EXISTENCE OF NORMAL COMPLEMENTS AND EXTENSION OF CHARACTERS IN FINITE GROUPS

Dedicated to Reinhold Baer on the occasion of his sixtieth birthday

BY

CHIH-HAN SAH[1]

The main purpose of this paper is to tie together the following problems and to find conditions under which they may be solved.

PROBLEM A. Given a finite group G and a Hall subgroup H, when is there a normal complement to H in G?

PROBLEM B. Given a finite group G and a Hall subgroup H, when is it possible to extend each of the irreducible characters of H to one of G?

Of course, a positive solution of Problem A for the groups G and H leads to a positive solution of Problem B.

In both problems, we may drop the restriction on H, but the example of an abelian group G shows that the extended problems are not equivalent.

Our main results are the following:

THEOREM 1. *Let G be a π-separated group. Then the following conditions are equivalent:*

(a) *G contains a normal π'-Hall subgroup.*

(b) *Each π-Hall subgroup of G is c-closed.*

(c) *At least one π-Hall subgroup of G is c-closed.*

THEOREM 2. *If H is a soluble Hall subgroup of G, then the following conditions are equivalent:*

(a) *G contains a normal complement to H.*

(b) *Each irreducible character of H may be extended to G.*

THEOREM 3. *Let H be a Hall subgroup of G such that at least one of the following conditions holds:*

(1) *H has a Sylow tower.*

(2) *The terminal member of the lower central series of H is nilpotent.*

Then, the following conditions are equivalent:

(a) *G contains a normal complement to H.*

(b) *H is c-closed in G.*

THEOREM 4. *If H and K are Hall subgroups of G of complementary orders, then the following conditions are equivalent:*

(a) *G is the direct product of H and K.*

(b) *Each irreducible character of H and of K may be extended to G.*

Received February 17, 1961.

[1] This research was supported in part by the Office of Ordnance Research, U.S. Army.

Problem A has been treated rather extensively in the literature; for example, see the papers of Baer [3], Frobenius [5], D. G. Higman [8], Thompson [10], and Wielandt [11]. In particular, Baer [3] had obtained necessary and sufficient conditions for the positive solution of Problem A. Our condition (c) in Theorem 1 looks very much like the condition of Baer [3] of π'-homogeneity. However, the equivalence of these two conditions does not seem to be obvious.

It is natural to raise the question, "Is the condition of solubility needed in Theorem 2?" There does not seem to be any elementary example to indicate the necessity of the solubility, although our proof uses it rather strongly.

Theorem 3 may be considered as a generalization of a theorem of Frobenius [5; Theorem I, p. 1324], which states that G contains a normal p-complement if and only if elements of G may induce only automorphisms of order a power of p in any p-subgroup of G. Another proof of this theorem may be found in [6; Theorem 14.4.7, p. 217].

In case H is a Sylow subgroup of G, Theorem 3 is a special case of a theorem of D. G. Higman [8; Theorem 3.6, p. 488].

Theorem 4 may be considered as the "character" analogue of theorems of Baer [2], Frobenius [5; Theorem II, p. 1220] and Ludwig [9].

In the last section, we shall describe some examples in connection with Theorem 1 and Theorem 3.

1. Notations

$H \leqq G$, $H < G$, $H \lhd G$ shall mean, respectively, that H is a subgroup, a proper subgroup, a normal subgroup of G.

$N_H(S)$, $C_H(S)$ shall mean, respectively, the normalizer and the centralizer of the subset S in $H \leqq G$; S does not have to be part of H.

$\{S\}$ = subgroup generated by the subset S in G.

$S^x = x^{-1}Sx$, $[x, y] = x^{-1}x^y$, $[H, K] = \{[x, y]\}$ with $x \in H$, $y \in K$.

$[G:H]$ = index of the subgroup H in G.

π, π' denote complementary sets of primes.

G is called a π-group if all the prime divisors of $[G:1]$ occur in π. G is called π-separated if its composition factors are either π-groups or π'-groups.

H is a Hall subgroup of G if $([G:H], [H:1]) = 1$. It is called a π-Hall subgroup of G if H is a π-subgroup and all the prime divisors of $[G:H]$ occur in π'.

2. c-closure

DEFINITION 2.1. Let $H \leqq G$. H is said to be *c-closed* in G if and only if two elements of H conjugate in G are already conjugate in H, i.e., if $x \in H$ and $y \in G$ are such that $x^y \in H$, then there exists $z \in H$ such that $x^z = x^y$. Equivalently, conjugate classes of H fall into different conjugate classes of G.

LEMMA 2.2. *Let H be c-closed in G. Then,*

(a) *$H \leqq K \leqq G$ implies that H is c-closed in K;*

(b) *$K \leqq H \leqq G$ and K is c-closed in H imply that K is c-closed in G;*
(c) *if $K \leqq H$ and $K \lhd G$, then H/K is c-closed in G/K;*
(d) *if $N \lhd G$ and $([N:1], [H:1]) = 1$, then HN/N is c-closed in G/N.*

Proof. The verifications of (a), (b), and (c) follow from the definition.

(d) Let xN and yN be elements of HN/N conjugate under G/N. Thus, we may assume that x and y lie in H and there exists $z \in G$ such that $x^z = yu$ with $u \in N$. Since $[N:1]$ and $[H:1]$ are relatively prime, we see that x and y must have the same order; in fact, their common order coincides with the order of the element xN in HN/N. Thus, $\{y\}$ and $\{yu\}$ are complements of N in $\{N, y\}$. Now, by the Theorem of Schur-Witt-Zassenhaus [12; Theorem 27, p. 132], $\{y\}$ and $\{yu\}$ are conjugate subgroups of $\{N, y\}$. Hence we can find $v \in \{N, y\}$ such that $y^r = (yu)^v$ for a suitable integer r. Since $\{yu\}$ is a complement of N in $\{N, y\}$, we may assume that $v \in N$. Hence,

$$y^{r-1} = v^{-1}yuvy^{-1} \in H \cap N = 1,$$

since $uv \in N \lhd G$ and $([N:1], [H:1]) = 1$. Thus, $y = (yu)^v = x^{zv}$. Since H is c-closed in G, we may assume that $zv \in H$. Thus, xN and yN are conjugate in HN/N, Q.E.D.

LEMMA 2.3. *Let $N \lhd G$, $H \leqq G$ be such that $NH = G$ and $N \cap H = 1$; then H is c-closed in G.*

Proof. Let $x, y \in H$, $z \in N$ be such that $x^{yz} \in H$. Then, $u = x^y \in H$ and $u^z \in H$. Hence, $u^{-1}u^z = [u, z] \in H \cap N = 1$, since $N \lhd G$ and $H \cap N = 1$. Thus, $x^{yz} = x^y$, Q.E.D.

LEMMA 2.4. *If each irreducible character of the subgroup H of G may be extended to one of G, then,*

(1) *H is c-closed in G;*

(2) *for each $N \lhd H$, there exists $K \lhd G$ such that $K \cap H = N$, and each irreducible character of HK/K may be extended to one of G/K.*

Proof. (1) From the orthogonality relations [6; Theorem 16.6.9, p. 274], the irreducible characters of H separate the H-classes in H. Since these characters may be extended to G, distinct H-classes of H must fall into distinct G-classes in G.

(2) For each irreducible character of H, trivial on N, we select an extension to G. The kernel of the extension must contain N. Let K be the intersection of all such kernels. Then, clearly, $K \lhd G$, and $K \cap H$ contains N. By the same argument as in (1), we see that $N = K \cap H$. Each irreducible character of HK/K gives rise to an irreducible character of

$$H/(H \cap K) = H/N$$

by way of the natural inclusion map of H into HK. By the choice of K, this character of H/N may be extended to one of G which is trivial on K, i.e., a character of G/K, Q.E.D.

3. Reduction lemmas

DEFINITION 3.1. Let A be a group of automorphisms of the group G, and let $H \leqq G$ be invariant under A. Then, H is said to be *A-closed* in G provided that elements of H conjugate under G are conjugate under A. In case A is the group of inner automorphisms of G induced by elements of H, then A-closure of H is equivalent to c-closure defined previously.

THEOREM 3.2. *Let H be a Hall subgroup of G, let A be a group of automorphisms of G leaving H invariant, and let H be A-closed in G. Let $H^* = \{x^{-1}x^a\}$, $x \in H$ and $a \in A$, that is, the "A-commutator subgroup" of H. Then,*

(1) $[H, H] \leqq H^*$.

(2) *The composition of the transfer map from G to H with the natural inclusion map from $H/[H, H]$ to H/H^* carries G onto H/H^*; the kernel of this homomorphism is A-invariant.*

Proof. (1) Since H is A-closed, we see that $x^y = x^a$ for suitable $a \in A$, where x, $y \in H$. Thus, $[x, y] = x^{-1}x^a \in H^*$, where x, $y \in H$.

(2) Let $x \in H$. By the formulae given in [12; p. 138], the transfer map from G to H carries x onto the element

$$[H, H] \prod y_i x^{f(i)} y_i^{-1},$$

where $\sum f(i) = n = [G:H]$, $y_i x^{f(i)} y_i^{-1} \in H$, and y_i are suitable elements of G. Since H is A-closed, the map described in (2) carries x onto H^*x^n, where $n = [G:H]$. Since H is a Hall subgroup of G, we see that the image is all of H/H^*. By the definition of H^*, we see that A preserves the homomorphism described; thus the kernel is A-invariant, Q.E.D.

COROLLARY 3.3. *Let H be a Hall subgroup of G. If H is c-closed in G, then,*

(1) *the transfer map from G to H carries G onto $H/[H, H]$;*

(2) *there exists $N_i \lhd G$ such that $N_i H = G$ and $N_i \cap H = H_i$, where H_i is the i^{th} term of the lower central series of H.*

Proof. (1) If we let A be the group of inner automorphisms of G induced by H, then Theorem 3.2 is applicable, and $H^* = [H, H]$. Thus, (1) holds.

(2) If we let N_1 be the kernel in the homomorphism described by Theorem 3.2, then $N_1 \lhd G$, $N_1 H = G$, and $N_1 \cap H = H_1 = [H, H]$. Thus, by induction, let $N_i \lhd G$ be such that $N_i H = G$ and $N_i \cap H = H_i$. It is easy to see that H_i is a Hall subgroup of N_i. The inner automorphism group A induced by H keeps N_i and H_i invariant. By the hypothesis of c-closure of H in G, we see that H_i is A-closed in N_i. Thus, by Theorem 3.2, the transfer map from N_i to H_i followed by the inclusion from $H_i/[H_i, H_i]$ to $H_i/[H, H_i] = H_i/H_{i+1}$ carries N_i onto H_i/H_{i+1}. Let N_{i+1} be the kernel of this homomorphism. Then, $N_{i+1} \lhd N_i$ and N_{i+1} is A-in-

variant by Theorem 3.2. Thus, $G = N_i H$ implies that $N_{i+1} \lhd G$. Clearly, $N_{i+1} H_i = N_i$ and $N_{i+1} \cap H_i = H_{i+1}$. Thus, $N_{i+1} H = N_{i+1} H_i H = N_i H = G$, and $N_{i+1} \cap H = N_{i+1} \cap N_i \cap H = N_{i+1} \cap H_i = H_{i+1}$, Q.E.D.

The following result has been communicated to us by Professor R. Baer. We would like to thank him for the permission to reproduce it here.

THEOREM 3.4. *G contains a normal π'-Hall subgroup with soluble quotient group if, and only if,*

(a) *there exists a soluble π-Hall subgroup of G, and*

(b) *S is a c-closed subgroup of C whenever C is a characteristic subgroup of G and S is a π-Hall subgroup of C.*

Proof. Let $N \lhd G$ be a π'-Hall subgroup such that G/N is soluble. Then, by Schur's Theorem [12; Theorem 25, p. 130], G contains a complement H to N. Since H is isomorphic to G/N, we see that H satisfies (a). If C is a characteristic subgroup of G, then $C \cap N \lhd C$ is a π'-Hall subgroup of C. Thus, S is a complement of $C \cap N$ in C. By Lemma 2.3, S is c-closed in C; hence (b) holds.

Conversely, assume that (a) and (b) hold in G. We proceed by complete induction on $[G:1]$. By (a), we can find a soluble π-Hall subgroup H in G. By (b), taking C to be G, we see that H is c-closed in G. By Corollary 3.3, the kernel N_1 of the transfer map from G to H satisfies

$$N_1 H = G \quad \text{and} \quad N_1 \cap H = H_1 = [H, H].$$

Since $N_1/[G, G]$ is the π'-Hall subgroup of $G/[G, G]$, N_1 is a characteristic subgroup of G. Since H is soluble, $[H, H] < H$; otherwise, there is nothing to prove. Thus, $N_1 < G$. It is clear that H_1 is a soluble π-Hall subgroup of N_1; thus (a) holds in N_1. Since characteristic subgroups of N_1 are also characteristic subgroups of G, the condition (b) in G is inherited by N_1. Thus, by induction, N_1 contains a normal π'-Hall subgroup N. It is clear that N is the normal complement to H desired, Q.E.D.

We now give a generalization of a theorem of Burnside [4; Theorem II, p. 89] as well as a specialization of the First Theorem of Grün, [12; Theorem 5, p. 140].

LEMMA 3.5. *Let $H \leqq G$ be a Hall subgroup of G, and let A be a group of automorphisms of G which leaves H invariant. Suppose that*

(a) $([A:1], [G:H]) = 1$,

(b) *H is A-closed in G,*

(c) *at least one of the following conditions holds:*

(B) $H \lhd G$.

(G) *H is a p-Sylow subgroup of G.*

Then, there exists a normal complement of H in G.

Proof. (B). Let $H \lhd G$. Let $x \in H$; let K, L, and M be the H-class, G-class, and A-class of x respectively. Thus, conditions (B), (b), and the

A-invariance of H and G imply that $K \subseteq L \subseteq M \subseteq H$. Now, L is the disjoint union of H-classes each of which is conjugate to K under G, and M is the disjoint union of G-classes each of which is conjugate to L under A. If we let r, s, and t be the number of group elements in K, L, and M respectively, then $r \mid s \mid t$. Let $s = qr$. Since G permutes the H-classes of L transitively and H leaves eacd of them fixed, we see that $q \mid [G:H]$. Now, A acts as a transitive permutation group on the elements of M; hence $t \mid [A:1]$. Thus, $q \mid ([A:1], [G:H]) = 1$. Hence $r = s$ and $K = L$. Thus, H-classes in H coincide with G-classes in H. By Schur's Theorem, [12; Theorem 25, p. 132], G contains a complement N to H. By a theorem of Burnside [4; Theorem II, p. 89], the condition that H-classes in H coincide with G-classes in H together with $([N:1], [H:1]) = 1$ imply that N induces the trivial automorphism in H, i.e., $[N, H] = 1$. Thus, G is the direct product of H and N.

(G). Let H be a p-Sylow subgroup of G. By (B), we may assume that H is not normal in G. Thus, $H \leqq N_G(H) < G$. G and H are invariant under A implies that $N_G(H)$ is also invariant under A. Thus, by induction on $[G:1]$, we may assume that $N_G(H)$ has a normal p-complement. Hence, the p-factor commutator group of $N_G(H)$ is isomorphic to $H/[H, H]$. It is then easy to see that

$$[H, H] = H \cap [N_G(H), N_G(H)].$$

Let $x \in H \cap [H, H]^z$, where $z \in G$; then $x = u^z$ with $u \in [H, H]$. By (b), $x = u^z = u^a$, where $a \in A$. Since H is A-invariant and $[H, H]$ is a characteristic subgroup of H, we see that $[H, H]$ is also A-invariant. Thus,

$$x = u^a \in [H, H], \quad \text{and} \quad H \cap [H, H]^z \leqq [H, H]$$

for each $z \in G$. By the First Theorem of Grün, [12; Theorem 5, p. 140], the transfer map from G to H leads to an isomorphism of G/G_p with H/H^*, where H^* is the subgroup generated by $H \cap [N_G(H), N_G(H)]$ and $H \cap [H, H]^z$ with z ranging over G. Our computation then shows that $H^* = [H, H] < H$. Now G_p is the p-commutator subgroup of G, hence a characteristic subgroup of G; therefore G_p is A-invariant. Moreover,

$$G_p H = G \quad \text{and} \quad G_p \cap H = H^* = [H, H] < H.$$

Thus, G_p, $[H, H]$, and A satisfy the same hypotheses. Hence, by induction, G_p contains a normal p-complement N to $[H, H]$. N is a characteristic subgroup of $G_p \lhd G$; hence N is a normal p-complement to H in G, Q.E.D.

4. Proofs of theorems

Proof of Theorem 1. By Lemma 2.3, (a) implies (b), since π-separated groups have π-Hall subgroups by [7; Corollary E2.2, p. 291]. (b) implies (c) trivially.

Let (c) hold in the π-separated group G. We will proceed to prove (a) by induction on $[G:1]$. Thus, let H be a π-Hall subgroup of G such that H

is c-closed in G. Let M be a minimal normal subgroup of G. Since G is π-separated, M is either a π'-group or a π-group.

Case 1. M is a π'-group. Then, by Lemma 2.2(d) and induction, G/M contains a normal π'-Hall subgroup N/M. Thus, N is a normal π'-Hall subgroup of G.

Case 2. M is a π-group. Thus, $M \leqq H$. Hence, by Lemma 2.2(b) and induction, G/M contains a normal π'-Hall subgroup L/M. The triple L, M, and H now satisfies the hypotheses of Lemma 3.5 (B). Thus, L contains a normal complement N to M. N is easily seen to be a normal π'-Hall subgroup of G, Q.E.D.

Proof of Theorem 2. It is obvious that (a) implies (b). Thus, let H be a soluble Hall subgroup of G such that each irreducible character of H may be extended to one of G.

Let M be a minimal normal subgroup of H. By Lemma 2.4(2) and induction, we can find $L \lhd G$ such that $LH = G$ and $L \cap H = M$. Since M is a minimal normal subgroup of a soluble group H, it is a p-group. Since G/L is isomorphic to H/M and H is a Hall subgroup of G, we see that M is a p-Sylow subgroup of L. Now L, M, and H satisfy the hypotheses of Lemma 3.1 (G); thus, L contains a normal p-complement N. It is then easy to see that N is a normal complement to H in G, Q.E.D.

Proof of Theorem 3. Again, it suffices to show that (b) implies (a).

(1) Let H be a Hall subgroup of G, and let $1 = H_0 \lhd H_1 \lhd \cdots \lhd H_t = H$ be a Sylow tower for H, i.e., each H_i is a normal Hall subgroup of H, and $[H_i:H_{i-1}]$ is a prime power for $i = 1, \cdots, t$. Finally, suppose that H is c-closed in G. We now proceed by induction on t.

The case $t = 0$ is trivial. Thus, we may assume that the theorem is true for $t-1$. By the theorem of Schur, [12; Theorem 25, p. 132], H_1 has a complement C_1 in H. By Lemma 2.2(b) and Lemma 2.3, C_1 is c-closed in G. H/H_1 is isomorphic to C_1; thus C_1 has a Sylow tower of length $t-1$. Hence, by induction, we may find $N_1 \lhd G$ such that $N_1 H = G$ and $N_1 \cap H = C_1$, namely a normal complement to C_1. Now, N_1, H_1, and H satisfy the hypotheses of Lemma 3.5 (G); thus, N_1 contains a normal complement N to H_1. It is easy to see that N is a normal complement to H in G.

(2) Let H be a Hall subgroup of G, and let C be the terminal member of the lower central series of H so that C is nilpotent and H is c-closed in G. We now proceed by induction on $[G:1]$.

Let K be the largest normal subgroup of G contained in H. If $1 < K$, then by Lemma 2.2(c), we see easily that G/K and H/K satisfy the same hypothesis. Thus, by induction, we can find $M \lhd G$ such that $MH = G$ and $M \cap H = K$. It is clear that K is a Hall subgroup of M. If we let A be the group of inner automorphisms induced by H, then M, K, and A satisfy the hypotheses of Lemma 3.5 (B); hence M contains a normal complement N to K. It is then easy to see that N is a normal complement to H in G. Thus, we may assume that $K = 1$.

By Corollary 3.3(2), we can find $M \lhd G$ such that $MH = G$ and $M \cap H = C$. Let $N^* \leqq M$ be minimal with respect to the following properties:

(A) $N^* \lhd G$.

(B) $N^*H = G$.

Thus, $N^* \cap H = C^* \leqq M \cap H = C$, and C^* is nilpotent. We let A be the group of inner automorphisms induced by H.

If $C^* \neq 1$ is a prime power group, then N^*, C^*, and A satisfy the hypotheses of Lemma 3.5 (G); hence N^* contains a normal complement N to C^*. Therefore N would be a normal complement to H in G; this contradicts the minimality of N^*.

If $C^* \neq 1$ is not a prime power group, then from C^* nilpotent, we may take $P \neq 1$ to be a p-Sylow subgroup of C^* for an odd prime p. Since C^* is nilpotent and A-invariant, we see that P is A-invariant. Let $1 < P_1 \lhd P$ be such that P_1 is A-invariant. Thus, $H \leqq N_G(P_1) < G$, where the first inclusion is a consequence of the A-invariance of P_1 and the second strict inclusion is a consequence of the hypothesis that H contains only the trivial normal subgroup of G. Clearly, $N_G(P_1)$ and H satisfy (2); thus, by induction, $N_G(P_1)$ contains a normal complement to H. Hence, $N_G(P_1)$ may induce only automorphisms in P_1 of orders dividing $[H:1]$. Therefore, $N_{N^*}(P_1)$ may induce only automorphisms in P_1 of orders dividing $[C^*:1]$. Since C^* is nilpotent and $C^* \leqq N_{N^*}(P_1)$, we see that $N_{N^*}(P_1)$ may induce only automorphisms of orders dividing $[P:1]$, i.e., $N_{N^*}(P_1)/C_{N^*}(P_1)$ is a p-group. Thus, by the Theorem of Thompson, [10; Theorem A, p. 332], N^* contains a normal p-complement N_1. Thus, N_1 would have properties (A) and (B) and $N_1 < N^*$. This again contradicts the minimality of N^*.

Thus, $C^* = 1$ and N^* is a normal complement to H in G, Q.E.D.

Proof of Theorem 4. It suffices to verify that (b) implies (a).

Let H and K be Hall subgroups of complementary orders in G such that each irreducible character of H and of K may be extended to one of G respectively. Let π be the set of prime divisors of $[H:1]$. We shall show that G is π-separated.

First, let $M \lhd G$. Then HM/M and KM/M are Hall subgroups of G/M with complementary orders. Thus, $[G:M] = [H:M \cap M][K:K \cap M]$ and $[M:1] = [H \cap M:1][K \cap M:1]$. Hence $H \cap M$ and $K \cap M$ are Hall subgroups of complementary orders in M.

Now, let $M \lhd G$ be minimal with respect to the property that G/M is π-separated. From the preceding paragraph, we have

$$M = (M \cap H)(M \cap K).$$

By the minimality of M, we see that $M = [M, M]$, and M is the normal closure in G of $M \cap H$, as well as of $M \cap K$.

By way of contradiction, let $1 < M$. Then

$$1 < M \cap H \lhd H \quad \text{and} \quad 1 < M \cap K \lhd K.$$

Thus, since the irreducible characters of a group separate the conjugate classes, we see that there exist characters of H and of K which are respectively nontrivial on $M \cap H$ and $M \cap K$. Among all such characters, take one of minimal degree. By symmetry, let this be χ, an irreducible character of H, with degree f. Thus, $f \mid [H:1]$; cf. [6; Theorem 16.8.4, p. 288].

Case 1. $f = 1$. Thus, we may extend χ to G, then restrict it to M. Since $M = [M, M]$, this restriction must be the trivial character on M. Thus, the restriction of χ to $M \cap H$ is trivial; this contradicts the choice of χ.

Case 2. $f > 1$. We now extend χ to G, then restrict it to K. This restriction to K is the direct sum of irreducible characters of K (cf. [6; Theorem 16.3.2, p. 255]), each of which must have degree dividing $[K:1]$. Since f is minimal and $([H:1], [K:1]) = 1$, we see that each of these components must be trivial on $M \cap K$. Thus, the restriction to K of an extension of χ to G is trivial on $M \cap K$. Hence, the extension of χ to G is trivial on $M \cap K$, therefore trivial on the normal closure in G of $M \cap K$, i.e., trivial on M. This contradicts the nontriviality of χ on $M \cap H$.

Thus, G is π-separated. Now, by Lemma 2.4(1) and Theorem 1(c), G contains a normal π'-Hall subgroup. This must be the only π'-Hall subgroup of G; i.e., it is K. By symmetry, we see that G is the direct product of H and K, Q.E.D.

5. Remarks

We may summarize the relations between Problem A and Problem B by the following:

THEOREM 5. *Let H be a π-Hall subgroup of G. Then Problem* A *and Problem* B *are equivalent in the following cases*:

(a) *H is soluble.*

(b) *G is π-separated.*

(c) *H has a Sylow tower.*

(d) *The terminal member of the lower central series of H is nilpotent.*

In cases (b), (c), *and* (d), *both problems are equivalent to the following*:

(C) *H is c-closed in G.*

We shall now describe some examples concerning Theorem 1 and Theorem 3.

Let S_n be the symmetric group of degree n. Let S_{n-1} be embedded in it in the obvious manner. In case n is a prime p, then S_{p-1} is a Hall subgroup of S_p. In all cases, S_{n-1} is c-closed in S_n. The verification of this is an elementary exercise in transformation of permutations. When $n > 4$, S_n does not contain a normal complement to S_{n-1}. Thus, the condition of π-separation in Theorem 1 cannot be dropped. Moreover, we cannot replace the conditions (1) and (2) of Theorem 1 by solubility plus the existence of a "nice" invariant series. In particular, the case of $n = 5$ shows that S_4 has an invariant series $1 \lhd V_4 \lhd A_4 \lhd S_4$ with invariant factors of orders 4, 3, and 2 respectively; furthermore, S_4 splits over each member of the series. But S_5 does not have a normal complement to S_4.

The conditions (1) and (2) in Theorem 3 are not equivalent, but are complementary in some sense. For example the direct product of A_4 with S_3 satisfies (2) but not (1); let S_3 operate faithfully on a finite vector space of characteristic not equal to 2 or 3 and form the split extension; then the resulting group will satisfy (1) but not (2).

It would be interesting to know if the condition of extendability of characters in Theorem 4 may be replaced by c-closure plus solubility. In particular, does there exist a nonsoluble group G such that G contains c-closed soluble Hall subgroups of complementary orders?

BIBLIOGRAPHY

1. R. BAER, *Closure and dispersion of finite groups*, Illinois J. Math., vol. 2 (1958), pp. 619-640.
2. ——, *Direkte Produkte von Gruppen teilerfremder Ordnung*, Math. Zeitschrift, vol. 71 (1959), pp. 454-457.
3. ——, *Die Existenz Hallscher Normalteiler*, Arch. Math., vol. 11 (1960), pp. 77-87.
4. W. BURNSIDE, *Theory of groups of finite order*, 2nd ed., New York, Dover Publications, 1955.
5. G. FROBENIUS, *Über auflösbare Gruppen, IV, V*, Sitzungber. Akad. Wiss. Berlin, 1901, pp. 1216-1230, 1324-1329.
6. M. HALL, *The theory of groups*, New York, Macmillan, 1959.
7. P. HALL, *Theorems like Sylow's*, Proc. London Math. Soc. (3), vol. 6 (1956), pp. 286-304.
8. D. G. HIGMAN, *Focal series in finite groups*, Canadian J. Math., vol. 5 (1953), pp. 477-497.
9. K. LUDWIG, *Direkte Produkte von π- und π'-Gruppen*, Arch. Math., vol. 11 (1960), pp. 171-175.
10. J. G. THOMPSON, *Normal p-complements for finite groups*, Math. Zeitschrift, vol. 72 (1960), pp. 332-354.
11. H. WIELANDT, *Über die Existenz von Normalteilern in endlichen Gruppen*, Math. Nachr., vol. 18 (1958), pp. 274-280.
12. H. ZASSENHAUS, *The theory of groups*, New York, Chelsea Publishing Co., 1949.

HARVARD UNIVERSITY
CAMBRIDGE, MASSACHUSETTS