UNIQUENESS OF INVARIANT WEDDERBURN FACTORS

BY

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1. Introduction

In this note, an affirmative answer is given to a conjecture which the author made in the last section of [4]. Let A denote a finite-dimensional associative algebra over a field Φ. Let N denote the radical of A, and let A/N be a separable algebra. Then it is well known (the Wedderburn principal theorem) that A possesses a separable subalgebra S such that $A = S + N$, $S \cap N = \{0\}$, and $S \cong A/N$. In [4], we showed that if G is a finite group, each of whose elements is either an automorphism or an antiautomorphism of A, and whose order is not divisible by the characteristic of Φ, then the subalgebra S described above may be chosen to be invariant under the operators in G, i.e., a G-subalgebra. In general, the Malcev theorem [3] states that if S and T are two such separable subalgebras (called Wedderburn factors), then there exists an (inner) automorphism of A which carries S onto T. In [4], we conjectured that if S and T are two G-invariant Wedderburn factors, then there exists an automorphism of A, carrying S onto T, which commutes with each operator in G, i.e., a G-automorphism. In Section 4 of [4], this was proved for the special case of characteristic Φ equals zero, and G consisting of an involution of A and the identity mapping of A. Here we establish the conjecture for an arbitrary finite group G for the case of characteristic Φ equals zero.

2. Preliminaries

We assume familiarity with the notions of a nilpotent derivation, and the adjoint mapping of A into its Lie algebra of derivations. In particular, if $z \in N$, then $\exp z$ is regular (in A_1, the algebra obtained from A by adjunction of an identity, if necessary), and $\exp(\operatorname{Ad} z)$ is the inner automorphism determined by conjugation by $\exp z$.

We first note that if G contains an element which is both an automorphism and an antiautomorphism, then A is commutative. In this case, since the automorphism given by the Malcev theorem is inner, there is a unique Wedderburn factor, so that the desired result is trivial. Hence we now assume that A is not commutative, and that each element of G is either an automorphism of A or an antiautomorphism of A, but not both.

If $\tau \in G$, we extend τ to A_1 by setting $\tau(\alpha 1) = \alpha 1$ for $\alpha \in \Phi$. If $z \in A_1$, we call z G-symmetric if $\tau z = z$ for $\tau \in G$, τ an automorphism of A, and $\tau z = -z$ for $\tau \in G$, τ an antiautomorphism of A. It is easy to verify that

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the G-symmetric elements of A form a Lie algebra, i.e., they form a linear space closed under the commutator operation $[z_1, z_2] \equiv z_1 z_2 - z_2 z_1$.

Let z be a G-symmetric element of N, and $a \in A$. If $\tau \in G$, τ an automorphism of A, then

$$\begin{aligned}(\exp(\mathrm{Ad}\, z)\tau)(a) &= (\exp(-z))(\tau a)(\exp z),\\ (\tau \exp(\mathrm{Ad}\, z))(a) &= \tau(\exp(-z)(a)\exp(z))\\ &= \exp(-z)(\tau a)\exp(z).\end{aligned}$$

If $\tau \in G$, τ an antiautomorphism of A, then

$$\begin{aligned}(\exp(\mathrm{Ad}\, z)\tau)(a) &= \exp(-z)(\tau a)(\exp z),\\ (\tau \exp(\mathrm{Ad}\, z))(a) &= \tau(\exp(z)(a)\exp z)\\ &= \exp(\tau z)(\tau a)\exp(-\tau z)\\ &= \exp(-z)(\tau a)\exp z.\end{aligned}$$

Hence $\exp(\mathrm{Ad}\, z)$ is a G-automorphism of A.

If S is a G-invariant Wedderburn factor of A, and z is a G-symmetric element of N, then $\exp(\mathrm{Ad}\, z)S = \exp(-z)S(\exp z)$ is another G-invariant Wedderburn factor. It is the converse which we wish to prove. Hence we make the following definitions.

Definition. An automorphism of A which is determined by conjugation by the exponential of a G-symmetric element of N is called a *G-symmetry* of A. Two subalgebras S and T are *G-symmetric* if there is a G-symmetry of A carrying S onto T.

It is clear that the identity mapping is a G-symmetry ($I = \exp(\mathrm{Ad}\, 0)$), and that the inverse of a G-symmetry is also a G-symmetry, since $(\exp(\mathrm{Ad}\, z))^{-1} = \exp(\mathrm{Ad}(-z))$. If z_1, z_2 are G-symmetric elements of N, then by using the Baker-Hausdorff formula (see [1]), we can express the product of $\exp(\mathrm{Ad}\, z_1)$ and $\exp(\mathrm{Ad}\, z_2)$ in the form $\exp(\mathrm{Ad}\, z_3)$, where z_3 is in the Lie algebra generated by z_1 and z_2. Since the G-symmetric elements of N form a Lie algebra, it follows that the product of two G-symmetries is also a G-symmetry. Hence the G-symmetries of A form a group, and the relation of being G-symmetric is an equivalence relation among the G-subalgebras of A.

3. The uniqueness theorem

Theorem. *Let A be a finite-dimensional associative algebra over a base field Φ of characteristic zero. Let G be a finite group, each of whose elements is either an automorphism or an antiautomorphism of A. Let N be the radical of A. Let S be a separable G-invariant subalgebra of A, and let $A = T + N$ be a Wedderburn decomposition of A such that T is a G-invariant Wedderburn factor of A. Then S is G-symmetric to a G-invariant subalgebra of T.*

Proof. The result is proved on pages 570–572 of [4] for the special case of a group G of order two consisting of the identity mapping of A and an involution $a \to a^*$. The proof given there may be extended to the more general case described here, and the necessary changes will now be indicated.

The term "self-adjoint" in [4] is to be replaced by "G-invariant," "skew" by "G-symmetric," "orthogonal conjugacy" by "G-symmetry," and "orthogonally conjugate" by "G-symmetric." In the discussion pertaining to equations (3) and (5), s^* should be replaced by τs, for $\tau \in G$. Equation (6) is to be replaced by

$$\delta(\tau \bar{z}) = \begin{cases} \delta \bar{z} & \text{if} \quad \tau \in G, \tau \text{ an automorphism,} \\ -\delta \bar{z} & \text{if} \quad \tau \in G, \tau \text{ an antiautomorphism.} \end{cases} \tag{6}$$

In the proof of (6), s^* is to be replaced by either τs or $\tau^{-1} s$, whichever is needed. Finally, one replaces $z'_{k+1} = \frac{1}{2}(z - z^*)$ of [4] by

$$z'_{k+1} = (1/r) \sum_{\tau \in G} (\text{sign } \tau)(\tau z),$$

where r is the order of G, and the sign of τ is $+1$ if τ is an automorphism, and -1 if τ is an antiautomorphism. Then $\delta(\overline{z'_{k+1}}) = \delta \bar{z}$, and the result follows from equation (7) as in [4].

This theorem has the following corollaries:

COROLLARY 1. *Let S and T be two G-invariant Wedderburn factors of a finite-dimensional associative algebra A over a field Φ of characteristic zero, where G is a finite group each of whose elements is an automorphism or an antiautomorphism of A. Then there exists an (inner) automorphism of A which commutes with each element of G, and which carries S onto T. This automorphism may be chosen to be a G-symmetry of the form* exp (Ad z), *where z is a G-symmetric element of the radical of A.*

COROLLARY 2. *Let A and G be as described in Corollary* 1. *Then any G-invariant separable subalgebra of A may be embedded in a G-invariant Wedderburn factor of A.*

4. G-orthogonality

If G is any group of automorphisms and antiautomorphisms of A over a field Φ of arbitrary characteristic, then we call an element w in A_1 *G-orthogonal* if $\tau w = w$ for $\tau \in G$, τ an automorphism of A, and $\tau w = w^{-1}$ for $\tau \in G$, τ an antiautomorphism of A. The collection of G-orthogonal elements forms a multiplicative group.

DEFINITION. A *G-orthogonal conjugacy* is an inner automorphism determined by conjugation by a G-orthogonal element. Two subalgebras of A are said to be *G-orthogonally conjugate* if there exists a G-orthogonal conjugacy of A carrying one onto the other.

It is easy to verify that any G-orthogonal conjugacy commutes with each element of G.

For the case of characteristic zero as discussed in Section 2, the element $\exp z$, where z is a G-symmetric element in the radical of A, and conjugation by which yields a G-symmetry, has the property of being G-orthogonal. Hence a G-symmetry is a G-orthogonal conjugacy, and Corollary 1 may now be stated in the following form.

COROLLARY 3. *Let A and G be described as in Corollary* 1. *Then any two G-invariant Wedderburn factors of A are G-orthogonally conjugate.*

Concerning the case of characteristic $\Phi = p$, one might conjecture that Corollary 3 holds, perhaps subject to a condition on the order of G, for example, that the order of G should not be a multiple of the prime p.

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