CONVERGENCE OF QUANTIFIERS AND MARTINGALES

BY

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1. Introduction

The theory of quantifiers in a Boolean algebra has wide application in mathematics. As introduced by Halmos [4], quantifiers were designed to give a precise algebraic version of some fundamental results in formal logic. Using this concept in formulating the notion of a polyadic Boolean algebra, Halmos was able to show that the semantic completeness theorem of Gödel is the statement that a polyadic algebra is semisimple, a fact which is remarkably easy to prove [5], [14]. Incompleteness theorems, on the other hand, take the form that certain polyadic algebras are not simple [6].

Quantifiers can be used elsewhere. In ergodic theory, for example, the Poincaré recurrence theorem holds for a measurable transformation if and only if the transformation generates a quantifier [16]. Starting from duality theory, the notion of a quantifier can be linearized to apply to real functions. This linearized form has an intimate connection with the Weierstrass-Stone theorem. The logical concept of a constant for a quantifier linearizes into the notion of a generalized mean, which is a kind of interpolating operator. The linearized version and its Boolean prototype meet in the study of bounded measurable functions, yielding some generalizations of the notion of conditional expectation [17].

In this paper, we consider an application of these ideas to the convergence of martingales in probability theory. A class of mappings of a Boolean algebra even more general than quantifiers will be considered. We show that the underlying nature of martingale convergence is as much order-theoretic as it is measure-theoretic. The convergence of conditional expectations is, in fact, a consequence of the convergence of certain existential and universal quantifiers.

2. Mappings of partially ordered sets

Let S be any partially ordered set, and let f be a mapping of S into itself. The mapping f is called *extensive* if $p \leqq fp$ for each p in S, *idempotent* if $ffp = fp$ for each p in S, and *isotone* if $p \leqq q$ implies $fp \leqq fq$ for any p, q in S. An extensive, idempotent, and isotone mapping is called an *extensional closure* on the set S.

These concepts originated with E. H. Moore [11], who considered such mappings in the Boolean algebra of all subsets of a given set. For general lattices, they have been considered by M. Ward [13]. Extensional closures

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are usually called, simply, closures. We prefer to use the latter term for a more restricted class of mappings which are closer in nature to the properties of a closure in the topological sense. In this usage we follow Tarski [7], [10] and Halmos [4].

A subset B of a partially ordered set S is called *relatively complete* if, for each element p of S, the set $B(p) = \{q \in B : p \leqq q\}$ has a smallest member. This terminology is from Halmos [4], who considered the relatively complete subalgebras of a Boolean algebra. The following extends theorems of E. H. Moore, M. Ward, and Halmos; the obvious proof may be omitted.

THEOREM. *Let f be an extensional closure of a partially ordered set S, and let $B = \{p \in S : fp = p\}$ be the range of f. Then B is a relatively complete subset of S. Conversely, if B is a relatively complete subset of a partially ordered set S, then the mapping f which assigns to each p in S the smallest member of $B(p)$ is an extensional closure of S. This correspondence is one-one.*

Let S be a partially ordered set containing a smallest element 0 (that is, $0 \leqq p$ for each p in S). A mapping f of S into itself will be called *normalized* if $f0 = 0$. A subset B of S will be called normalized if 0 is a member of B. It is clear that an extensional closure is normalized if and only if its range is a normalized subset.

A *semilattice* is a partially ordered set S in which every pair p, q of elements of S has a greatest lower bound, or infimum, denoted by $p \wedge q$. A *complete semilattice* is a partially ordered set in which every nonempty subset P has a greatest lower bound, or infimum, which will be denoted either as $\bigwedge P$ or as $\inf P$. A subset B of a semilattice S is called a *subsemilattice* of S if B is not empty and if $p, q \in B$ imply $p \wedge q \in B$. A subsemilattice B of a complete semilattice S is called a *complete subsemilattice* of S if, for each nonempty subset P of B, the element $\inf P$, as computed in S, belongs to B. Note that a complete semilattice always has a least element 0.

THEOREM. *If S is a complete semilattice, a subset B of S is a relatively complete subset of S if and only if B is a complete subsemilattice of S.*

Proof. If S is a complete semilattice, and if B is a relatively complete subset of S, let P be any nonempty subset of B, and let $a = \inf P$. Let f be the extensional closure defined by B. Since $a \leqq p$ for each p in P, then $fa \leqq fp = p$ for each p in P. Then $fa \leqq a$, and since $a \leqq fa$, we have $a = fa$. Therefore a is in B. The converse is trivial.

In any complete semilattice S, the intersection of an arbitrary collection of complete subsemilattices forms a complete subsemilattice. Hence any subset of S is contained in a least complete subsemilattice of S, which we shall call the complete subsemilattice generated by the set.

Manifestly, the notion of a semilattice has a dual, in which the greatest lower bound (infimum) is replaced by the least upper bound (supremum), and where the least element 0 is replaced by the greatest element 1. We shall refer to such an object as a *dual semilattice*. Suprema will be denoted by $p \vee q$, and by $\bigvee P$ or sup P. A *lattice* is a partially ordered set which is both a semilattice and a dual semilattice. We need waste no time in explaining terminology for lattices.

One word of caution for the expert in lattice theory: if S is a complete semilattice, and if f is an extensional closure on S, then the range B of f is not only a complete subsemilattice of S, but is also a complete lattice in a natural manner. In lattice theory, it is customary to consider B as a lattice; in this note we explicitly avoid doing this. The reason is that if S is a complete lattice, and if B is a complete subsemilattice, B need not even be a sublattice of S, although it is a complete lattice in its own right.

The set of all mappings of a partially ordered set S into itself may be regarded as a partially ordered set. We define $f \leqq g$ to mean $fp \leqq gp$ for each p in S. If S is a semilattice, dual semilattice, or lattice, then the set of all mappings of S into itself is likewise a semilattice, dual semilattice, or lattice, under the usual pointwise rules. If S is complete, then the set of all mappings is also complete.

A mapping f of a dual semilattice S into itself is called *additive* if

$$f(p \vee q) = fp \vee fq$$

for each p, q in S.

THEOREM. *Let L be a lattice, f an extensional closure in L, and B the range of f. Then f is additive if and only if B is a sublattice of L.*

Proof. Suppose B is a sublattice of L. For any two elements p, q in L, we have fp and fq in B, so that $fp \vee fq$ is in B. Since $p \leqq fp$, $q \leqq fq$, then $p \vee q \leqq fp \vee fq$, so that $fp \vee fq \in B(p \vee q)$. Thus $f(p \vee q) \leqq fp \vee fq$. On the other hand $p \leqq p \vee q$ implies that $fp \leqq f(p \vee q)$, and $q \leqq p \vee q$ implies that $fq \leqq f(p \vee q)$. Hence $fp \vee fq \leqq f(p \vee q)$. Then $fp \vee fq = f(p \vee q)$, and f is additive. Conversely, if f is additive and if $p, q \in B$, then $p \vee q = fp \vee fq = f(p \vee q) \in B$.

A dual semilattice can be characterized as a commutative semigroup in which every element is idempotent. The greatest element of a dual semilattice, if one exists, is a "zero" for the semigroup; the least element, if one exists, is an identity element for the semigroup. An additive mapping of a dual semilattice is an endomorphism of the semigroup. If the dual semilattice has a least element 0, the requirement that a mapping be normalized is merely the requirement that the identity element of the semigroup be mapped onto itself. The notion of a normalized additive mapping [7] of a dual semilattice is consequently a natural one in the semigroup setting; we shall call such a mapping a *hemimorphism*. This word was coined by Halmos [4] to describe mappings of a Boolean algebra which are roughly "half of a homomorphism."

A mapping h of a partially ordered set S into itself is called *antitone* if $p \leqq q$ implies $hq \leqq hp$.

THEOREM. *Let h be an antitone mapping of a partially ordered set S, and let f be an extensional closure in S, with range B. Then the following are equivalent statements*: (a) $fhf \leqq h$; (b) $fhf = hf$; (c) $q \in B$ *implies* $hq \in B$.

Proof. Suppose (a) is true; let q belong to B. Then $fq = q$, so that $hfq = hq$, and hence $fhfq = fhq$. By (a) we thus have $fhq \leqq hq$. But $hq \leqq fhq$, because f is extensive, so that $fhq = hq$; this means that hq belongs to B. Therefore (a) implies (c). If (c) is true, let p be any element of S; then fp is in B, so that hfp is also in B, and hence $fhfp = hfp$. Therefore (c) implies (b). That (b) implies (a) is trivial.

Let L be a lattice with least element 0. By a *closure* in L is meant an extensive, idempotent hemimorphism f of L. A closure therefore satisfies the following: $f0 = 0$, $p \leqq fp$, $ffp = fp$, $f(p \vee q) = fp \vee fq$. It follows easily that $p \leqq q$ implies $fp \leqq fq$, so that every closure is an extensional closure; the converse is not true in general.

The most important lattices are the Boolean algebras. If A is a Boolean algebra, we denote the complement of an element p of A by p'. The mapping of an element onto its complement is an antitone mapping of a Boolean algebra.

By an *existential quantifier* f in a Boolean algebra A is meant a closure in A which satisfies the condition $f((fp)') \leqq p'$ for each p in A. This definition [15] of an existential quantifier is equivalent to the original one given by Halmos [4]. We have at once the following facts about closures and quantifiers.

THEOREM. *Let f be an extensional closure, with range B, in a lattice L. Then f is a closure if and only if B is a normalized sublattice of L.*

THEOEM. *Let f be an extensional closure with range B in a Boolean algebra A. Then f is an existential quantifier if and only if B is a Boolean subalgebra of A.*

3. Martingales

By a *directed set* we shall mean a set T with a transitive and reflexive relation $\geqq$ such that, for any s and t in T, there is an element r in T with $r \geqq s$ and $r \geqq t$. By a *net* in a set S we shall mean a pair consisting of a directed set T and a mapping $t \to p_t$ of T into S; we denote such a net by $\{p_t : t \in T\}$. A net $\{p_t : t \in T\}$ in a partially ordered set S is said to be *increasing* if $s \geqq t$ implies $p_s \geqq p_t$, and *decreasing* if $s \geqq t$ implies $p_s \leqq p_t$.

A net $\{f_t : t \in T\}$ of mappings of any set S into itself will be called a *martingale* if $s \geqq t$ implies $f_s f_t p = f_t p$ for each p in S.

THEOREM. *Let $\{f_t : t \in T\}$ be a net of mappings of a partially ordered set S.* (a) *If $\{f_t : t \in T\}$ is a martingale, each f_t is idempotent.* (b) *If $\{f_t : t \in T\}$ is a martingale of isotone and extensive mappings, then $\{f_t : t \in T\}$ is a decreasing net.* (c) *If $\{f_t : t \in T\}$ is a decreasing net of idempotent and extensive mappings, then $\{f_t : t \in T\}$ is a martingale.* (d) *If $\{f_t : t \in T\}$ is a net of* *extensional closures, then the net is a martingale if and only if it is a decreasing net.*

Proof. (a) Since $t \geqq t$, we have $f_t f_t p = f_t p$. (b) Since $p \leqq f_t p$, and since f_s is isotone, we have $f_s p \leqq f_s f_t p$. If $s \geqq t$, then $f_s f_t p = f_t p$, so that $f_s p \leqq f_t p$. (c) Since f_s is extensive, we obtain $f_t p \leqq f_s f_t p$. If $s \geqq t$, we have $f_s q \leqq f_t q$ for any q in S; set $q = f_t p$, and get $f_s f_t p \leqq f_t f_t p = f_t p$. Hence $s \geqq t$ implies $f_s f_t p = f_t p$. (d) This follows at once from the previous parts of the theorem.

COROLLARY. *Let $\{f_t : t \in T\}$ be a net of extensional closures in a partially ordered set S, and let B_t be the range of f_t. Then $\{f_t : t \in T\}$ is a martingale if and only if the net $\{B_t : t \in T\}$ of subsets of S is increasing.*

4. Order convergence

If $\{p_t : t \in T\}$ is a net in a complete lattice L, the limits superior and inferior are defined by

$$\limsup_t p_t = \inf_s \sup_{t \geqq s} p_t \quad \text{and} \quad \liminf_t = \sup_s \inf_{t \geqq s} p_t .$$

The inequalities

$$\inf_t p_t \leqq \liminf_t p_t \leqq \limsup_t p_t \leqq \sup_t p_t$$

always hold. The net $\{p_t : t \in T\}$ is said to *converge* if

$$\liminf_t p_t = \limsup_t p_t ;$$

we refer to convergence in this sense as *order-convergence*. If a net converges, the common value of its lim sup and lim inf is denoted by $\lim_t p_t$.

A *conditionally complete lattice* is a lattice in which any bounded set has an infimum and a supremum. If $\{p_t : t \in T\}$ is a bounded net in a conditionally complete lattice, the considerations of the preceding paragraph may be applied to the net.

THEOREM. *If $\{p_t : t \in T\}$ is a bounded decreasing net in a conditionally complete lattice L, then* $\lim_t p_t = \inf_t p_t$; *if $\{p_t : t \in T\}$ is a bounded increasing net in a conditionally complete lattice L, then* $\lim_t p_t = \sup_t p_t$.

Proof. If the net is decreasing, then for $t \geqq s$, $p_t \leqq p_s$, so that $\sup_{t \geqq s} p_t = p_s$, and therefore $\inf_s \sup_{t \geqq s} p_t = \inf_s p_s$. That is,

$$\limsup_t p_t = \inf_s p_s \leqq \liminf_t p_t ,$$

and hence $\lim_t p_t = \inf_t p_t$. The other half of the theorem is similar.

If $\{f_t : t \in T\}$ is a decreasing net of mappings of a complete lattice L into itself, then for each element p of L, the net $\{f_t p : t \in T\}$ is a decreasing net in L, and consequently $\lim_t f_t p = \inf_t f_t p$. In other words, a decreasing net of mappings of a complete lattice converges pointwise. The convergence theorems for martingales, proved below, identify the limit mappings so defined, for certain special martingales.

THEOREM 1. *If* $\{f_t : t \in T\}$ *is a martingale of extensional closures in a complete lattice* L, *and if* $\lim_t f_t p = fp$ *for each* p *in* L, *then* f *is an extensional closure. Moreover, if* B_t *is the range of* f_t, *then* f *is the extensional closure associated with the complete subsemilattice generated by all the* B_t, $t \in T$.

Proof. We know that $fp = \lim_t f_t p = \inf_t f_t p$. Since $p \leqq f_t p$ for each t, then $p \leqq fp$. If $p \leqq q$, then $f_t p \leqq f_t q$, so that $fp \leqq fq$. Finally, $fp \leqq f_t p$ for any t, so that $f_t fp \leqq f_t f_t p = f_t p$. Hence $ffp \leqq fp$; since $fp \leqq ffp$, we have $ffp = fp$. Therefore f is an extensional closure. Let B be the complete subsemilattice generated by all the B_t, and let C be the range of f. If p belongs to B_t, then $f_t p = p$, so that $fp = p$, and hence p belongs to C. Therefore $B_t \subset C$; since this holds for each t, then $B \subset C$. On the other hand, if p belongs to C, then $p = fp = \inf_t f_t p$; since $f_t p$ belongs to B_t, the infimum belongs to the least complete subsemilattice B generated by all the B_t. Therefore $B = C$.

A complete lattice L will be called a *continuous lattice* if it satisfies the two conditions: (i) for any increasing net $\{p_t : t \in T\}$ in L and any element q in L, $q \wedge \lim_t p_t = \lim_t (q \wedge p_t)$; (ii) for any decreasing net $\{p_t : t \in T\}$ in L and any element q in L, $q \vee \lim_t p_t = \lim_t (q \vee p_t)$. (This terminology is that used by von Neumann [12]; it is a concise way of saying that L is a topological lattice in its order topology [1, page 63].) There exist complete lattices which are not continuous.

THEOREM 2. *Let* $\{f_t : t \in T\}$ *be a martingale of closures in a continuous lattice* L. *If* $fp = \lim_t f_t p$ *for each* p *in* L, *then* f *is a closure.*

Proof. That $f0 = 0$ is obvious. We need to show that $f(p \vee q) = fp \vee fq$ for each p and q in L. We have $fp \leqq f_t p \leqq f_t p \vee f_t q = f_t(p \vee q)$, so that $fp \leqq f(p \vee q) = \inf_t f_t(p \vee q)$. Interchanging p and q yields

$$fq \leqq f(p \vee q),$$

so that $fp \vee fq \leqq f(p \vee q)$. Since L is a continuous lattice, we have $fp \vee fq = fp \vee \inf_r f_r q = \inf_r (fp \vee f_r q)$. Also,

$$\begin{aligned} fp \vee f_r q &= (\lim\inf_t f_t p) \vee f_r q \\ &= (\sup_s \inf_{t \geqq s} f_t p) \vee f_r q \geqq (\inf_{t \geqq r} f_t p) \vee f_r q \\ &= \inf_{t \geqq r}(f_t p \vee f_r q) = \inf_{t \geqq r}(f_t p \vee f_t f_r p) \\ &= \inf_{t \geqq r} f_t(p \vee f_r q) \geqq \inf_{t \geqq r} f_t(p \vee q) \\ &\geqq \inf_t f_t(p \vee q) = f(p \vee q). \end{aligned}$$

Thus $fp \vee f_r q \geqq f(p \vee q)$ for any r in T, and therefore $fp \vee fq \geqq f(p \vee q)$. The proof is complete.

Any complete Boolean algebra is a continuous lattice (von Neumann [12]; see also [1, page 165]). Consequently the limit of a martingale of closures in a complete Boolean algebra is a closure. We turn next to the question of convergence of quantifiers, and we shall use the traditional notation $\exists$ to denote an existential quantifier.

THEOREM 3. *Let* $\{\exists_t : t \in T\}$ *be a martingale of existential quantifiers in a complete Boolean algebra* A. *If* $fp = \lim_t \exists_t p$ *for each* p *in* A, *then* f *is an existential quantifier.*

Proof. For this we need the terminology and results of [15] concerning duality in Boolean algebras and spaces. Since each $\exists_t$ is a quantifier, its adjoint $\exists_t^*$ is an equivalence relation, and therefore $\bigcap_t \exists_t^*$ is an equivalence relation. To show that f is a quantifier, it suffices to show that $f^* = \bigcap_t \exists_t^*$. Since $fp \leqq \exists_t p$ for each t and p, then $f^* \subset \exists_t^*$ for each t [15, Theorem 4]; thus $f^* \subset \bigcap_t \exists_t^*$. Applying [15, Theorems 1, 6], we have

$$f = f^{**} \leqq (\bigcap_t \exists_t^*)^*.$$

Since $\bigcap_t \exists_t^* \subset \exists_s^*$, then $(\bigcap_t \exists_t^*)^* \leqq \exists_s^{**} = \exists_s$. Then $(\bigcap_t \exists_t^*) \leqq f$, and hence $(\bigcap_t \exists_t^*)^* = f$. In particular, $\bigcap_t \exists_t^*$ is a Boolean relation [15, Theorem 2], so that $\bigcap_t \exists_t^* = (\bigcap_t \exists_t^*)^{**} = f^*$. This proves the theorem.

Associated with every existential quantifier $\exists$ is its dual *universal quantifier* $\forall$, defined by $\forall p = (\exists(p'))'$. If $\{\exists_t : t \in T\}$ is a martingale of existential quantifiers, then the net $\{\forall_t : t \in T\}$ is also a martingale. For any p in A, the net $\{\forall_t p : t \in T\}$ is an increasing net. We may now state the full convergence theorem for martingales of quantifiers.

THEOREM 4. *Let* A *be a complete Boolean algebra, let* $\{C_t : t \in T\}$ *be an increasing net of complete subalgebras of* A, *and let* C *be the complete subalgebra of* A, *generated by all the* C_t. *Let* $\exists_t$ *and* $\forall_t$ *be the existential and universal quantifiers associated with* C_t, *and let* $\exists$ *and* $\forall$ *be the quantifiers associated with* C. *Then, for any element* p *of* A,

$$\exists p = \lim_t \exists_t p = \inf_t \exists_t p \quad \textit{and} \quad \forall p = \lim_t \forall_t p = \sup_t \forall_t p.$$

These convergence theorems have a variety of applications. If, for instance, a set X has defined on it certain operations making it an abstract algebra, then an extensional closure can be defined in the Boolean algebra 2^X of all subsets of X. The closure of any subset of X is the smallest subalgebra of X containing that set. Theorem 1 is therefore connected with the notion of "species of algebraic structures" [3]. A closure in 2^X arises whenever X is a topological space; Theorem 2 is concerned with the convergence of topologies which tend toward the discrete. In the remainder of this paper, we shall be interested in situations involving Theorems 3 and 4. We will see below that, in probability theory, Theorem 4 is nearly equivalent to the martingale convergence theorem of probability theory.

5. Convergence of extremal and intermediate values

The concept of a quantifier for a Boolean algebra has an analogue in the theory of continuous real-valued functions [17]. In this section, we extend

Theorem 4 above to this analogue. Let X be a compact, extremally disconnected, Hausdorff space, let $R(X)$ denote the algebra of all continuous real-valued functions on X, and let A denote the Boolean algebra of all idempotents in $R(X)$. Then A is a complete Boolean algebra. For any complete Boolean subalgebra B of A, let $[B]$ denote the smallest uniformly closed subalgebra of $R(X)$ which contains B. Then, for any φ in $R(X)$, there exist functions $M\varphi$ and $m\varphi$ in $[B]$ such that $m\varphi \leqq \varphi \leqq M\varphi$, and such that, if ψ_1, ψ_2 belong to $[B]$ and satisfy $\psi_1 \leqq \varphi \leqq \psi_2$, then $\psi_1 \leqq m\varphi$ and $M\varphi \leqq \psi_2$. The mappings M and m of $R(X)$ into $[B]$ are extensions of the quantifiers $\exists$ and $\forall$ which map A into B. We shall call M and m the *extremal operators* associated with B, or with $[B]$. Note that if X is a compact, extremally disconnected, Hausdorff space, then $R(X)$ is a conditionally complete lattice.

THEOREM 5. *Let X be an extremally disconnected, compact, Hausdorff space, let $R(X)$ be the algebra of all continuous, real-valued functions on X, and let A be the complete Boolean algebra of idempotents in $R(X)$. Let $\{B_t : t \in T\}$ be an increasing net of complete Boolean subalgebras of A, and let B be the complete subalgebra of A generated by all the B_t. Let M_t, m_t be the extremal mappings associated with B_t, and let M, m be the extremal mappings associated with B. Then, for any element φ of $R(X)$,*

$$M\varphi = \lim_t M_t \varphi = \inf_t M_t \varphi, \quad \text{and} \quad m\varphi = \lim_t m_t \varphi = \sup_t m_t \varphi,$$

where the limit is understood to be in the sense of order-convergence in $R(X)$.

Proof. For any real $a \geqq 0$ and any φ in $R(X)$, we have $M(a\varphi) = aM(\varphi)$ and $m(a\varphi) = am(\varphi)$. It therefore suffices to consider the case $\| \varphi \| \leqq 1$; in this case we also have $\| M\varphi \| \leqq 1$ and $\| m\varphi \| \leqq 1$, by [17, Theorem 4.4]. Recalling that $\| \varphi \| \leqq 1$ if and only if $-1 \leqq \varphi \leqq 1$, we see that it is sufficient to consider the complete lattice $L = \{\varphi \in R(X) : -1 \leqq \varphi \leqq 1\}$, and the mappings M_t and m_t of L into itself. It is well-known, and easily seen, that L is a continuous lattice. By [17, Theorem 4.3], the mapping M is a closure in the lattice L. The range of M_t is clearly $L \cap [B_t]$. By Theorem 2, the mapping f of L into itself defined by $f\varphi = \lim_t M_t \varphi$ is again a closure on L, whose range is $L \cap [B]$. Since the range of M is the same thing, we have $f\varphi = M\varphi$ for each φ in L. The other half of the theorem follows from the identity $M\varphi = -m(-\varphi)$ for extremal operators.

If B is a complete Boolean subalgebra of A, by an *intermediate evaluation* for $R(X)$ with respect to B we shall mean any mapping h of $R(X)$ into $[B]$ satisfying $m\varphi \leqq h\varphi \leqq M\varphi$ for each φ in $R(X)$. We do not require h to be linear; if it is linear, it is a generalized mean in the sense of [17].

THEOREM 6. *Let X be an extremally disconnected, compact, Hausdorff space, and let $\{\exists_t : t \in T\}$ be a martingale of existential quantifiers in the Boolean algebra A of idempotents in $R(X)$. Let h_t be an intermediate evaluation for $R(X)$ with respect to the range B_t of $\exists_t$. Then $\{h_t : t \in T\}$ is a martingale of mappings of $R(X)$. If $\lim_t \exists_t p = \exists p$, if B is the range of $\exists$, and if h is an intermediate evaluation of $R(X)$ with respect to B, such that $h_t h\varphi = h_t \varphi$*

for each t in T and each φ in $R(X)$, then $\lim_t h_t \varphi = h\varphi$, *in the sense of order-convergence in* $R(X)$.

Proof. Since $s \geqq t$ implies $B_s \supset B_t$, then $s \geqq t$ implies $[B_s] \supset [B_t]$. This clearly implies that the nets $\{M_t : t \in T\}$ and $\{m_t : t \in T\}$ are martingales. Moreover $M_t h_t \varphi = h_t \varphi$ and $m_t h_t \varphi = h_t \varphi$ for any t and φ. Then, if $s \geqq t$, we have $m_s h_t \varphi = m_s m_t h_t \varphi = m_t h_t \varphi = h_t \varphi$; similarly

$$M_s h_t \varphi = h_t \varphi.$$

Hence, for $s \geqq t$, we have $h_t \varphi = m_s h_t \varphi \leqq h_s h_t \varphi \leqq M_s h_t \varphi = h_t \varphi$, so that $h_s h_t = h_t$. This says that $\{h_t : t \in T\}$ is a martingale.

Now suppose that the intermediate evaluation h described in the hypothesis exists. For any φ in $R(X)$, set $\psi = h\varphi$. Then ψ belongs to $[B]$, $h\psi = \psi = h\varphi$, and $h_t \psi = h_t h\varphi = h_t \varphi$. It suffices, then, to show that $\lim_t h_t \psi = \psi$. But we have $m_t \psi \leqq h_t \psi \leqq M_t \psi$, and since ψ belongs to $[B]$, Theorem 5 implies $\sup_t m_t \psi = \psi = \inf_t M_t \psi$. Hence $\lim_t h_t \psi = \psi$, and the proof is complete.

In the special case of Theorem 6 in which $\lim_t \exists_t p = p$ for each p in A, we have $B = A$, and hence $[B] = R(X)$. The only intermediate evaluation for $R(X)$ in this case is the identity mapping of $R(X)$. This fulfils the condition imposed on h in the theorem, so that $\lim_t h_t \varphi = \varphi$ for each φ in $R(X)$.

6. Probability theory

Let $(\Omega, \mathcal{A}, P)$ be a probability space. For any subfield $\mathcal{B}$ of $\mathcal{A}$, let $E(\varphi \mid \mathcal{B})$ denote the conditional expectation of the integrable function φ with respect to the subfield $\mathcal{B}$. A directed family $\{\varphi_t : t \in T\}$ of integrable functions is called a (closed) martingale if there are an increasing net $\{\mathcal{B}_t : t \in T\}$ of subfields of $\mathcal{A}$ and an integrable function ψ such that $\varphi_t = E(\psi \mid \mathcal{B}_t)$ for each t in T; cf. [2].

Let A be the measure algebra of the probability space, and let X be the Boolean representation space of A. Then the algebra $L^\infty(\mathcal{A})$ of all essentially bounded real functions which are measurable $(\mathcal{A})$ is isomorphic and isometric with $R(X)$. If $\mathcal{B}$ is any subfield of $\mathcal{A}$, then $\mathcal{B}$ defines a complete subalgebra of A, and *vice versa*. The conditional expectation with respect to $\mathcal{B}$ clearly defines an intermediate evaluation in the sense of this note. For an increasing net $\{\mathcal{B}_t : t \in T\}$ of subfields of $\mathcal{A}$, generating the subfield $\mathcal{B}$, we have the identity $E(E(\varphi \mid \mathcal{B}) \mid \mathcal{B}_t) = E(\varphi \mid \mathcal{B}_t)$. We can therefore apply Theorem 6 at once, and assert that $\lim_t E(\varphi \mid \mathcal{B}_t) = E(\varphi \mid \mathcal{B})$, in the sense of order-convergence.

The extremal mappings of $R(X)$ defined by a complete subalgebra of A can be carried over to $L^\infty(\mathcal{A})$ via the isomorphism. If $\mathcal{B}$ is a subfield of $\mathcal{A}$, the extremal mappings are denoted by $\max(\varphi \mid \mathcal{B})$ and $\min(\varphi \mid \mathcal{B})$, and are called the conditional maximum and minimum of φ with respect to $\mathcal{B}$ [17, Definition 8.2]. Applying Theorem 5, we may assert that

$$\lim_t \max(\varphi \mid \mathcal{B}_t) = \max(\varphi \mid \mathcal{B}) \quad \text{and} \quad \lim_t \min(\varphi \mid \mathcal{B}_t) = \min(\varphi \mid \mathcal{B}),$$

in the sense of order convergence.

In probability theory, however, we are concerned primarily with convergence in mean and convergence pointwise almost everywhere. Convergence in mean follows at once from order-convergence, by virtue of the generalized monotone convergence theorem. In fact, the mapping which assigns to the real, integrable function φ on Ω the real number $\int_\Omega \varphi\, dP$ is an isotone, positive valuation, in the sense of [1, page 74]. This makes the space L^1 of integrable functions a complete metric lattice, and the generalized Lebesgue monotone convergence theorem [1, page 82, ex. 4(a) and Theorem 16] asserts that order-convergence of increasing (or decreasing) nets in L^1 is equivalent to metric convergence. This leads at once to the following.

THEOREM 7. *Let $\{\mathcal{B}_t : t \in T\}$ be an increasing net of subfields of the field $\mathcal{A}$ in the probability space $(\Omega, \mathcal{A}, P)$, and let $\mathcal{B}$ be the subfield generated by all the $\mathcal{B}_t$. For any φ in $L^\infty(\mathcal{A})$,*

$$\lim_t \int_\Omega \mid \max(\varphi \mid \mathcal{B}_t) - \max(\varphi \mid \mathcal{B}) \mid dP = 0,$$

$$\lim_t \int_\Omega \mid \min(\varphi \mid \mathcal{B}_t) - \min(\varphi \mid \mathcal{B}) \mid dP = 0,$$

$$\lim_t \int_\Omega \mid E(\varphi \mid \mathcal{B}_t) - E(\varphi \mid \mathcal{B}) \mid dP = 0.$$

Proof. The order convergence given by Theorem 5 and the monotone convergence theorem cited above yield the first two assertions. To prove convergence of conditional expectations, we observe that it suffices to consider the case where φ belongs to $L^\infty(\mathcal{A})$, as in the proof of Theorem 6, and to show that $E(\varphi \mid \mathcal{B}_t) \to \varphi$ in mean. Since

$$\min(\varphi \mid \mathcal{B}_t) \leqq \varphi \leqq \max(\varphi \mid \mathcal{B}_t)$$

and

$$\min(\varphi \mid \mathcal{B}_t) \leqq E(\varphi \mid \mathcal{B}_t) \leqq \max(\varphi \mid \mathcal{B}_t),$$

the mean convergence of the conditional maxima and minima implies the mean convergence of the conditional expectations.

For convergence in mean of order $p > 1$, it suffices to note that the space L^p is an example of the UMB-lattices of G. Birkhoff, and that order-convergence implies metric convergence [1, pages 248–249]. We omit the details.

Pointwise convergence a.e. presents more difficulties, and involves the nature of the directed set T. For considerations of this kind, see Doob [2, page 51 et seq.] and Krickeberg [8]. If T is the set of natural numbers in the usual ordering, then we are dealing with the convergence of ordinary sequences. For this case, the order-convergence of Theorem 5 at once implies the almost sure pointwise convergence of the conditional maxima and minima, and the order-convergence of intermediate evaluations given by Theorem 6 implies the pointwise convergence of conditional expectations. (Convergence in mean, in this case, is a consequence of convergence a.e. and the usual Lebesgue monotone convergence and bounded convergence theorems.) We therefore have the following result.

THEOREM 8. *If $\{\mathcal{B}_n\}$ is an increasing sequence of subfields of the field $\mathcal{A}$ of the probability space $(\Omega, \mathcal{A}, P)$, generating the subfield $\mathcal{B}$, then, for any φ in $L^\infty(\mathcal{A})$,*

$$\max(\varphi \mid \mathcal{B}_n) \to \max(\varphi \mid \mathcal{B}) \quad \text{a.e.},$$

$$\min(\varphi \mid \mathcal{B}_n) \to \min(\varphi \mid \mathcal{B}) \quad \text{a.e.},$$

and

$$E(\varphi \mid \mathcal{B}_n) \to E(\varphi \mid \mathcal{B}) \quad \text{a.e.}$$

The convergence of conditional expectations, as given in Theorems 7 and 8, represents a typical case of the martingale convergence theorems of Lévy [9], Doob [2], and Krickeberg [8]. If we consider the special case of a characteristic function $\varphi = \chi_F$ of a set F in $\mathcal{A}$, we obtain the original theorem of Lévy. In this case, we can be more explicit about the nature of $\max(\chi_F \mid \mathcal{B})$ and $\min(\chi_F \mid \mathcal{B})$. It is easily seen that both of these are characteristic functions of sets F_* and F^* in $\mathcal{B}$; F_* and F^* represent the existentially and universally quantified versions of F, using the quantifiers determined in the measure algebra by the subfield $\mathcal{B}$. If $P(F \mid \mathcal{B}) = E(\chi_F \mid \mathcal{B})$ is the conditional probability of F with respect to $\mathcal{B}$, then

$$F^* = \{\omega \in \Omega : P(F \mid \mathcal{B})(\omega) > 0\}$$

and

$$F_* = \{\omega \in \Omega : P(F \mid \mathcal{B})(\omega) = 1\}.$$

This means that the convergence of $\max(\chi_F \mid \mathcal{B}_n)$ and $\min(\chi_F \mid \mathcal{B}_n)$, as given in Theorem 8, can be deduced from the convergence of $P(F \mid \mathcal{B}_n)$, as given by Lévy's theorem. Thus Theorem 8 and Lévy's theorem are equivalent for characteristic functions. (Theorem 8, restricted to characteristic functions, is the measure-theoretic specialization of the sequential version of Theorem 4.) I am indebted to Professor Doob for these observations.

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