THE RADIUS OF UNIVALENCE OF CERTAIN ENTIRE FUNCTIONS

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It was shown in [1] (see also [5]) that the radius of univalence, $R_U(\nu)$, of the function $z^{1-\nu}J_\nu(z)$, where $J_\nu(z)$ is the usual Bessel function ($\nu > 0$), is the smallest positive zero of its derivative, and two-sided inequalities were obtained for $R_U(\nu)$. In this note we give a short proof of a more general result, which delineates a rather broad class of entire functions for which the same conclusion holds. Further, we refine the inequalities mentioned above to sharper ones which give asymptotic equalities for $\nu \to \infty$. The basic idea is simply that whereas the radius of univalence is quite troublesome to deal with directly, the radius of starlikeness is obtainable almost immediately from Hadamard's factorization.

Let $\mathfrak{F}$ be a Montel compact [2] family of functions

$$f(z) = z + a_2 z^2 + \cdots, \tag{1}$$

regular in $|z| < 1$, and put $\gamma_n = \max_{f \in \mathfrak{F}} |a_n|$ $(n = 2, 3, \cdots)$. If

$$g(z) = z + b_2 z^2 + \cdots \tag{2}$$

is a given entire function, then the $\mathfrak{F}$-radius, $R_\mathfrak{F}$, of $g(z)$ is

$$R_\mathfrak{F} = \sup\{R \mid R^{-1}g(Rz) \in \mathfrak{F}\}. \tag{3}$$

The inequalities $|b_n| R^{n-1} \leqq \gamma_n$ $(n = 2, 3, \cdots)$ which must hold for all $R \leqq R_\mathfrak{F}$, show first that either $R_\mathfrak{F} < \infty$ or $g(z) \equiv z$, and second that

$$R_\mathfrak{F} \leqq \min_{n \leqq 2} \{\gamma_n / |b_n|\}^{1/(n-1)} \tag{4}$$

We consider the families (T) of typically real functions, (U) of univalent functions, (S) of starlike univalent functions, and (C) of convex univalent functions. If $g(z)$ in (2) has real coefficients, then plainly

$$R_C \leqq R_S \leqq R_U \leqq R_T \tag{5}$$

since a univalent function with real coefficients is typically real.

Now let G denote the class of entire functions of either of the following two forms:

$$\text{(6)}\quad \begin{aligned} &\text{(a)} \quad g(z) = z e^{\beta z} \textstyle\prod_{n=1}^{\infty} (1 + z/a_n), \\ &\text{(b)} \quad \beta \geqq 0; \quad 0 < a_1 \leqq a_2 \leqq \cdots; \quad \textstyle\sum |a_n|^{-1} < \infty, \end{aligned}$$

or

$$\text{(7)}\quad \begin{aligned} &\text{(a)} \quad g(z) = z \textstyle\prod_{n=1}^{\infty} (1 - z^2/a_n^2), \\ &\text{(b)} \quad 0 < a_1 \leqq a_2 \leqq \cdots; \quad \textstyle\sum |a_n|^{-2} < \infty. \end{aligned}$$

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THEOREM 1. *Let $g(z) \in G$, and let α denote the smallest of the moduli of the zeros of $g'(z)$. Then*

$$R_C \leqq R_S = R_U = \alpha \leqq R_T \leqq \min_{n \geqq 2} \{n/|b_n|\}^{1/(n-1)}. \tag{8}$$

Proof. The rightmost inequality in (8) follows from (4) and Rogosinski's theorem [3] that $\gamma_n = n$ in (T). In view of (5) and the obvious fact that $R_U \leqq \alpha$ we need only show that $R_S = \alpha$. But R_S is the radius of the smallest circle on which

$$\operatorname{Re}\{zg'(z)/g(z)\} > 0 \tag{9}$$

fails at some point. If, e.g., $g(z)$ is of the form (6), then for $|z| = r < a_1$ and $\arg z = \theta$ we have

$$\begin{aligned}
\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} &= 1 + \operatorname{Re}\left\{\beta z + \sum_{n=1}^{\infty} \frac{z}{z + a_n}\right\} \\
&= 1 + \beta r \cos\theta + r \sum_{n=1}^{\infty} \left\{\frac{r + a_n \cos\theta}{r^2 + a_n^2 + 2a_n r \cos\theta}\right\} \\
&\geqq 1 - \beta r + r \sum_{n=1}^{\infty} \frac{r - a_n}{r^2 + a_n^2 - 2a_n r} \\
&= \frac{(-r)g'(-r)}{g(-r)}.
\end{aligned}$$

The last quantity clearly remains positive until the first zero of $g'(-r)$ is reached, i.e., as long as $r \leqq \alpha$. The proof in the case (7) is virtually identical.

THEOREM 2. *For the function $z^{1-\nu} J_\nu(z) \in G$ we have*

$$R_U(\nu) = \sqrt{2\nu}\{1 + 1/4\nu + O(\nu^{-2})\} \qquad (\nu \to \infty). \tag{10}$$

Proof. Let us define

$$h_\nu(z) = 2^\nu \Gamma(\nu + 1) z^{1-\nu} J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m+1}}{m!\, 4^m (\nu + 1) \cdots (\nu + m)},$$

and then

$$\phi_\nu(z) = h_\nu'(2i\sqrt{\nu z}) = \sum_{m=0}^{\infty} \frac{(2m + 1)(\nu z)^m}{m!\,(\nu + 1) \cdots (\nu + m)}. \tag{11}$$

Since $h_\nu(z)$ has only real zeros, so has $h_\nu'(z)$, and thus $\phi_\nu(z)$ has only negative real zeros. Being of order $\frac{1}{2}$, it is of the form

$$\phi_\nu(z) = \prod_{n=1}^{\infty} (1 + z/a_n) \qquad (a_j > 0;\ j = 1, 2, \cdots).$$

Following the method of Euler ([4], p. 500), let us write

$$\sigma_j = \sum_{n=1}^{\infty} a_n^{-j} \qquad (j = 1, 2, \cdots).$$

We then find that

$$\phi_\nu'(z)/\phi_\nu(z) = \sum_{j=0}^{\infty} (-1)^j \sigma_{j+1} z^j \qquad (|z| < a_1). \tag{12}$$

By matching coefficients in (11) and (12) the first few σ_j are easily calculated (we omit the somewhat lengthy details), and then the relation

$$\sigma_3^{-1/3} \leqq a_1 \leqq \sigma_3/\sigma_4$$

gives the result (10).

References

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