POINT REALIZATIONS OF TRANSFORMATION GROUPS

BY
GEORGE W. MACKEY

Introduction

Let G be a group. By a G-space we shall mean a set S together with a mapping β of $S \times G$ into S such that, if we write $\beta(x, y) = [x]y$, we have

(a) $[x]y_1 y_2 = [[x]y_1]y_2$ for all x, y_1, y_2 in $S \times G \times G$.
(b) $[x]e = x$ for all x in S where e is the identity of G.

Now let S be a measure space as well as a G-space, and suppose that for each y the mapping $x \to [x]y$ preserves measurability and null sets. Then the σ-Boolean algebra B of measurable sets modulo null sets is itself a G-space in a natural way, and each $b \to [b]y$ is an automorphism of B. We shall call B a Boolean G-space, and S a point realization of B. Given an abstract Boolean G-space one can ask whether or not it has point realizations, and to what extent these realizations can differ. These questions are of interest because in dealing with measurable transformation groups it is usually more elegant to take the Boolean-algebra point of view but often very convenient to have points available. In this note we shall show that very satisfying answers can be given when S is a standard Borel space in the sense defined in [8],[1] G is separable and locally compact, and β satisfies a natural measurability condition.

For the special case in which G is countable our results follow easily from known ones in [4]. Moreover when G is the group of integers, a result for highly nonseparable S has been given by Maharam in [10]. Von Neumann in his classic paper [11] treats the case in which G is the additive group of the real line and there is an invariant measure. Our main theorem in that case is a refinement of his in that no exceptional null sets occur. Indeed that is just the point of our contribution—getting rid of the exceptional null sets in the continuous case. This has been done by Doob [3] when G is the real line, but his space S is much too big to be a standard Borel space. A central construction in Doob's proof is analogous to a similar construction in ours, but we get a smaller space by identifying functions which are equal almost everywhere, and we use quite different arguments to complete the proof.

Our main result is stated in Section 3. The associated uniqueness theorem is stated and proved in Section 5. Sections 6 and 7 deal with some simple applications.

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[1] It has been called to our attention that ideas very close to, if not identical with, those developed in the first few sections of [8] were found independently by Blackwell and published in [2]. However the applications of these ideas given in [8] and [2] are quite distinct.

1. Borel G-spaces and their associated Boolean G-spaces

Let G be a separable locally compact group. We shall call the G-space S a Borel G-space if S is equipped with a Borel structure (that is, a σ-field of subsets which separates points) such that the mapping taking x, y into $[x]y$ is a Borel function. Let μ be a finite measure defined on all the Borel sets in the Borel G-space S. Suppose that μ is quasi-invariant in the sense that $\mu(N) = 0$ implies $\mu([N]y) = 0$ for all $y \in G$ and all Borel subsets N of S. For each Borel subset E of S let $\tilde{E}$ denote the set of all Borel sets which differ from E by a null set, and let B_μ denote the σ-Boolean algebra of all $\tilde{E}$. Setting $[\tilde{E}]y = [[E]y]^{\sim}$ we convert B_μ into a G-space. In general a G-space B which is also a σ-Boolean algebra will be said to be a Boolean G-space if the map taking E into $[E]y$ is a Boolean-algebra automorphism for each $y \in G$, and if $\mu([E]y)$ is a Borel function of y for each $E \in B$ and each finite measure μ on B.

LEMMA 1. *The Boolean algebra B_μ is a Boolean G-space under the action described above.*

Proof. It is obvious that $\tilde{E} \to [\tilde{E}]y$ is an automorphism of B_μ. Let ν be any finite measure on B. For each Borel set E in S let $\nu_1(E) = \nu(\tilde{E})$. Then ν_1 is a finite measure on the Borel sets of S, and $\nu([\tilde{E}]y) = \nu_1([E]y)$. Let $T(x, y) = [x]y, y$, and let $T_1(x, y) = [x]y^{-1}, y$. Then T and T_1 are Borel maps of $S \times G$ onto $S \times G$, and each is the inverse of the other. Hence each is a Borel automorphism of $S \times G$. Let ψ denote the characteristic function of the Borel set $T(E \times G)$. By one form of the Fubini theorem $\int \psi(x, y)\, d\nu_1(x)$ is a Borel function of y. But

$$T(E \times G) \cap (S \times y) = [E]y \times y.$$

Hence

$$\int \psi(x, y)\, d\nu_1(x) = \nu_1([E]y).$$

Hence $\nu([\tilde{E}]y) = \nu_1([E]y)$ is a Borel function of y, and the proof is complete.

We shall call B_μ the Boolean G-space associated with S and the quasi-invariant measure μ.

Let S_1 and S_2 be Borel G-spaces, and let T be a Borel isomorphism of S_1 on S_2 (as Borel spaces). If $T([x]y) = [T(x)]y$ for all $x, y \in S_1 \times G$, we shall say that T is an equivalence of S_1 on S_2, and that S_1 and S_2 are equivalent as Borel G-spaces. Similarly if B_1 and B_2 are Boolean G-spaces and T is a Boolean-algebra isomorphism of B_1 onto B_2, we shall say that T is an equivalence if $T([E]y) = [T(E)]y$ for all $E, y \in B_1 \times G$, and that B_1 and B_2 are equivalent as Boolean G-spaces.

Let E be a Borel subset of the Borel G-space S such that $[E]y = E$ for all $y \in G$. Then E is itself a Borel G-space. We shall say that it is a sub-G-space of S.

2. Universal Borel G-spaces

For each separable locally compact group G let F_G denote the set of all real-valued functions on G which are square summable on compact sets and where functions which are almost everywhere equal have been identified.[2] For each compact subset K of G let

$$\| f \|_K^2 = \int_K | f(x) |^2 \, dx.$$

Then $\| \quad \|_K$ is a pseudo norm for F_G regarded as a vector space. The collection of all of these pseudo norms defines a topology in F_G under which it is a complete separable metric space. In particular F_G is a standard Borel space. Setting $[f]y(x) = f(yx)$ makes F_G into a G-space. That $[f]y$ is a Borel function of both variables and hence that F_G is a Borel G-space follows from the obvious continuity in f and an evident minor modification of the proof of Lemma 9.2 of [7]. We shall call this G-space the universal G-space. This terminology is justified by

LEMMA 2. *Any Borel G-space S, such that S as a Borel space is analytic, is equivalent to a subspace of the universal G-space.*

Proof. Without loss of generality we may suppose that S is an analytic subset of the unit interval. Then for each fixed x in S the mapping $y \to [x]y$ will be a Borel function f_x from G to the unit interval. Let $\tilde{f}_x$ denote the corresponding member of F_G, that is, the equivalence class of locally square summable functions which contains f_x. We shall show that $x \to \tilde{f}_x$ sets up the desired equivalence. It is obvious that the mapping is one-to-one and that it commutes with the action of G. Thus we have only to show that it is a Borel mapping and has a Borel inverse. Since S is analytic and F_G is standard, it will suffice ([8], Theorem 4.2) to show that the mapping is Borel. Let $\tilde{g}$ be any member of F_G such that g has compact support. Then the mapping taking $\tilde{f}$ into $\int f(y)g(y)\,dy$ is a continuous function on F_G, and there exists a countable sequence of such functions which separates the points of F_G. On the other hand, for each g, the composite mapping

$$x \to f_x \to \int f_x(y)g(y)\,dy = \int [x]yg(y)\,dy$$

is a Borel function of x by the Fubini theorem.

3. Statement of main theorem and first part of proof

It is well known that the following properties that a σ-Boolean algebra might have are equivalent: (a) It is isomorphic to a complete Boolean algebra of projections in a separable Hilbert space. (b) It is countably generated

[2] The use of this space was suggested to us by Section 4 of Part A of [1].

and admits a nowhere-zero finite measure. (c) It is the algebra of Borel sets modulo null sets for a finite measure μ in a standard Borel space. A σ-Boolean algebra with any and hence all of these properties will be said to be standard.

THEOREM 1. *Let G be a separable locally compact group, and let B be a Boolean G-space which is standard as a σ-Boolean algebra. Then there exist a Borel G-space S and a finite quasi-invariant measure μ in S such that B is equivalent to the Boolean G-space associated with S and μ.*

The theorem is trivial for the atomic part, if any, of B, so we may assume that B is atom-free. However, as is well known, any two atom-free complete Boolean algebras of projections in a separable infinite-dimensional Hilbert space are isomorphic. Thus we may assume without loss of generality that B is the Boolean algebra of Borel sets mod Lebesgue null sets on the unit interval I. Let μ_0 denote Lebesgue measure in I. In this section we shall construct a map of I into the universal G-space F_G and use it to map μ_0 into a measure μ on F_G. In Section 4 we shall complete the proof of Theorem 1 by showing that μ is quasi-invariant and that the Boolean G-space associated with F_G and μ is equivalent to B.

For each y in G let α_y denote the mapping $E \to [E]y$. If α_y preserves μ_0, then by a well known result of Halmos and von Neumann [4] there exists a one-to-one map β_y' of almost all of I onto almost all of I which is measurable and measure-preserving in both directions and which induces the mapping α_y. Actually, as is made quite explicit in the proof, one can choose the exceptional null sets M and N so that they are Borel sets, and so that β_y' is a Borel isomorphism of $I - M$ with $I - N$. It is obviously possible to choose M and N so that neither is finite or countable, and it then follows from a result of Kuratowski (Remark 1 on page 358 of [5]) that M and N are Borel isomorphic. Thus β_y' may always be extended so as to be a Borel automorphism β_y of I. If α_y does not preserve μ_0, an easy argument (see Lemma 4 of [10]) shows that β_y' (and hence β_y) still exists but of course only preserves the null sets of μ_0. It follows at once from the almost-everywhere uniqueness of each β_y (Lemma 5 of [10]) that for each y and z we have

$$(*) \qquad \beta_{yz}(x) = \beta_z(\beta_y(x)) \qquad \text{for almost all } x \text{ in } I.$$

If we knew that $(*)$ held for all triples x, y, z and that $\beta_y(x)$ were a Borel function on $I \times G$, the proof would now be complete. However we do not, and this is just the difficulty which necessitates the involved argument which follows.

Let $g'(x, y) = \beta_y(x)$. Our next main step will be to prove the existence of a Borel function g on $I \times G$ such that for almost all y, $g(x, y) = g'(x, y)$ for almost all x. Since I is the unit interval, g and g' are real-valued functions. Let E be any Borel subset of I, and let ϕ_E denote its characteristic function. Let ν be any Borel measure in I which is finite and absolutely continuous with respect to μ_0. Then since $\nu(\alpha_y(E)) = \nu(\beta_y(E))$ and the former is a Borel function of y, it follows that $\int \phi_E(g'(x, y))\, d\nu(x)$ is a Borel function of y. Taking $\nu(E)$ to be $\mu_0(E \cap F)$ we see that $\int_F \phi_E(g'(x, y))d\mu_0(x)$ is a Borel function of y for all Borel subsets E and F of I. Hence by Lemma 3.1 of [6] there exists for each Borel set E a Borel function f_E on $I \times G$ such that for almost all y in G we have $\phi_E(g'(x, y)) = f_E(x, y)$ for almost all x in I. Let $E_1, E_2, \cdots$ be a sequence of Borel sets such that $\phi_{E_j}(x) = \phi_{E_j}(x')$ for all j implies $x = x'$. Let T_1 denote the Borel mapping of I into the space of all sequences of 0's and 1's (with the product Borel structure) which takes x into $\phi_{E_1}(x), \phi_{E_2}(x), \cdots$, and let T_2 denote the Borel mapping of $I \times G$ into this sequence space which maps x, y into $f_{E_1}(xy), f_{E_2}(xy), \cdots$. Then T_1 is one-to-one and hence has a Borel inverse. Moreover for almost all y,

$$T_1(g'(x, y)) = T_2(x, y)$$

for almost all x. Hence for almost all y, $g'(x, y) = T_1^{-1}T_2(x, y)$ for almost all x. Let T_3 coincide with T_1^{-1} on the Borel subset on which T_1^{-1} is defined, and let T_3 be zero elsewhere. Then the function g such that $g(x, y) = T_3\, T_2(x, y)$ clearly has the required properties.

Now let $f_x(y) = g(x, y)$, so that f_x is a bounded real-valued function on G for each x, and let $\tilde{f}_x$ denote the member of F_G which it defines. Then the mapping π which takes x into $\tilde{f}_x$ is a Borel mapping from I to F_G, as can be seen by repeating the argument contained in the last few lines of the proof of Theorem 1. We define μ to be the measure in F_G which assigns the measure $\mu_0(\pi^{-1}(F))$ to each Borel set F.

4. Completion of proof of main theorem

To show that μ is quasi-invariant we begin by writing equation (*) of the preceding section in the form

$$g'(x, yz) = g'(\beta_y(x), z).$$

Now for each y and almost all z we have $g'(x, yz) = g(x, yz)$ for almost all x, and $g'(\beta_y(x), z) = g(\beta_y(x), z)$ for almost all x. Hence for each y and almost all z we have $g(x, yz) = g(\beta_y(x), z)$ for almost all x. But for each y, each side of this last equation is a Borel function on $I \times G$. Hence for each y and almost all x we have the same equation for almost all z. Hence for each y and almost all x, $[\tilde{f}_x]y = \tilde{f}_{\beta_y(x)}$. Let N be any Borel subset of F_G with $\mu(N) = 0$, and let $N' = \pi^{-1}(N)$ so that $\mu_0(N') = 0$. Given any y in G let N'' denote the set of all x with $\tilde{f}_x \in [N]y^{-1}$; that is, $[\tilde{f}_x]y \in N$. Because of the above equation N'' differs by a set of μ_0-measure zero from the set N''' of all x with $\tilde{f}_{\beta_y(x)} \in N$. But $x \in N'''$ if and only if $\beta_y(x) \in N'$; that is, $x \in \beta_y^{-1}(N')$. Thus $\mu_0(N''') = 0$. Hence $\mu_0(N'') = 0$. Hence $\mu([N]y^{-1}) = 0$. Thus μ is quasi-invariant as was to be proved.

Let $\tilde{B}$ denote the Boolean G-space associated with F_G and the quasi-invariant measure μ. We shall show that the mapping $E \rightarrow \pi^{-1}(E)$ from Borel subsets of F_G to Borel subsets of I defines an isomorphism between the σ-Boolean algebras $\tilde{B}$ and B which sets up an equivalence between them as G-spaces. It is routine to verify that if E and F are Borel subsets of F_G which differ by a μ-null set, then $\pi^{-1}(E)$ and $\pi^{-1}(F)$ differ by a null set, so that π^{-1} induces a mapping π^* from $\tilde{B}$ into B. It is equally easy to see that π^* is one-to-one and preserves the countable Boolean operations. To complete the proof we must show that π^* maps B onto all of B and that it commutes with the action of G.

Since π is a Borel function from one standard Borel space to another, its graph is a Borel set ([5], p. 291). Moreover $\pi(I)$ is an analytic subset of F_G and hence differs from a Borel subset of F_G by a Borel null set. Thus we may apply the reformulation of the von Neumann selection lemma given as Theorem 6.3 of [8] and deduce the existence of a Borel set N in F_G and a Borel subset A of $\pi^{-1}(F_G - N) = I - \pi^{-1}(N)$ such that π is one-to-one on A, $\mu(N) = 0$, and $\pi(A) = \pi(I - \pi^{-1}(N))$. Given any Borel set $E \subseteq I - \pi^{-1}(N)$ we see that $\pi(E) = \pi(E \cap A)$ and hence is a Borel set. Thus $\pi^{-1}(\pi(E))$ is a Borel set, and so is $E' = \pi^{-1}(\pi(E)) - E$. For each x in E' let $\theta(x)$ be the unique member of $E \cap A$ with $\pi(x) = \pi(\theta(x))$. The case in which E' is empty is of no interest to us because then $E = \pi^{-1}(\pi(E))$, and the corresponding Boolean-algebra element is accordingly in the range of π^*. θ is obviously a Borel function. Let $g^0(x, y) = g(x, y)$ whenever x is in E, and let $g^0(x, y) = g(\theta(x), y)$ whenever x is in E'. Then g^0 is a Borel function on $\pi^{-1}(\pi(E)) \times F_G$ and for almost all x is equal to $g(x, y)$ for almost all y. Hence g and g^0 are almost everywhere equal. But for almost all y, $g(x, y)$ is almost everywhere equal to $\beta_y(x)$ and hence is one-to-one outside of a null set. Hence for almost all y, $g^0(x, y)$ is one-to-one outside of a null set. Choose y_0 and a Borel null set E_0 such that $g^0(x, y_0)$ is a one-to-one function of x on $(E \cup E') - E_0$. Then we have

$$\theta(E' - (E_0 \cap E')) \subseteq E_0 \cap E$$

since $g^0(\theta(x), y_0) = g^0(x, y_0)$ for all x in E'. Thus

$$\pi^{-1}(\pi(E - E \cap E_0)) \subseteq E \cup (E' \cap E_0).$$

Thus $E - E \cap E_0$ differs by a null set from $\pi^{-1}(\pi(E - E \cap E_0))$. Thus E differs by a null set from $\pi^{-1}(\pi(E - E \cap E_0))$. Thus the Boolean-algebra element defined by E is in the range of π^*. Thus π^* is onto.

To show that π^* defines an equivalence between $\tilde{B}$ and B as Boolean G-spaces, let E be any Borel set in F_G. Then $\pi^{-1}([E]y)$ is the set of all x with $[\pi(x)]y^{-1} \in E$. But for each y in G we have $[\pi(x)]y^{-1} = \pi(\beta_y^{-1}(x))$ for almost all x. Thus $\pi^{-1}([E]y)$ differs by a null set from the set of all x with $\beta_y^{-1}(x) \in \pi^{-1}(E)$ and hence differs by a null set from the set of all x with $x \in \beta_y(\pi^{-1}(E))$. Thus $\pi^{-1}([E]y)$ and $\beta_y(\pi^{-1}(E))$ define the same Boolean-algebra element. But this shows that π^* sets up the desired equivalence.

5. Essential uniqueness of the Borel G-space associated with a Boolean G-space

In this section we shall show that the system S of Theorem 1 is "as unique" as one could hope.

THEOREM 2. *Let S_1 and S_2 be standard Borel G-spaces where G is separable and locally compact. Let μ_1 and μ_2 be quasi-invariant finite Borel measures in S_1 and S_2, and let B_1 and B_2 be the associated Boolean G-spaces. Let ϕ be an isomorphism of B_1 on B_2 as Boolean algebras which sets up an equivalence between B_1 and B_2 as Boolean G-spaces. Then there exist invariant Borel subsets S_1' and S_2' of S_1 and S_2 respectively and an equivalence θ of S_1' and S_2' as Borel G-spaces such that $S_1 - S_1'$ and $S_2 - S_2'$ have measure zero, and such that the mapping of B_1 on B_2 defined by θ is equal to ϕ.*

Proof. As in the proof of Theorem 1 we may restrict attention to the case in which B_1 and B_2 are free of atoms. By the argument given at the beginning of the proof of Theorem 1 there exists a Borel isomorphism θ of S_1 on S_2 which defines the given Boolean-algebra isomorphism. Let ψ_2 be a Borel isomorphism of S_2 with the unit interval, and let $\psi_1 = \psi_2 \circ \theta$ where $\circ$ denotes composition of mappings. For each $x \in S_i$ let $\tilde{f}_x^i$ denote the member of F_G defined by the function which takes y into $\psi_i([x]y)$. Then

$$\psi_1([x]y) = \psi_2(\theta([x]y)) = \psi_2([\theta(x)]y) \qquad \text{for almost all } x.$$

Hence $\tilde{f}_x^1 = \tilde{f}_{\theta(x)}^2$ for almost all $x \in S_1$. Moreover the mapping π_i which takes x in S_i into $\tilde{f}_x^i$ sets up a G-space equivalence between S_i and a subspace of F_G. Let R denote the intersection of the ranges of π_1 and π_2, and let $S_i' = \pi_i^{-1}(R)$. Then S_i' is a sub-G-space of S_i. Moreover S_i' contains the set of all x with $\tilde{f}_x^1 = \tilde{f}_{\theta(x)}^2$ and hence is the complement of a null set. Since π_1 restricted to S_1' and π_2 restricted to S_2' are equivalences of S_1' and S_2' with the same invariant subspace of F_G, it is clear that θ restricted to S_1' is an equivalence of S_1' with S_2'. This completes the proof.

6. Ergodicity and irreducibility

It is natural to define a Boolean G-space to be irreducible if $[E]y = E$ for all y in G implies that $E = 0$ or $E = 1$. Suppose that our Boolean G-space B is that associated with a quasi-invariant measure in a Borel G-space S. According to a common definition of ergodicity (metric transitivity) S is ergodic if and only if every measurable subset of S which is invariant under the action of G is either a null set or the complement of a null set. Since it is not clear that an invariant Boolean-algebra element can be defined by an invariant measurable set, one has the unpleasant possibility that ergodicity for S in this sense could differ from irreducibility of B. Our next theorem shows that when S is standard, this unpleasantness does not occur.

THEOREM 3. *Let S be a standard Borel G-space where G is separable and locally compact. Let μ be a finite quasi-invariant measure defined on the Borel subsets of S. Let E be a Borel set in S such that the corresponding Boolean-algebra element is invariant under G. Then E differs by a null set from a Borel set which is invariant under G.*

Proof. Let B denote the Boolean G-space defined by S and μ. Let B_1 and B_2 denote the Boolean subspaces defined by E and $S - E$. By Theorem 1 there exist standard Borel spaces S_1 and S_2 and quasi-invariant measures in them μ_1 and μ_2 such that B_i is equivalent as a Boolean G-space to the Boolean G-space associated with S_i and μ_i. We may suppose that S_1 and S_2 are disjoint as sets and convert their point-set sum into a G-space in the obvious way. Clearly the Boolean G-space associated with $S_1 \cup S_2$ and the measure ν which coincides in S_i with μ_i is equivalent to B. By Theorem 2 there exist invariant Borel null sets N and N' in S and $S_1 \cup S_2$ respectively, so that $S - N$ is equivalent as a G-space to $(S_1 \cup S_2) - N'$. Let E' be the Borel set in $S - N$ corresponding to $S_1 - (S_1 \cap N')$. Then E' will be invariant under G and differs by a null set from E.

7. An application to systems of imprimitivity

Let $x \to U_x$ denote a strongly continuous unitary representation of the separable locally compact group G acting in the Hilbert space $H(U)$. Let S be a Borel G-space. A system of imprimitivity for U based on S is ([9], p. 278) a projection-valued measure $E \to P_E$ defined on the Borel sets in S such that

$$U_y P_E U_y^{-1} = P_{[E]y^{-1}} \qquad \text{for all } E \text{ and } y.$$

Let R_P denote the range of P, that is, the set of all P_E for E ranging over the Borel subsets of S. Clearly R_P is a complete Boolean algebra of projections which is invariant under U in the sense that $Q \in R_P$ implies $U_y Q U_y^{-1} \in R_P$ for all y in G.

THEOREM 4. *Let G and U be as above, and let B be any complete Boolean algebra of projections which is invariant under U. Then B is the range of a system of imprimitivity P for U based on a standard Borel G-space.*

Proof. Make B into a G-space by setting $[Q]y = U_y^{-1} Q U_y$. It is well known that given any finite measure μ in B we may find a vector ϕ in $H(U)$ such that $\mu(Q) = (Q(\phi), \phi)$ for all Q in B. Thus

$$\mu([Q]y) = (QU_y(\phi), U_y(\phi)) = \sum_{j=1}^{\infty} (U_y(\phi), Q(\theta_j))(\theta_j, U_y(\phi))$$

where $\theta_1, \theta_2, \cdots$ is a complete orthonormal system for $H(U)$. Thus $\mu([Q]y)$ is a Borel function of y, and B is a Boolean G-space. By Theorem 1 there exist a standard Borel G-space S and a quasi-invariant measure ν in S such that B is equivalent as a Boolean G-space to that associated with S and ν. For each Borel subset E of S let P_E be the member of B associated with the

Boolean-algebra element defined by E by the given equivalence. Clearly the mapping $E \to P_E$ is the desired system of imprimitivity.

THEOREM 5. *Let G and U be as in Theorem 4, and let P^1 and P^2 be systems of imprimitivity for U based on standard Borel G-spaces S_1 and S_2. Suppose that P^1 and P^2 have the same range B. Then there exist invariant Borel subsets S_1' and S_2' of S_1 and S_2 respectively and an equivalence θ of S_1' and S_2' such that P^j maps $S_j - S_j'$ into the zero projection and $P_E^2 = P_{\theta(E)}^1$ for all E.*

Proof. Let μ be a measure in B which is zero only at the zero element. Let μ_j be the measure in S_j taking E into $\mu(P_E^j)$. Then μ_1 and μ_2 are quasi-invariant measures in S_1 and S_2 respectively, and $E \to P_E^j$ sets up an equivalence between the Boolean G-space associated with S_j and μ_j on the one hand, and the Boolean G-space B on the other. These two equivalences define an equivalence between the Boolean G-space associated with S_1 and μ_1 and that associated with S_2 and μ_2. The θ, S_1', and S_2' supplied by Theorem 2 clearly have the required properties.

It follows from Theorems 4 and 5 that the notion of system of imprimitivity for group representations is essentially equivalent to the notion of an invariant complete Boolean algebra of projections.

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HARVARD UNIVERSITY
CAMBRIDGE, MASSACHUSETTS