, and furthermore $K(x, y_n)$ is a Cauchy sequence of numbers for each $x \in X$.

The completion we call $X \cup B$, B being the new points. $X \cup B$ is a complete separable metric space, and X has the discrete topology. $X \cup B$ need not be compact, as is shown by the same example as that which worked for $X \cup \Gamma$. (This fact was pointed out to me by the referee.)

It is shown in [4] that a.e. for $\omega \in \Omega$, either $\tau(\omega) < \infty$ or else $\xi_n(\omega)$ converges to a point of B. If $\tau(\omega) = \infty$, define $\xi_\infty(\omega)$ to be this limit. Then ξ_τ is defined on almost all of Ω. A measure is defined on the Borel sets of $X \cup B$, by assigning to a subset A the mass $\Pr_\varepsilon\{\xi_\tau \in A\}$. By μ will be meant the restriction of this measure to subsets of B. So μ has total mass $\Pr_\varepsilon\{\tau = \infty\}$.

The functions $K(x, \cdot)$ extend continuously to $X \cup B$. The extended function will still be denoted by $K(x, \cdot)$. So, for $x \in X$ and $y \in B$, the meaning of $K(x, y)$ is $\lim_{z \to y, z \in X} K(x, z)$. Then $K(\cdot, y)$, $y \in X \cup B$, is a nonnegative function, and

$$\int P(x, dz)K(z, y) \leqq K(x, y)$$

for each $y \in X \cup B$, the inequality being strict if $y \in X$. Further,

$$\int K(z, y)\varepsilon(dz) \leqq 1$$

for each $y \in X \cup B$, the inequality being an equality if $y \in X$.

Let $\mathcal{H}$ be the normed linear space of harmonic functions f in $\mathcal{L}_1(\varepsilon)$, i.e.,

$$\int P(x, dy)f(y) = f(x) \qquad \text{for all} \quad x \in X,$$

and let $\mathcal{H}_+$ be the positive functions in $\mathcal{H}$. Let $\mathcal{E}$ be the extreme points of the unit sphere of $\mathcal{H}_+$. Let $E = \{y \in B \mid K(\cdot, y) \in \mathcal{E}\}$. This is a Borel subset of B. Then it is shown in [4] that

(1) $\mu(B - E) = 0$,

(2) there is a 1-1 correspondence between functions $f \in \mathcal{H}_+$ and finite Borel measures ν on B with $\nu(B - E) = 0$, given by the correspondence

$$\nu \longrightarrow \int K(\cdot, y)\nu(dy) = K\nu.$$

In this correspondence, $\int K\nu\, d\xi = \nu(B) = \nu(E)$. In particular, all the functions of E are given in the form $K(\cdot, y)$, $y \in E$.

4. Relations between the boundaries

LEMMA 4.1. *If* $f \geqq 0$, $Pf \leqq f$, $g \in \mathcal{B}$, *and* $\lim_{n\to\infty}(f(\xi_n) - g(\xi_n)) \geqq 0$, *a.e. on* $\{\tau = \infty\}$, *then* $f \geqq g$.

Proof. Let $S = \{x \mid f(x) < g(x) - 1/m\}$. Then

$$\begin{aligned} f(x) \geqq P^n f(x) &= \int_S P^n(x, dy)f(y) + \int_{S^\perp} P^n(x, dy)f(y) \\ &\geqq \int_{S^\perp} P^n(x, dy)g(y) - \frac{1}{m} \\ &= \int P^n(x, dy)g(y) - \int_S P^n(x, dy)g(y) - \frac{1}{m} \\ &\geqq g(x) - \| g \|_\infty \Pr_x \{\xi_n \in S\} - \frac{1}{m} \\ &= g(x) - \| g \|_\infty \Pr_x\left\{f(\xi_n) < g(\xi_n) - \frac{1}{m}\right\} - \frac{1}{m}. \end{aligned}$$

If we let $n \to \infty$, $f(x) \geqq g(x) - 1/m$, for all m, and so $f(x) \geqq g(x)$.

The key point in seeing the relation between the two boundaries is the following simple fact.

THEOREM 4.1. $K\mu = e$.

Proof. First, $\lim_{n\to\infty} K\mu(\xi_n) = 1$ a.e. on $\{\tau = \infty\}$, by Theorem 4.2 of [4]. Next, $\lim_{n\to\infty} e(\xi_n) = 1$ a.e. on $\{\tau = \infty\}$. To prove this, let $\Phi = \lim_{n\to\infty} e(\xi_n)$, on $\{\tau = \infty\}$. This limit exists, by the martingale convergence theorem. So

$$e(x) = \Pr_x \{\tau = \infty\} \geqq \int \Phi \, d\Pr_x = \lim_{n\to\infty} \int e(\xi_n) \, d\Pr_x = \lim_{n\to\infty} P^n e(x) = e(x).$$

Then equality holds throughout. Since $e \leqq 1, \Phi \leqq 1$ on $\{\tau = \infty\}$. So Φ must equal 1 on $\{\tau = \infty\}$.

Combining these two facts, and applying Lemma 4.1, $K\mu \geqq e$. But

$$\int K\mu \, d\varepsilon = \iint K(x, dy)\mu(dy)\varepsilon(dx) \leqq \int \mu(dy) = \Pr_x \{\tau = \infty\} = \int e \, d\varepsilon.$$

Thus $K\mu = e$.

Now define $k: \mathcal{L}_\infty(\mu) \to \mathcal{H}$ by $k\phi = K\nu$, where $d\nu = \phi \, d\mu$.

THEOREM 4.2. *k is an order-preserving linear isometry from $\mathcal{L}_\infty(\mu)$ onto $\mathcal{B}$.*

Proof. Let $f = K\nu$ for a finite Borel measure ν on E. Then $f \in \mathcal{B}$, and, for any nonnegative b, we have

$$\begin{aligned} \| f \|_\infty \leqq b \quad &\Leftrightarrow \quad -be \leqq f \leqq be \quad \Leftrightarrow \quad -b\mu \leqq \nu \leqq b\mu \\ &\Leftrightarrow \quad f = k\phi \quad \text{with } -b \leqq \phi \leqq b. \end{aligned}$$

COROLLARY. *$j \circ k$ is an isomorphism from the ordered normed algebra $\mathfrak{L}_\infty(\mu)$ onto the ordered normed algebra $\mathcal{C}(\Gamma)$.*

Now let C be the set of all points $y \in B$ for which every neighborhood of y in B has positive μ-measure. C is easily seen to be closed. It can also be described as the minimal closed set whose complement in B has zero μ-measure. For each Borel set $A \subset B$, define a closed and open set $\Gamma(A)$ in Γ by the formula

$$1_{\Gamma(A)} = j \circ k(1_A).$$

Finally, for each $y \in C$, set $\Gamma_y = \bigcap \Gamma(A)$, where A ranges over all open neighborhoods of y in B.

THEOREM 4.3. *The sets Γ_y, $y \in C$, are closed, nonempty, and disjoint, and their union is Γ.*

Proof. If $y \in C$ and $A_1, \cdots, A_n$ are open neighborhoods of y in B, then $A = A_1 \cap \cdots \cap A_n$ is again an open neighborhood of y in B, and $\mu(A) > 0$. Thus $\Gamma(A_1) \cap \cdots \cap \Gamma(A_n) = \Gamma(A)$ is a nonempty set. Then by compactness, Γ_y is a closed nonempty set. Notice that the fact that each neighborhood of y has positive measure was needed for nonemptiness.

If $y_1 \neq y_2$, then we can choose disjoint open neighborhoods A_j of y_j in B. Then $\Gamma(A_1) \cap \Gamma(A_2) = \Gamma(A_1 \cap A_2)$ is empty. Therefore $\Gamma_{y_1} \cap \Gamma_{y_2}$ is empty.

It is in proving that $\bigcup_{y \in C} \Gamma_y = \Gamma$ that the facts that C is closed and that $\mu(B - C) = 0$ are used. Suppose $\gamma \notin \bigcup_{y \in C} \Gamma_y$. Then for each $y \in C$ there is an open set A_y of B such that $y \in A_y$ and $\gamma \notin \Gamma(A_y)$. Let $A_{y_1} \cup \cdots \cup A_{y_n}$ cover C (since C is compact). Then

$$\Gamma(A_{y_1}) \cup \cdots \cup \Gamma(A_{y_n}) = \Gamma(A_{y_1} \cup \cdots \cup A_{y_n}) \supset \Gamma(C) = \Gamma,$$

since $\mu(B - C) = 0$. Thus $\gamma \in$ some $\Gamma(A_y)$, contradiction.

Now we define a projection $\pi : \Gamma \to C$ by $\pi(\gamma) = y$ if $\gamma \in \Gamma_y$.

THEOREM 4.4. *The topology of C is precisely the topology induced on C from Γ by the projection π.*

Proof. Since Γ is compact and C is Hausdorff, it suffices to show that π is continuous. That is, given an open subset A of B, we wish to show that $\pi^{-1}(A \cap C)$ is open in Γ.

For any open subset O of B, clearly $\pi^{-1}(O \cap C) \subset \Gamma(O)$. On the other hand, for a closed subset F of B we have $\pi^{-1}(F^\perp \cap C) = \pi^{-1}(F \cap C)^\perp$, and $\Gamma(F^\perp) = \Gamma(F)^\perp$, so that $\pi^{-1}(F \cap C) \supset \Gamma(F)$.

Let $\gamma \in \pi^{-1}(A)$, where A is an open subset of B, and let F be a closed neighborhood of $\pi(\gamma)$ with $F \subset A$. Then

$$\pi^{-1}(A) \supset \pi^{-1}(F) \supset \Gamma(F) \supset \Gamma(F^0) \supset \pi^{-1}(F^0),$$

which contains the point γ. But $\Gamma(F)$ is open. Thus $\pi^{-1}(A)$ is open.

The question arises, would it have been possible to construct the sets Γ_y in

Γ without having the map k on hand, but just from knowledge of $X \cup \Gamma$ and the map j? The answer is yes. This can be thought of as reconstructing C from $X \cup \Gamma$ and j, as mentioned in paragraph 5 of the introduction. To perform this, we proceed as follows.

Let J be any function whose domain X_J is a finite subset of X, and whose values are nonempty open intervals $J(x)$. We define

$$S_J = \{y \in B \mid K(x, y) \in J(x) \text{ for all } x \in X_J\},$$

and Δ_J = the closure in Γ of the union of all closed and open sets Δ in Γ with the following property:

Property (J). If $\emptyset \neq \Theta \subset \Delta$, Θ being closed and open, and if s is the function $j^{-1}(1_\Theta)$, then for each $x \in X_J$ we have

$$\left(\int s \, d\varepsilon\right)^{-1} s(x) \in J(x).$$

Δ_J is itself the closure of an open set in an extremely disconnected space, and so is also open.

LEMMA 4.2. *Δ_J itself has property (J).*

Proof. Let $\{\Delta_n\}$ be a maximal family of disjoint nonempty closed and open subsets of Δ_J, each having property (J). There are only countably many Δ_n, since

$$\sum_n \int j^{-1}(1_{\Delta_n}) \, d\varepsilon = \int j^{-1}(1_{\overline{\cup_n \Delta_n}}) \, d\varepsilon < \infty.$$

If Δ is closed and open in Δ_J and has property (J), then $\Delta - \overline{\cup_n \Delta_n}$ is closed and open, and has property (J), so, by maximality of $\{\Delta_n\}$, we have $\Delta \subset \overline{\cup_n \Delta_n}$, and thus $\Delta_J = \overline{\cup_n \Delta_n}$. Now let Θ be a nonempty closed and open subset of Δ_J. Let $\Theta_n = \Delta_n \cap \Theta$, and let $s_n = j^{-1}(1_{\Theta_n})$. Then

$$\left(\int s_n \, d\varepsilon\right)^{-1} s_n(x) \in J(x) \qquad \text{for all} \quad x \in X_J,$$

(provided $\int s_n \, d\varepsilon \neq 0$). Set

$$\lambda_n = \left(\int s \, d\varepsilon\right)^{-1} \left(\int s_n \, d\varepsilon\right) \quad \text{if} \quad \int s \, d\varepsilon \neq 0,$$

$$\lambda_n = 0 \qquad\qquad \text{if} \quad \int s \, d\varepsilon = 0.$$

Then

$$\left(\int s \, d\varepsilon\right)^{-1} s(x) = \sum_n \lambda_n \left(\int s_n \, d\varepsilon\right)^{-1} s_n(x) \in J(x),$$

since $\lambda_n \geqq 0$ and $\sum_n \lambda_n = 1$.

LEMMA 4.3. *The sets S_J generate the topology of B.*

Proof. By construction, the topology on B is precisely that of pointwise convergence of the functions $K(\cdot, y)$.

LEMMA 4.4. $\Gamma(S_J) = \Delta_J$.

Proof. The nonemty closed and open subsets Δ of $\Gamma(S_J)$ are all of the form $\Gamma(T)$, where T is some n-nonnull Borel subset of S_J. But if $s = j^{-1}(\Gamma(T))$, then $s = k1_T$, and

$$\left(\int s\,d\varepsilon\right)^{-1} s(x) = \frac{1}{\mu(T)} \int_T K(x, y)\mu(dy),$$

and since each $K(x, y) \in J(x)$ for $x \in X_J$ and $y \in T$, the same holds for the integral. So $\Gamma(S_J)$ has property (J), and $\Gamma(S_J) \subset \Delta_J$.

To prove the opposite inclusion, suppose the closed and open set $\Delta_J \cap \Gamma(S_J)^{\perp}$ is nonempty. Then it has the form $\Gamma(T)$, where $\mu(T) > 0$ and $\mu(T \cap S_J) = 0$. Thus there is some $x_0 \in X_J$ such that, denoting by (a, b) the interval $J(x_0)$, either

$$V = \{y \in T \mid K(x_0, y) \leqq a\} \quad \text{or} \quad W = \{y \in T \mid K(x_0, y) \geqq b\}$$

has positive μ-measure. Thus, either $\Gamma(V)$ or $\Gamma(W)$ is a closed and open nonempty subset of Δ_J which does not have property (J), contradiction.

THEOREM 4.5. *The sets Γ_y, $y \in C$, are precisely the nonempty intersections of the sets Δ_J.*

Proof. This follows immediately from Lemmas 4.3 and 4.4.

5. Topologies in $X \cup C$

The projection π from Γ onto C can be extended to a projection ρ from $X \cup \Gamma$ onto $X \cup C$, by defining

$$\rho(x) = x \quad \text{if} \quad x \in X,$$

$$\rho(\gamma) = \pi(\gamma) \quad \text{if} \quad \gamma \in \Gamma.$$

Thus we have two topologies on $X \cup C$: the original topology induced from $X \cup B$, which will be called the *Martin topology*; and the topology induced from $X \cup \Gamma$ by the projection ρ, which will be called the *projective topology*. Both induce the same topology on X and C.

THEOREM 5.1. *For any continuous function ϕ on C, we have $\phi \circ \rho = j \circ k\phi$.*

Proof. Given $y_0 \in C$ and $\varepsilon > 0$, let A be an open neighborhood of y_0 in B such that ϕ varies from y_0 in $A \cap C$ by less than ε, i.e.,

$$\|(\phi(\cdot) - \phi(y_0))1_A\|_\infty < \varepsilon.$$

Since $j \circ k$ is an isometric isomorphism from $\mathcal{L}_\infty(\mu)$ to $\mathcal{C}(\Gamma)$, we get

$$\| j \circ k((\phi(\cdot) - \phi(y_0))1_A \|_\infty = \| (j \circ k\phi(\cdot) - \phi(y_0))1_{\Gamma(A)} \|_\infty < \varepsilon.$$

So $j \circ k\phi(\gamma) = \phi(y_0)$ for any γ in Γ_{y_0}.

COROLLARY. *For any continuous function ϕ on C, the function on $X \cup C$ defined as $k\phi$ on X and ϕ on C is continuous in the projective topology.*

THEOREM 5.2. *$X \cup C$ is a normal space in the projective topology.*

Proof. The proof is the same as that of Remark 2.2, once we have established that $X \cup C$ is Hausdorff. This will now be proved.

Let y_1, y_2 be different points in C, and let N_j be an open neighborhood of y_j such that $\bar{N}_1 \cap \bar{N}_2$ is empty. Such N_j exist, since C is a compact Hausdorff space. The compact sets $\pi^{-1}(\bar{N}_j)$ in $X \cup \Gamma$ are disjoint. Thus they can be separated by open sets S_j in $X \cup \Gamma$, since $X \cup \Gamma$ is Hausdorff; i.e., $S_j \supset \pi^{-1}(\bar{N}_j)$, and $S_1 \cap S_2 = \emptyset$. Also, $\pi^{-1}(N_j)$ is open in Γ, so there is some subset Y_j of X such that $\pi^{-1}(N_j) \cup Y_j$ is open. Now let $T_j = (\pi^{-1}(N_j) \cup Y_j) \cap S_j$. This is open in $X \cup \Gamma$, contains $\pi^{-1}(N_j)$, and $T_1 \cap T_2 = \emptyset$. Furthermore, it is precisely $\rho^{-1}(N_j \cup (Y_j \cap S_j))$, since $\pi^{-1}(N_j) \subset S_j \cap \Gamma$. Thus, the images $N_j \cup (Y_j \cap S_j)$ are disjoint open sets in $X \cup C$ separating y_1, y_2.

Remark 5.1. The Martin topology and the projective topology on $X \cup C$ are, of course, not identical. To see this, a counterexample will be constructed where x_n can converge to $y_0 \in C$ in the Martin topology without $k\phi(x_n)$ converging to $\phi(y_0)$ for all continuous ϕ on C. Thus, in the Martin topology, a harmonic function need not assume its boundary values continuously, in contrast to the projective topology, which inherits the good behavior of $X \cup \Gamma$ with respect to boundary values of bounded harmonic functions.

Example. Let X be the set of pairs (j, k) of nonnegative integers. Let $P((j, k), \{(j, k + 1)\}) = 1$ if $k \geqq 1$, $P((j, 0), \{(j, 1)\}) = \frac{1}{2}$, $P(x, \{y\}) = 0$ in all other cases. Let ε be a finite measure positive on nonempty sets, and let $a(m, n) = \varepsilon(\{(m, n)\})$. Set

$$b(m, n) = \tfrac{1}{2}a(m, 0) + \textstyle\sum_{1 \leqq k \leqq n} a(m, k), \quad \text{and}$$

$$b(m) = \lim_{n \to \infty} b(m, n).$$

Then

$$K((j, k), (j, n)) = 1/b(j, n) \quad \text{if } 1 \leqq k \leqq n,$$

$$K((j, 0), (j, n)) = 1/2b(j, n) \quad \text{if } 1 \leqq n,$$

$$K((j, 0), (j, 0)) = 1/a(j, 0),$$

$$K(x, y) = 0 \quad \text{in all other cases.}$$

There are two types of sequences $x_i = (m_i, n_i)$ which converge to a point of B:

(a) $m_i \to \infty$, n_i arbitrary. In this case, $K(x, x_i) \to 0$ for all $x \in X$. Thus there is associated with such sequences a single limit point $\bar{y}$ in B, and $K(x, \bar{y}) = 0$ for all $x \in X$.

(b) m_i is equal to some fixed m from some point on, while $n_i \to \infty$. In this case,

$$\begin{aligned} K((m, 0), x_i) &\to 1/2b(m), & \\ K((m, k), x_i) &\to 1/b(m) & \text{if } k \geqq 1, \\ K(x, x_i) &\to 0 & \text{otherwise.} \end{aligned}$$

This gives a point $y_m \in B$.

The topology on B is the one-point compactification of the points y_m, with $\bar{y}$ playing the role of ∞. The measure μ assigns mass $b(m)$ to y_m. Thus, $C = B$.

Finally, consider the function $e = K\mu = k1$, and the sequence $x_i = (i, 0)$. Then $e(x_i)$ is $\frac{1}{2}$ for all i, while the sequence x_i converges in the Martin topology to $\bar{y}$. This is precisely the phenomenon which it was required to exhibit.

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