ON THE HOMOTOPY-COMMUTATIVITY OF SUSPENSIONS

BY

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1. Introduction

It is well known that the loop space of an H-space always is homotopy-commutative. Recently, M. Sugawara [5; Theorem 8.1] has proved the following partial converse of this fact: If the loop space of a CW-complex X such that $\pi_q(X) = 0$ for $q < n$ and $q > 3n - 2$ is homotopy-commutative, then X is an H-space. In a sense, this result is the best possible since, with $n = 2$, the CW-complex obtained by killing off the homotopy groups in dimensions $\geqq 6$ of the complex projective plane fails to be an H-space even though its loop space is homotopy-commutative [1; §3.10].

It will be shown below that the suspension over a reasonable space of Lusternik-Schnirelmann category $\leqq 2$ always is homotopy-commutative,[1] and our main result consists of a partial converse of this fact:

Theorem 1. *Let X be an $(n - 1)$-connected CW-complex of dimension less than or equal to $3n - 2$ $(n \geqq 1)$. If the suspension ΣX is homotopy-commutative, then* $\operatorname{cat} X \leqq 2$.

Theorem 1 is an immediate consequence of Lemma 3.2 below and of

Theorem 2. *Let X be an $(n - 1)$-connected CW-complex of dimension less than or equal to $(k + 1)n - 2$ $(n \geqq 1)$. If* $\operatorname{conil} \Sigma X \leqq k - 1$, *then* $\operatorname{cat} X \leqq k$.

Theorem 2 is, in turn, an immediate consequence of Theorems 3 and 4 which will be stated and proved in the next sections.

As above, Theorem 1 yields the best possible result. For, let X denote the CW-complex obtained by attaching a $(3n - 1)$-cell to the wedge $S^n \vee S^n$ by means of a map in the class of the triple Whitehead product $[i_1, [i_1, i_2]]$, where i_1 and i_2 are the homotopy classes of the left and right embeddings $S^n \to S^n \vee S^n$. Evidently, X is $(n - 1)$-connected and $\dim X = 3n - 1$; according to [2; p. 450] one has $\operatorname{cat} X = 3$ but $\operatorname{w cat} X = 2$, so that, by Corollary 3.3 below, the suspension of X is homotopy-commutative. Finally, since an H-space is a space with multiplication, whereas a space has a co-multiplication if and only if it has category $\leqq 2$, Theorem 1 is the dual in the sense of Eckmann-Hilton [3] of the above result by Sugawara. However, our proofs are not dual to those given by Sugawara.

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[1] All the necessary definitions are given in the next two sections.

2. Category and weak category

All spaces, maps, and homotopies in this paper are assumed to possess or preserve a base-point, generally denoted by $*$; in a CW-complex, $*$ will always be a 0-cell. We write $f \simeq g$ if f and g are homotopic maps, and denote by 0 the constant map. For any integer $k \geqq 1$ and any space X we shall denote by X^k the k-fold Cartesian power of X, and by $T = T(X, k)$ the subset of X^k consisting of all points $(x_1, \cdots, x_k)$ such that $x_q = *$ for some q with $1 \leqq q \leqq k$. We write $j: T \to X^k$ for the inclusion map, and $\triangle: X \to X^k$ for the diagonal map which is given by $\triangle(x) = (x, \cdots, x)$. Finally, let $X^{(k)}$ and $\eta: X^k \to X^{(k)}$ denote the identification space and identification map resulting by pinching the subset T of X^k to a point, which will serve as base-point in $X^{(k)}$.

The Lusternik-Schnirelmann category cat X of any space X is the least integer $k \geqq 1$ such that X may be covered by k open subsets which are contractible in X; if no such integer exists, cat $X = \infty$. It has been pointed out by G. W. Whitehead [6] that, for a large class of spaces including all connected CW-complexes, this is equivalent to saying that cat $X \leqq k$ if and only if there is a map $\phi: X \to T(X, k)$ such that $j \circ \phi \simeq \triangle$. As in [2] we say that X has weak category $\leqq k$ and write w cat $X \leqq k$ if and only if $\eta \circ \triangle \simeq 0$. For any connected CW-complex X one has w cat $X \leqq$ cat X, but, as mentioned in the Introduction, the converse inequality may fail to hold. Nevertheless, we prove

THEOREM 3. *Let X be an $(n-1)$-connected CW-complex of dimension less than or equal to $(k+1)n - 2$ $(n \geqq 1)$. If* w cat $X \leqq k$, *then also* cat $X \leqq k$.

Proof. The result being trivial if $n = 1$, we shall assume that $n \geqq 2$. Let Y^I denote the space of all paths in any space Y. Consider the diagram

$$\begin{array}{ccccc} T & \xrightarrow{\ j\ } & X^k & \xrightarrow{\ \eta\ } & X^{(k)} \\ \Big\downarrow g & & r\Big\uparrow\Big\downarrow f & & \Big\downarrow \mathrm{id} \\ F & \xrightarrow{\ i\ } & E & \xrightarrow{\ p\ } & X^{(k)} \end{array}$$

in which

$$E = \{((x_1, \cdots, x_k), \lambda) \mid \lambda(0) = \eta(x_1, \cdots, x_k)\} \subset X^k \times (X^{(k)})^I,$$

$$f(x) = (x, \lambda_x) \qquad \text{with} \quad x = (x_1, \cdots, x_k) \quad \text{and} \quad \lambda_x(t) = \eta(x),$$

$$r((x_1, \cdots, x_k), \lambda) = (x_1, \cdots, x_k), \qquad p((x_1, \cdots, x_k), \lambda) = \lambda(1),$$

$$F = p^{-1}(*), \quad \text{and} \quad i \text{ is the inclusion map.}$$

One has $p \circ f = \eta$, so that, since $\eta \circ j(T) = *$, f defines a map g which may be regarded as an inclusion map. It is well known that

(1) *f is a homotopy equivalence with $r \circ f = \mathrm{id}$,*

(2) *the lower row in the preceding diagram is a fibration.*

In the commutative diagram

$$\begin{array}{ccccccccccc} H_{q+1}(X^k) & \to & H_{q+1}(X^k, T) & \to & H_q(T) & \to & H_q(X^k) & \to & H_q(X^k, T) & \xrightarrow{\eta'_*} & H_q(X^{(k)}, *) \\ \downarrow f_* & & \downarrow f'_* & & \downarrow g_* & & \downarrow f_* & & \downarrow f'_* & & \downarrow \mathrm{id} \\ H_{q+1}(E) & \to & H_{q+1}(E, F) & \to & H_q(F) & \to & H_q(E) & \to & H_q(E, F) & \xrightarrow{p'_*} & H_q(X^{(k)}, *) \end{array}$$

η'_* is isomorphic for all $q \geqq 0$ (see for instance [4; Lemma 1.6]). Since X and hence X^k are $(n-1)$-connected, so is E by (1); also, by the relative Künneth and the Hurewicz theorems, $X^{(k)}$ is $(kn-1)$-connected. Therefore, by (2), F is $(n-1)$-connected, and, according to a well known result by Serre, p'_* and hence f'_* are isomorphic for $q \leqq (k+1)n - 1$. The "five lemma" now implies that g_* is isomorphic for $q \leqq (k+1)n - 2$, and standard arguments yield

(3) $\pi_q(F, T) = 0$ *for* $q \leqq (k+1)n - 2.$

Since w cat $X \leqq k$, we have $\eta \circ \triangle \simeq 0$, and hence $p \circ f \circ \triangle \simeq 0$. It follows from (2) that there exists a map $\psi: X \to F$ such that $i \circ \psi \simeq f \circ \triangle$. By (3) and since dim $X \leqq (k+1)n - 2$, a standard deformation argument yields a map $\phi: X \to T$ such that $g \circ \phi \simeq \psi$. We have

$$j \circ \phi = r \circ f \circ j \circ \phi = r \circ i \circ g \circ \phi \simeq r \circ i \circ \psi \simeq r \circ f \circ \triangle = \triangle,$$

and Theorem 3 is proved.

3. Weak category and co-nilpotency of suspensions

Let X be an arbitrary space with base-point $*$. The reduced suspension ΣX is the identification space obtained by pinching to a point the subset $0 \times X \cup 1 \times X \cup I \times *$ of the Cartesian product $I \times X$; the image in ΣX of $(s, x) \in I \times X$ will be denoted by $\langle s, x \rangle$. The co-multiplication and co-inversion maps

$$\sigma: \Sigma X \to \Sigma X \vee \Sigma X \quad \text{and} \quad \tau: \Sigma X \to \Sigma X$$

given by

$$\begin{aligned} \sigma\langle s, x \rangle &= (\langle 2s, x \rangle, *) && \text{for } 0 \leqq 2s \leqq 1, \\ &= (*, \langle 2s - 1, x \rangle) && \text{for } 1 \leqq 2s \leqq 2, \\ \tau\langle s, x \rangle &= \langle 1 - s, x \rangle && \text{for } 0 \leqq s \leqq 1, \end{aligned}$$

convert ΣX into an H'-space, the dual in the sense of Eckmann-Hilton [3] of a homotopy-associative H-space with homotopy inversion. For any H'-space Y and any $k \geqq 1$ we have defined inductively in [1; Definition 1.4] a co-commutator map ψ_k of weight k; ψ_1 is the identity map of Y, ψ_{k+1} is the composition

$$Y \xrightarrow{\psi} Y \vee Y \xrightarrow{\psi_k \vee \mathrm{id}} {}^kY \vee Y = {}^{k+1}Y$$

in which ${}^1Y = Y$, and, in case $Y = \Sigma X$, ψ is given by

$$\begin{aligned} \psi\langle s, x\rangle &= (\langle 4s, x\rangle, *) && \text{for } 0 \leqq 4s \leqq 1, \\ &= (*, \langle 4s - 1, x\rangle) && \text{for } 1 \leqq 4s \leqq 2, \\ &= (\langle 3 - 4s, x\rangle, *) && \text{for } 2 \leqq 4s \leqq 3, \\ &= (*, \langle 4 - 4s, x\rangle) && \text{for } 3 \leqq 4s \leqq 4. \end{aligned}$$

The co-nilpotency class conil ΣX is the least integer $k \geqq 0$ such that $\psi_{k+1} \simeq 0$; if no such integer exists, conil $\Sigma X = \infty$ [1; Definition 1.8].

DEFINIITION 3.1. *The suspension ΣX is homotopy-commutative if $\varepsilon \circ \sigma \simeq \sigma$, where $\varepsilon: \Sigma X \vee \Sigma X \to \Sigma X \vee \Sigma X$ is given by*

$$\varepsilon(y, *) = (*, y) \quad \textit{and} \quad \varepsilon(*, y) = (y, *) \qquad \textit{for} \quad y \in \Sigma X.$$

LEMMA 3.2. *The suspension ΣX is homotopy-commutative if and only if* conil $\Sigma X = 1$.

Proof. The set $\pi(\Sigma X, Y)$ of based homotopy classes of maps of ΣX into an arbitrary space Y may be converted into a (non-Abelian) group by setting

$$\{f\} + \{g\} = \{\nabla \circ (f \vee g) \circ \sigma\} \quad \text{and} \quad -\{f\} = \{f \circ \tau\};$$

here, $\{h\}$ is the homotopy class of the map $h: \Sigma X \to Y$ and $\nabla: Y \vee Y \to Y$ is given by $\nabla(y, *) = \nabla(*, y) = y$. The zero of the group $\pi(\Sigma X, Y)$ is the homotopy class of the constant map. With $Y = \Sigma X \vee \Sigma X$, it is easy to check that $\psi = \nabla \circ (\sigma \vee \varepsilon \circ \sigma \circ \tau) \circ \sigma$; therefore,

$$\{\psi\} = \{\sigma\} - \{\varepsilon \circ \sigma\},$$

so that $\{\psi\} = 0$ if and only if $\{\sigma\} = \{\varepsilon \circ \sigma\}$.

As an immediate consequence of Lemma 3.2 and of [1; Theorem 6.13] we have the following corollary which, in fact, is valid for a more general class of spaces.

COROLLARY 3.3. *Let X be a connected CW-complex. If* w cat $X \leqq 2$ (*or if* cat $X \leqq 2$), *then ΣX is homotopy-commutative.*

We now prove

THEOREM 4. *Let X be an $(n - 1)$-connected CW-complex of dimension less than or equal to $2kn - 2$ $(n \geqq 1)$. If* conil $\Sigma X \leqq k - 1$, *then* w cat $X \leqq k$.

Proof. Let ΩY denote the loop space of any space Y with base-point. Since conil $\Sigma X \leqq k - 1$, the co-commutator map ψ_k of weight k is nullhomotopic. As ${}^k\Sigma X = \Sigma({}^kX)$, this implies that the map

$$\phi: X \to \Omega\Sigma({}^kX),$$

given by $\phi(x)(s) = \psi_k\langle s, x\rangle$, also is nullhomotopic. As shown by [1; Proposition 2.17], in the diagram below there is a map γ such that $\phi = \gamma \circ \triangle$.

$$\begin{array}{ccccc} X & \xrightarrow{\triangle} & X^k & \xrightarrow{\gamma} & \Omega\Sigma({}^kX) \\ & & \Big\downarrow{\scriptstyle\eta} & & {\scriptstyle\rho}\Big\uparrow\Big\downarrow{\scriptstyle r} \\ & & X^{(k)} & \xrightarrow{e} & \Omega\Sigma(X^{(k)}) \end{array}$$

Also, by [1; Lemma 6.9], we have $\gamma \mid T(X, k) \simeq 0$, and there results a map $d: X^{(k)} \to \Omega\Sigma({}^kX)$ such that $d \circ \eta \simeq \gamma$. As is easily seen, there is a map ρ such that $\rho \circ e = d$, where e is the natural embedding given by $e(y)(s) = \langle s, y\rangle$ for $y \in X^{(k)}$. It follows from results by Milnor[2] (the construction FK) that there is a map r such that $r \circ \rho \simeq \mathrm{id}$. Therefore,

$$e \circ \eta \circ \triangle \simeq r \circ \rho \circ e \circ \eta \circ \triangle \simeq r \circ \gamma \circ \triangle = r \circ \phi \simeq 0. \tag{4}$$

Since X is $(n-1)$-connected, $e_q: \pi_q(X^{(k)}) \to \pi_q(\Omega\Sigma(X^{(k)}))$ is monomorphic for $q \leqq 2kn - 2$ and epimorphic for $q \leqq 2kn - 1$. Since X is a CW-complex of dimension $\leqq 2kn - 2$, it follows now from (4) that $\eta \circ \triangle \simeq 0$, i.e.,

$$\mathrm{w\,cat}\, X \leqq k$$

as asserted.

Remark. As shown in [2; p. 450], the 5-dimensional polyhedron X obtained by attaching to S^2 a 5-cell by means of a map in the class generating $\pi_4(S^2)$ has vanishing 2-fold cup products but w cat $X = 3$. Therefore, by Theorem 4, conil $\Sigma X = 2$ so that $\smile$-long $X <$ conil ΣX; the general inequality

$$\smile\text{-long } X \leqq \text{conil } \Sigma X$$

is proved in [1; Theorem 5.8]. However, we know of no example X such that conil $\Sigma X <$ w cat $X - 1$.

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