SYMMETRIC HOMOLOGY SPHERES

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1. Introduction

In the course of thinking about a very suggestive conjecture [1], [2] concerning periodic transformations on the three-sphere, I ran across some interesting four-manifolds, W, which are of the homotopy type of S^4, but possibly not topologically equivalent to S^4. The conjecture claims that if a periodic transformation on the three-sphere has a circle as fixed-point set, then that circle must be unknotted. On these four-manifolds, W, that are constructed, one may exhibit an action of the circle group S, with fixed-point set a two-sphere Σ. The fundamental group of the complement, $\pi_1(W - \Sigma)$ is a split extension of the integers by a nontrivial group π, and therefore the two-sphere is knotted. (It cannot bound a flat disc.) The two-sphere Σ does however bound a one-parameter family of Poincaré cells (i.e., manifolds with trivial homology and with π as fundamental group) whose interiors are disjoint and which sweep out the space W.

The construction of these manifolds W involves the use of homology spheres with specific kinds of symmetries. Manifolds of that sort, I call symmetric homology spheres. The Poincaré icosahedral space is an example of such an object.

By employing a recent (as yet unpublished) characterization of Euclidean n-space ($n \geqq 5$) by Stallings, and using the above construction, an action of the circle on S^5 may be obtained, with a knotted three-sphere Σ^3 as fixed-point set, whose knot group is again a split extension of the group of integers by the group, π.

It should also be remarked that π may be taken to be the icosahedral group, thus exhibiting a phenomenon which cannot occur with knotted imbeddings of S^1 in S^3: the knot group of Σ^3 contains elements of finite order.

2. Terminology

All manifolds and maps in this paper will be combinatorial. Thus homeomorphism will mean combinatorial homeomorphism.

I denotes the unit interval, D^n the n-cell, S^n the n-sphere. If M is an n-manifold, ∂M is its boundary and int M its interior. M^* will denote M with a point removed; M_0 will denote M with the interior of a closed n-cell removed. A flat disc D^k in M^n is one which may be thrown onto the standard k-cell in a closed n-cell $D^n \subset M^n$ by a global automorphism of M^n. If X, Y are spaces, $f: X \to Y$ a map, $f: \pi_1(X) \to \pi_1(Y)$ will be the induced homo-

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morphism on the fundamental groups. R, Z will stand for the real numbers and integers respectively.

A Poincaré cell will be a manifold with trivial homology and nontrivial fundamental group, whose boundary is a sphere. The icosahedral group will be denoted by Δ, and the Poincaré icosahedral space is the homogeneous space S^3/Δ where S^3 is considered as the quaternions of norm 1, and $\Delta \subset S^3$ is considered as a subgroup in a natural manner. More useful for our purposes will be the description of S^3/Δ given in [3], or by its Heegard diagram (Figure 1).

DEFINITION 1. *A symmetric homology n-sphere is a manifold M such that*
(i) $H_g(M) \approx H_g(S^n)$ *for all g.*
(ii) *There is an orientation-preserving automorphism*

$$\tau: M \to M$$

with the following properties:
(a) *It is periodic (i.e., $\tau^\lambda = 1$ for some λ).*
(b) *It leaves some point $p \in M$ fixed:*

$$\tau(p) = p.$$

(c) *It trivializes the fundamental group of M. That is, if $N \subset \pi_1(M)$ is the normal subgroup generated by elements of the form*

$$\{\tau_* \alpha \cdot \alpha^{-1} \mid \alpha \in \pi_1(M)\},$$

where $\tau: \pi_1(M) \to \pi_1(M)$ is the homomorphism induced by τ, then

$$N = \pi_1(M).$$

Given a symmetric homology n-sphere, M, one may construct a manifold W which is of the homotopy type of S^{n+1}, with rather interesting properties.

3. The construction of W

Let M be a symmetric homology n-sphere, and $\tau: M \to M$ its automorphism. Let

$$S^1 \times_\tau M$$

be the space $I \times M$ after one identifies $O \times M$ with $1 \times M$ via the identification

$$\hat{\tau}: (O, m) \to (1, \tau(m)).$$

Let $\beta: I \times M \to S^1 \times_\tau M$ be the identification map. One may calculate $\pi_1(S^1 \times_\tau M)$ simply by observing that $S^1 \times_\tau M$ is a fibre bundle over S^1, possessing the cross-section

$$\Gamma = \{(t, p) \mid Z \in I\},$$

where p is the fixed point of τ (guaranteed to exist, by (ii b) of the definition of symmetric homology n-sphere). Therefore, by the homotopy exact sequence of a bundle,

$$(1) \qquad 0 \to \pi_1(M) \to \pi_1(S^1 \times_\tau M) \rightleftarrows \pi_1(S^1) \to 0,$$

and $\pi_1(S^1 \times_\tau M)$ is a split extension of $\pi_1(M)$ by $\pi_1(S^1) \approx Z$. Moreover, if ζ represents the appropriate generator of $\pi_1(S^1)$ in $\pi_1(S^1 \times_\tau M)$, one has the commutativity relations

$$\zeta\alpha\zeta^{-1} = \tau_* \alpha \qquad \text{for } \alpha \in \pi_1(M) \subset \pi_1(S^1 \times_\tau M).$$

One must now perform a bit of "surgery" on $S^1 \times_\tau M$. Let $\Gamma \subset S^1 \times_\tau M$ be the curve described before. Let $\Gamma \times D^n$ be a "thickening" of Γ (a tubular neighborhood). Such a tubular neighborhood may be considered $\Gamma \times D^n$ simply because (ii) τ is orientation-preserving. I may also assume that $\Gamma \times D^n$ is the image of $I \times D^n \subset I \times M$ for some $D^n \subset M$ under the identification map

$$\beta: I \times M \to S^1 \times_\tau M.$$

Let W_1 be the manifold $S^1 \times_\tau M$ with the interior of $\Gamma \times D^n$ removed. Thus

$$\partial W_1 = \Gamma \times S^{n-1}.$$

Consider a space $W_2 = D^2 \times S^{n-1}$, and a homeomorphism

$$\phi: \partial W_2 \to \partial W_1,$$

where

$$\phi: (\partial D^2 \times \theta) \to \Gamma \times \theta \qquad \text{for fixed } \theta \in S^{n-1}.$$

Call $W = W_1 \cup_\phi W_2$. This is the manifold which is of interest.

4. W is simply connected

It is obvious that $\pi_1(W_1) \approx \pi_1(S^1 \times_\tau M)$ since the removal of int $(\Gamma \times D^n)$ from $S^1 \times_\tau M$ does not affect the fundamental group as long as $n \geqq 3$. Adding W_2 to W_1 has the effect of trivializing ζ. Thus the injection $i: W_1 \to W$ induces an epimorphism $i_*: \pi_1(W_1) \to \pi_1(W)$ where $i_*(\zeta) = 1$. But then

$$i_*(\zeta\alpha\zeta^{-1}\alpha^{-1}) = 1 \qquad \text{for all } \alpha \in \pi_1(W_1),$$

and $i_*(N) = 1$, where N is as in (ii c), taking into account the fact that

$$\zeta\alpha\zeta^{-1} = \tau_* \alpha \qquad \text{for } \alpha \in \pi_1(W).$$

It follows from (ii c), formula (1), and the fact the i_* is onto, that

$$\pi_1(W) = \{1\}.$$

It is easy to check that W is a homology $(n + 1)$-sphere, and therefore it is a simple consequence of the Hurewicz theorem that W is of the homotopy type of S^{n+1}.

5. The "knotted" $(n-1)$-sphere Σ^{n-1} in W

Let Σ^{n-1} be the $(n-1)$-sphere

$$O \times S^{n-1} \subset D^2 \times S^{n-1} = W_2 \subset W,$$

where $O \in D^2$ is the center of the disc.

This Σ^{n-1} is knotted in W in the sense that its complement is homeomorphic with

$$S^1 \times_\tau M \subset \operatorname{int} W_1,$$

where M^* is M with a point removed. By formula (1),

$$\pi_1(S^1 \times_\tau M)$$

is an extension of $\pi_1(S^1)$ by the group, $\pi_1(M)$.

If $\pi_1(M)$ is nontrivial, therefore, Σ^{n-1} is knotted (i.e., Σ^{n-1} does not bound a flat disc in W). Notice, however, that Σ^{n-1} does bound a one-parameter family of homology cells homeomorphic with M_0, which span W.

6. The action of R on W

Let $\Phi_r : W \to W$ for $r \in R$ be the automorphism obtained from the automorphism $\phi_r : S^1 \times_\tau M \to S^1 \times_\tau M$ defined as follows:

$$\phi_r(t, \alpha) = (t + r - [r], \tau^{[r]}\alpha),$$

where $[r]$ is the greatest integer in r. It is clear that ϕ_r determines a continuous action of R on $S^1 \times_\tau M$.

Moreover, ϕ_r restricts to W_1 and may be extended to an action Φ_r of

$$W = W_1 \cup W_2$$

by radial extension of ϕ_r to $D^2 \times \theta \subset W_2$ for each fixed $\theta \in S^{n-1}$, ϕ_r being already defined on $\partial D^2 \times \theta \subset W_1 \cap W_2$. Moreover, it is also clear that the set of points fixed for all Φ_r $(r \in R)$ is Σ^{n-1}.

Since τ is periodic, of period λ, we have

$$\Phi_{r+\lambda} = \Phi_r,$$

and so Φ determines actually an action of the circle

$$S = R/\lambda Z$$

on the manifold W. The circle group S has Σ^{n-1} as fixed-point set, and M_0 as orbit space.

PROPOSITION 1. *The Poincaré manifold S^3/Δ is a symmetric homology three-sphere.*

Proof. There is a Heegard diagram of S^3/Δ on a five-holed torus T (see Figure 1). The torus T may be represented as a five-spoked wheel, the holes being arranged symmetrically in a regular pentagon about the center so that

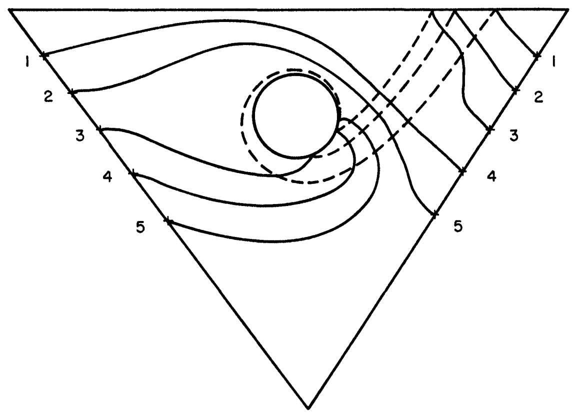

Figure 1

the entire Heegard diagram is brought into itself by the rotation of T about its center through the angle $2\pi/5$. One-fifth of the Heegard diagram of S^3/Δ is shown in Figure 1. The curves on the five-holed torus are given by linking together, in the indicated way, the five strands, which form five disjoint simple closed curves on T. I am thankful to Arnold Shapiro for this representation of S^3/Δ. This rotation induces an automorphism

$$\tau: S^3/\Delta \to S^3/\Delta$$

which is of period five, orientation-preserving, clearly leaves a point p (in fact an entire circle) fixed. Since τ permutes the five holes of T, it permutes the five generators of the representation of

$$\pi_1(S^3/\Delta) = \Delta$$

which this Heegard diagram induces. Thus

$$\tau_*: \pi_1(S^3/\Delta) \to \pi_1(S^3/\Delta)$$

is not the identity isomorphism. Using the fact that Δ is a simple group, it is immediate that τ satisfies condition (ii c) of Definition 1.

Actually, this could also be proved rather easily without resorting to the simplicity of Δ. Therefore, S^3/Δ is symmetric. In fact, any homology manifold obtainable via such a cyclically permutable Heegard diagram would be symmetric.

In this case, the constructed manifold W is four-dimensional, and Σ is the "common" two-sphere boundary of a one-parameter family of Poincaré cells which sweep out W. The circle group S acts on W by revolving the family of Poincaré cells. It leaves Σ fixed. The orbit of a "general" point $p \in W$ is a circle S_p which rotates "five times" about Σ. In connection with a remark made in the introduction, the commutator subgroup of $\pi_1(W - \Sigma)$

is finite and nontrivial, whereas knot groups of one-spheres in three-space contain no nontrivial elements of finite order [4]. I might remark that in this case, $\pi_1(W - \Sigma)$ is isomorphic with the direct product of the integers and the icosahedral group. This may be seen as follows:

The isosahedral group is a subgroup of index 2 in its automorphism group, which is the symmetric group on five letters. Therefore, any automorphism of period 5 (in particular, τ) must be an inner automorphism. Let $\rho \in \Delta$ be an element such that

$$\rho \cdot x \cdot \rho^{-1} = \tau(x) \qquad \text{for all } x \in \Delta.$$

Define, then, an isomorphism $\delta : Z \times \Delta \to \pi_1(W - \Sigma)$ by

$$\delta(z, x) = (\rho^{-1}\zeta, x),$$

where z is a generator of Z.

Let q be a point of Σ. Then the circle group acts on $W - \{q\}$. Extend this action to the space

$$L = (W - \{q\}) \times R$$

by letting the circle group act trivially on the R factor. This space is homeomorphic to R^5, according to Stallings. Taking the one-point compactification of L, L', one obtains a space homeomorphic to S^5 on which the circle group acts with fixed-point set, Σ^3, a three-sphere, such that

$$\pi_1(L' - \Sigma^3) \approx \pi_1(W - \Sigma).$$

This space L' can be given a combinatorial structure of S^5 which is almost the ordinary combinatorial structure of S^5, and with respect to which the circle group acts in a combinatorial manner. The combinatorial structure given to L' is such that there is a point x for which $L' - \{x\}$ is combinatorially equivalent to R^5.

References

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