RISES AND UPCROSSINGS OF NONNEGATIVE MARTINGALES

BY

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1. Introduction

Let f_n be a stochastic process such as one that describes the successive fortunes of a gambler, the successive prices of a stock, or the population of a particular species. Such processes are *nonnegative*. For each positive real number y, the process experiences a *rise* of size y if for some r and s with $r < s, f_s - f_r \geqq y$. Let x be a positive real number. If $f_0 \equiv x$, the process *begins* at x. In lieu of the semimartingale terminology we sometimes find it suggestive to call a process *subfair* or (*conditional*) *expectation-decreasing* if for all r and s with $r < s$, the conditional expectation of f_s given f_n for $n \leqq r$ does not exceed f_r.

(1.1) THEOREM. *Let f_n, $n = 0, 1, 2, \cdots$, be a nonnegative subfair process that begins at the positive real number x. Then, for each positive real number y, the probability that the process experiences a rise of size y is strictly less than $1 - e^{-x/y}$. Moreover, this bound is best possible.*

The main purpose of this paper is to prove Theorem (1.1), or rather, its generalization, Theorem (11.1), which gives sharp bounds to the probability that nonnegative expectation-decreasing processes experience several rises. Though the first eleven sections of this paper are needed for the proof of (11.1), some of the intermediate results are of interest in themselves. Some of the ideas used in proving the "concrete" results (1.1) and (11.1) have been isolated, and presented in a somewhat general and abstract form in Sections 3, 4, and 6. These same ideas and techniques are then easily applied in Sections 12, 13, and 14, to find sharp bounds to the probability that nonnegative lower semimartingales have k or more upcrossings or downcrossings. These latter sections make contact with earlier work of Doob and Hunt [4], [10].

2. The bound in Theorem (1.1) is best possible

Let f_n be the fortune at time n of a gambler who gambles according to a scheme about to be described. Consider a fair two-valued gamble g that wins $y > 0$ with probability W, and that loses $s > 0$ with probability L. Here $W + L = 1$. Since g is fair,

$$(2.1) \qquad W = \frac{s}{s + y}; \qquad L = \frac{y}{s + y} = \frac{1}{1 + s/y}.$$

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Suppose that our imaginary gambler divides his initial fortune x into a large number N of equal parts, and that he then plays at a sequence of N independent fair gambles, each with a stake s equal to x/N, and each with a gain of y. The probability L that any single stake is lost, is, from (2.1), seen to be

$$L = \frac{1}{1 + x/Ny}. \tag{2.2}$$

The probability that all N stakes are lost is, therefore, L^N. As $N \to \infty$, $L^N \to e^{-x/y}$. That is, the probability of winning at least once, or of experiencing a rise of size y, is arbitrarily close to $1 - e^{-x/y}$. This suggests how the inequality was conjectured and shows that the upper bound is best possible.

3. Transformations that preserve lower semimartingales

Let F be any interval of real numbers, perhaps unbounded, and let F^* be the set of all finite sequences $f_0, f_1, \cdots, f_n$ of elements of F. A real-vaued function Q with domain F^* will be said to be *nondecreasing* if (i) $Q(f_0)$ is nondecreasing in f_0, and (ii) for all n and all $(f_0, \cdots, f_{n-1})$, $Q(f_0, \cdots, f_{n-1}, f_n)$ is nondecreasing in f_n. Q will be said to be *coordinatewise concave* if $Q(f_0)$ is concave in f_0, and if for all n and all $f_0, \cdots, f_{n-1}$, $Q(f_0, \cdots, f_{n-1}, f_n)$ is concave in f_n. Q is *patient* if for all $n \geqq 1$, and all $f_0, \cdots, f_{n-1}, f_n$ such that $f_{n-1} = f_n$, $Q(f_0, \cdots, f_{n-1}) = Q(f_0, \cdots, f_{n-1}, f_n)$. We *assume henceforth* that Q is *bounded* and *continuous*, as it is in the applications, and thereby simplify tedious integrability and measurability justifications.

For random variables g, $E(g)$ designates the expected value of g.

(3.1) THEOREM. *Let Q be nondecreasing, coordinatewise concave, and patient. Let f_n, $n = 0, 1, 2, \cdots$, be a lower semimartingale. Then $Q(f_0, \cdots, f_n)$ is a lower semimartingale, and for all n, $E(Q(f_0, \cdots, f_n)) \leqq Q(E(f_0))$.*

Proof. $E[Q(f_0, \cdots, f_n, f_{n+1}) \mid f_0, \cdots, f_n]$, the conditional expectation of $Q(f_0, \cdots, f_n, f_{n+1})$ given $f_0, \cdots, f_n$, equals

$$\int q(z)\, dv(z), \tag{3.2}$$

where $v = v(f_0, \cdots, f_n)$ is the conditional probability distribution of f_{n+1} given $f_0, \cdots, f_n$, and $q(z) = Q(f_0, \cdots, f_n, z)$.

Since Q is concave in z, Jensen's inequality [9] implies that (3.2) is dominated by $q(\bar{v})$, where $\bar{v} = \bar{v}(f_0, \cdots, f_n)$ is the mean of v. Of course, $q(\bar{v}) = Q(f_0, \cdots, f_n, \bar{v})$. Since f_j is a lower semimartingale, $\bar{v} \leqq f_n$. Because Q is nondecreasing in its last coordinate,

$$Q(f_0, \cdots, f_n, \bar{v}) \leqq Q(f_0, \cdots, f_n, f_n).$$

Since Q is patient, this last expression equals $Q(f_0, \cdots, f_{n-1}, f_n)$. Thus, $Q(f_0, \cdots, f_n)$ is a lower semimartingale, as asserted. Therefore

$$E(Q(f_0, \cdots, f_n)) \leqq E(Q(f_0)).$$

Again by Jensen's inequality, $E(Q(f_0)) \leqq Q(E(f_0))$. This completes the proof.

4. Upper bound for the probability of irrevocable events

In this section, F may be any set, and F^* is the set of all finite sequences of elements of F. An element $(f_0, \cdots, f_k)$ of F^* is an *extension* of another element $(f_0', \cdots, f_n')$ of F^* if $k \geqq n$ and $f_i' = f_i$ for $0 \leqq i \leqq n$. A set A of finite sequences is *closed under extension* if every extension of every element of A is in A.

(4.1) LEMMA. *Let u_n be an increasing sequence of real numbers, and let q_n be decreasing, $n = 0, 1, \cdots$. Suppose that for all n, $u_n \leqq q_n$. Then for all i and j, $u_i \leqq q_j$.*

Proof. If $i \leqq j$, then $u_i \leqq u_j \leqq q_j$. If $i \geqq j$, then $u_i \leqq q_i \leqq q_j$. This completes the proof.

Let u_n be a real-valued stochastic process, $n = 0, 1, \cdots$. It is *increasing* if $u_n \leqq u_{n+1}$ for all n; *conditional-expectation-increasing* if $E[u_{n+1} \mid u_0, \cdots, u_n] \geqq u_n$ for all n; *expectation-increasing* if $E(u_{n+1}) \geqq E(u_n)$ for all n. Each of these three conditions obviously implies the next.

In each of the applications to be made in this paper of the results of this section and of Section 6, F will be an interval of real numbers, and, in each of these applications, it will be obvious that u and A are measurable in the usual Borel sense. Thus, for the purposes of this paper, the reader may interpret these results as having implicit measurability assumptions that he will have no trouble providing for himself. The reasons that no explicit measurability assumptions are made are twofold: first, the essential hypotheses are thus brought into sharper relief, and second, a foundation to the theory of stochastic processes [5] can be given that imparts wider validity to the results of these two sections. In particular, with such a foundation, it is possible to treat rigorously all nonmeasurable and non-countably-additive discrete time-parameter processes f_n, $n = 0, 1, \cdots$, such that the joint distribution of $(f_0, \cdots, f_{n+1})$ is obtained in the usual way from the joint distribution of $(f_0, \cdots, f_n)$ and a conditional distribution of f_{n+1} given $(f_0, \cdots, f_n)$, and, after this foundation is laid, it is easy to prove that even if such processes are admitted to the competition, the bounds obtained in this paper remain the best possible. The proofs of these stronger theorems just alluded to depend upon a more generous interpretation of the results of this section and of Section 6 than is allowed by the usual measurability constraints.

(4.2) LEMMA. *Let u and Q be defined for all finite sequences $(f_0, \cdots, f_n)$, and suppose that $u \leqq Q$. Suppose that f_n is a stochastic process, and that $u(f_0, \cdots, f_n)$ is an expectation-increasing process, and $Q(f_0, \cdots, f_n)$ is expectation-decreasing. Then, for all i and j,*

$$E(u(f_0, \cdots, f_i)) \leqq E(Q(f_0, \cdots, f_j)).$$

Proof. Let $u_n = E(u(f_0, \cdots, f_n))$, and let $q_n = E(Q(f_0, \cdots, f_n))$; a referral to (4.1) completes the proof.

(4.3) LEMMA. *Let A be a set of finite sequences of elements of F. Suppose that A is closed under extension. Let $u(f_0, \cdots, f_n) = 1$ or 0 according as $(f_0, \cdots, f_n) \in A$ or not. If $\{f_n, n = 0, 1, \cdots\}$ is any stochastic process with values in F, then $u(f_0, \cdots, f_n)$ is an increasing process.*

These lemmas are now applied to prove the following variant of a theorem in [5, Chapter 2, Section 12].

(4.4) THEOREM. *Let A be a subset of F^* that is closed under extension. Let f_n, $n = 0, 1, \cdots$, be a stochastic process with values in F, and let P be the probability that for some n, $(f_0, \cdots, f_n) \in A$. Let Q be nonnegative, and suppose that $Q(f_0, \cdots, f_n) \geqq 1$ whenever $(f_0, \cdots, f_n) \in A$. Suppose, too, that $Q(f_0, \cdots, f_n)$ is expectation-decreasing. Then for all n,*

$$P \leqq E(Q(f_0, \cdots, f_n)).$$

Proof. Let u be as in (4.3). Then $u(f_0, \cdots, f_n)$ is increasing and hence expectation-increasing. Clearly, $u \leqq Q$. Therefore by (4.2), for all i and n, $E(u(f_0, \cdots, f_i)) \leqq E(Q(f_0, \cdots, f_n))$. But $P = \lim E(u(f_0, \cdots, f_i))$. This completes the proof.

By restricting F to an interval of real numbers, (3.1) and (4.4) immediately yield the principal general theorem of this paper.

(4.5) THEOREM. *Let A be a set of finite sequences of real numbers that is closed under extension. Let f_n, $n = 0, 1, \cdots$, be conditional expectation-decreasing, and let P be the probability that for some j, $(f_0, \cdots, f_j) \in A$. Let Q be nonnegative, and suppose that $Q(f_0, \cdots, f_n) \geqq 1$ whenever $(f_0, \cdots, f_n) \in A$. Suppose, too, that Q is nondecreasing, coordinatewise concave, and patient. Then $P \leqq Q(E(f_0))$.*

5. Proof for the bound in Theorem (1.1)

Let y be 1, for this is only a change of scale. Let F^* be the set of all finite sequences $(f_0, \cdots, f_n)$, where each f_i is a nonnegative real number. Our immediate objective is to define a function Q on F^*. Suppose that a gambler's successive fortunes are $(f_0, \cdots, f_n)$, and that he then switches to a gambling strategy suggested by the scheme in Section 2. $Q(f_0, \cdots, f_n)$ is to represent the probability that he will ultimately achieve a rise of size 1. Of course, a rise of size 1 may have already occurred in the finite sequence $(f_0, \cdots, f_n)$. If so, define $Q(f_0, \cdots, f_n) = 1$. Having in mind the case that a rise of size 1 has not occurred in $(f_0, \cdots, f_n)$, define a function m as follows: If $n \geqq 1$, let $m = m(f_0, \cdots, f_{n-1})$ be the minimum of $f_0, \cdots f_{n-1}$. For formal convenience let $m = \infty$ when $n = 0$. There are three possibilities: Either $f_n \leqq m$, $m < f_n < m + 1$, or $m + 1 \leqq f_n$. In the last case, there is certainly a rise of size 1, and therefore $Q(f_0, \cdots, f_n) = 1$.

Suppose now that $f_n \leqq m$, and that no rise of size 1 has occurred in $(f_0, \cdots, f_n)$. Then, by gambling according to the scheme in Section 2, it is possible to achieve a rise of size 1 with a probability arbitrarily close to $1 - e^{-f_n}$. This suggests letting $Q(f_0, \cdots, f_n) = 1 - e^{-f_n}$ in this case. Consider now the case $m < f_n < m + 1$, and no rise of size 1 has occurred in $(f_0, \cdots, f_n)$. Suppose that the gambler selects a fair gamble that increases his fortune to $m + 1$ with probability $f_n - m$, or that decreases it to m with probability $1 - (f_n - m)$. If he wins that gamble, he has achieved a rise of size 1. If not, he can yet achieve his desired rise with a probability arbitrarily close to $1 - e^{-m}$. This suggests defining Q for such a sequence $(f_0, \cdots, f_n)$ by $Q(f_0, \cdots, f_n) = (f_n - m) + (1 - (f_n - m))(1 - e^{-m})$.

It is easy to verify that Q is nondecreasing, coordinatewise concave, and patient. Let A be the set of all finite sequences $(f_0, \cdots, f_n)$ of real numbers such that a rise of size 1 occurs in $(f_0, \cdots, f_n)$. Now let $\{f_n\}$ be a conditional-expectation-decreasing process. It is easy to verify that the hypotheses of (4.5) are satisfied. Therefore, $P \leqq Q(E(f_0))$. In particular, if $\{f_n\}$ begins at x, then $f_0 \equiv x$, and hence $Q(E(f_0)) = Q(x) = 1 - e^{-x}$. Thus, except for the proof that the inequality is strict, the proof of (1.1) is complete.

The next section is concerned with generalities preliminary to the proof that the inequality is strict.

6. Strict bounds for the probability of irrevocable events

Let A be a set of finite sequences of elements of a set F. Let $f_0, \cdots f_n, \cdots$ be a stochastic process. If B is the event, for some n, $(f_0, \cdots, f_n) \in A$, then B is said to *irrevocable*. Let A^* be the set of all extensions of elements of A. Then A^* is extensionally closed. Clearly, B is likewise the event that for some n, $(f_0, \cdots, f_n) \in A^*$. Thus, in the definition of *irrevocable*, there is no real difference if we restrict attention to extensionally closed sets A.

Using the notation of Section 3, Q is *strongly coordinatewise concave* if it is coordinatewise concave, and if, for all $n \geqq 0$, and all f_0, if $f_i = f_0$ for $0 \leqq i \leqq n$, then $Q(f_0, f_1, \cdots, f_n, f_{n+1})$ is strictly concave in f_{n+1} for $f_{n+1} < f_0$.

A stochastic process f_n, $n = 0, 1, \cdots$, is *advancing* if for some n and $\varepsilon > 0$ the probability that $f_n \geqq f_0 + \varepsilon$ is positive.

(6.1) Lemma (Variant of Jensen's inequality). *Let q be a concave function of a real variable. Let f be a real number, and suppose that for $z < f$, q is nondecreasing and strictly concave. Let v be a probability measure with a mean*

$\bar{v} \leqq f$. *Suppose that for some* $\varepsilon > 0$, $v[f + \varepsilon, \infty) > 0$. *Then,*

$$\int q(z)\, dv(z) < q(f).$$

The proof is easy and is omitted.

(6.2) LEMMA. *Let* f_n, $n = 0, 1, \cdots$, *be an advancing lower semimartingale. Let* Q *be nondecreasing, strongly coordinatewise concave, and patient. Then, for some* n, $E(Q(f_0, \cdots, f_n)) < Q(E(f_0))$.

Proof. Consider the least n such that for some $\varepsilon > 0$, with positive probability $f_n \geqq f_0 + \varepsilon$. Then $f_i = f_0$ for $0 \leqq i \leqq n - 1$.

$$E[Q(f_0, \cdots, f_{n-1}, f_n) \mid f_0, \cdots, f_{n-1}] = \int q(z)\, dv(z),$$

where $q(z) = Q(f_0, \cdots, f_{n-1}, z)$, and $v = v(f_0, \cdots, f_{n-1})$ is the conditional distribution of f_n given $f_0, \cdots, f_{n-1}$. By assumption on Q, $q(z)$ is concave, and, for $z < f_{n-1}$, it is strictly concave and strictly increasing. Since f_j is a lower semimartingale, $\bar{v}$, the mean of v, is less than or equal to f_{n-1}.

Since $f_n \geqq f_{n-1} + \varepsilon$ with positive probability, with positive probability v assigns positive probability to the set of $z > f_{n-1}$. In this event, by the preceding lemma,

$$\int q(z)\, dv(z) < q(f_{n-1}).$$

That is, with positive probability

$$\int q(z)\, dv(z) < q(f_{n-1}),$$

and with the remaining probability

$$\int q(z)\, dv(z) \leqq q(f_{n-1}).$$

Thus $E[Q(f_0, \cdots, f_{n-1}, f_n) \mid f_0, \cdots, f_{n-1}]$ is dominated by

$$Q(f_0, \cdots, f_{n-1}, f_{n-1}) = Q(f_0, \cdots, f_{n-1})$$

where the domination is strict with positive probability. Therefore, $E(Q(f_0, \cdots, f_{n-1}, f_n)) < E(Q(f_0, \cdots, f_{n-1})) = E(Q(f_0)) \leqq Q(E(f_0))$. This completes the proof.

From (3.1) and (4.4) it is simple to deduce

(6.3) LEMMA. *Suppose that the hypotheses of* (4.5) *hold. Then, for all* n, $P \leqq E(Q(f_0, \cdots, f_n))$.

From (6.2) and (6.3) we immediately obtain

(6.4) THEOREM. *Let* A *be a set of finite sequences of real numbers that is closed under extension. Let* f_n, $n = 0, 1, \cdots$, *be advancing but conditional-*

expectation-decreasing, and let P be the probability that for some n, $(f_0, \cdots, f_n) \in A$. *Let Q be nonnegative, and suppose that* $Q(f_0, \cdots, f_n) \geqq 1$ *whenever* $(f_0, \cdots, f_n) \in A$. *If Q is nondecreasing, strongly coordinatewise concave, and patient, then* $P < Q(E(f_0))$.

By applying (6.4) to the Q and A in Section 5, it is obvious that the inequality in (1.1) is strict. Thus, *the proof of* (1.1) *is now complete.*

7. A path counting problem

We now devote four sections, 7 through 10, to preparing for the proof of our main theorem in Section 11. The honor of priority for much, and perhaps all, of this preparatory material is due to others, for example, to Gani and Pyke (see [8], [11]). But, a prior publication adaptable to the needs of this paper has not been found. Moreover, I hope that something new on these topics will be said here.

A preliminary to the next section, and of some independent interest, is the following counting problem, that is similar to the "ballot box problem" treated in [6] and [7, page 66].

Let s be a positive real number, and let ε_i be either plus 1 or minus s, for $1 \leqq i \leqq N$ where N is a positive integer. A *path* (*with losses of size s and rises of size* 1) is a polygonal line in the plane whose vertices have abscissas $0, 1, \cdots N$, and ordinates $x_0, x_1, \cdots, x_N$, where $x_i = x_{i-1} + \varepsilon_i$, $1 \leqq i \leqq N$.

A path has *precisely k rises* if precisely k of the ε_i are plus 1. Of course $x_N - x_0 = \sum \varepsilon_i = k - (N - k)s$.

In particular, N is determined when x_N, x_0, k, and s are given. The path is said to *terminate near* 0 if $0 \leqq x_N < s$.

Thus, given x_0, k, and s, N is uniquely determined for paths that terminate near 0. The path is said to *begin at* x if $x_0 = x$. It is *proper* if it terminates near 0 and for all $i < N$, $x_i \geqq s$.

Let $M = M(k, x, s)$ be the number of proper paths beginning at x with losses of size s, that have precisely k rises of size 1. In order to express the asymptotic behavior of M as s approaches 0, define by induction a sequence of polynomials:

(7.1) $$\theta(0, x) \equiv 1.$$

(7.2) $$\theta(k + 1, x) = \int_1^{1+x} \theta(k, t)\, dt, \qquad \text{for } k \geqq 0.$$

These polynomials can be expressed in closed form [11].

(7.3) THEOREM. *For each* $k \geqq 0$ *and each* $x > 0$, $s^k M(k, x, s)$ *converges to* $\theta(k, x)$ *as s converges to* 0 *through positive values of s. Moreover, for each* $k \geqq 1$, *the convergence is uniform with respect to x, provided that x is confined to any finite interval of nonnegative real numbers.*

Proof. Proceed by induction on k. If $k = 0$, $s^k M(k, x, s) = 1 \cdot 1 = 1 = \theta(k, x)$ for all $x \geqq s$. Consider now a proper path beginning at x that has precisely $k + 1$ rises. For such a path there is at least one i such that $\varepsilon_i = +1$. Let j be the least such i. Thus, $x_{j-1} = x - (j - 1)s$, and $x_j = x - (j - 1)s + 1$. Since $x_{j-1} \geqq s, j \leqq (x/s)$. In fact, $j \leqq [x/s]$, where $[z]$ means the largest integer $\leqq z$. It is now easy to see that

$$(7.4)\qquad \begin{aligned} M(k + 1, x, s) &= \textstyle\sum_{j=1}^{[x/s]} M(k, x - (j - 1)s + 1, s) \\ &= \textstyle\sum_{j=0}^{[x/s]-1} M(k, (x - js + 1, s). \end{aligned}$$

Therefore, $s^{k+1}M(k + 1, x, s) = s\sum s^kM(k, x - js + 1, s)$.

As j varies between 0 and $[x/s]$, js remains in the interval $[0, x]$. Hence $x - js + 1$ is in the interval $[1, x + 1]$. Now let I be any bounded interval of nonnegative reals, say $I = [0, y]$. Then as x varies over I, and j varies between 0 and $[x/s]$, $x - js + 1$ remains in the interval $[1, y + 1]$. Let $\varepsilon > 0$. By the inductive hypothesis there is an $s_0 > 0$ such that for $0 < s_0 < s$, $s^kM(k, t, s)$ differs from $\theta(k, t)$ by less than ε for all t satisfying $[1 \leqq t \leqq y + 1]$. Therefore $\sum s^kM(k, x - js + 1)$ differs from $\sum\theta(k, x - js + 1)$ by less than $([x/s] - 1)\varepsilon < \varepsilon[x/s] \leqq \varepsilon x/s \leqq \varepsilon y/s$; consequently, $s\sum s^kM(k, x - js + 1, s)$ differs from $s\sum\theta(k, x - js + 1)$ by less than εy. Of course, $\theta(k, t)$ is bounded for $1 \leqq t \leqq 1 + y$, say by b, where b depends on k and y. Since $\theta(k, t)$ is uniformly continuous for $1 \leqq t \leqq 1 + y$, there is an s_1 such that if $| t - t' | < s_1$ then $| \theta(k, t) - \theta(k, t')| < \varepsilon$. Then, if s is less than s_1, $s\sum\theta(k, x - js + 1)$ differs from $\int_1^{1+x}\theta(k, t)\, dt$ by less than $\varepsilon x + bs < \varepsilon y + bs$. All in all, if s is less than both s_0 and s_1, and if x is confined to $[0, y]$, then $s^{k+1}M(k + 1, x, s)$ differs from $\int_1^{1+x}\theta(k, t)\, dt$ by less than $2\varepsilon y + bs$. That is, as s converges to 0, there is uniform convergence to $\theta(k + 1, x)$ for x in $[0, y]$. This completes the proof.

8. Rises for fair binomial processes

Let k be a positive integer, and y a positive real number. A stochastic process f_n is said to experience k or more *rises* of size (at least) y if there exist subscripts $r_1, \cdots, r_k$, and subscripts $s_1, \cdots, s_k$, such that

$$r_1 < s_1 \leqq r_2 < s_2 \leqq \cdots \leqq r_k < s_k \quad \text{and} \quad f_{s_i} - f_{r_i} \geqq y \quad \text{for} \quad 1 \leqq i \leqq k.$$

Our principal purpose is to obtain, for each k, sharp bounds to probability that a nonnegative expectation-decreasing process that begins at a positive real number x experiences k or more rises of size y. The case $k = 1$ was settled in the preceding sections. In this section we make further progress toward the solution for general k by studying the probability that certain particularly simple and promising processes experience k or more rises. Just as for $k = 1$, it suffices to treat the case $y = 1$.

Consider a gambler with an initial fortune of $x > 0$ engaged in a sequence of fair bets, each of stake $s < x$ and of gain 1. He terminates play the first time his fortune falls below s. It is well known that termination will occur with probability 1, [2], and by the Blackwell-Wald Theorem [1], the expected number of times that the gambler wins is infinite. Let $T(k, x, s)$ be the probability that this stopped *fair binomial process* experiences k or more rises of size 1. We are interested in the behavior of $T(k, x, s)$ as s approaches 0.

The probability that any particular gamble is lost is $L = 1/(1 + s)$ and the probability that it is won is $W = s/(1 + s)$.

Let $p(k, x, s)$ be the probability of precisely k rises prior to termination. This probability can be calculated as follows. Again let $[z]$ represent the largest integer not exceeding z. Any path that has precisely k rises and that terminates at a fortune in $[0, s)$ must have precisely $l = [(x + k)/s]$ losses. The probability of such a path is $W^k L^l$. Let $M = M(k, x, s)$ be the number of such paths. Therefore, $p(k, x, s) = MW^k L^l$.

Observe next that as s goes to 0, $L^l = (1 + s)^{-l}$ converges to $e^{-(x+k)}$. Observe too, that according to the preceding section, $W^k M(k, x, s) = s^k M(k, x, s)/(1 + s)^k$ converges to $\theta(k, x)$ as s converges to 0.

We now have established

(8.1) THEOREM. *Let f_n be a stopped fair binomial process that begins at x with stake s and gain* 1. *Then, for all $k > 0$, the probability that f_n experiences precisely k rises of size* 1 *converges to $\theta(k, x)e^{-(x+k)}$.*

Define

(8.2) $$T(k, x) = 1 - \sum_{j=0}^{k-1} \theta(j, x)e^{-(x+j)} \quad \text{for } k \geqq 1 \quad \text{and} \quad T(0, x) = 1.$$

(8.3) COROLLARY. *For all k, the probability that the stopped fair binomial process experiences k or more rises converges to $T(k, x)$.*

9. Rises for centered Poisson processes

Let x_t be a nice Poisson process of mean t, so that the centered process $y_t = x_t - t$ has mean 0. Let x be a positive number. Then the process $x + y_t$ begins at x. Let τ be the first time that $x + y_t = 0$. It follows from [2] that τ is finite with probability 1. Also, it is known that the probability that $\tau = x + k$ is $\theta(k, x)e^{-(x+k)}$ (see [8]). Consequently,

(9.1) $$1 = \sum_{j=0}^{\infty} \theta(j, x)e^{-(x+j)}.$$

This implies

(9.2) $$T(k, x) = \sum_{j=k}^{\infty} \theta(j, x)e^{-(x+j)}.$$

And it also implies the following known expansion for the exponential function [11]:

(9.3) $$e^x = \sum_{j=0}^{\infty} \theta(j, x)e^{-j}.$$

Let $f_t = x + y_t$ if $t \leqq \tau$, and let $f_t = 0$ if $t \geqq \tau$. Then f_t is a continuous-parameter nonnegative martingale that begins at x (see [3]). It is easy to see that the events, f_t has k or more rises of size 1 and $\tau \geqq x + k$, are equivalent. Therefore, we have

(9.4) THEOREM. *The probability that the stopped Poisson process experiences k or more rises is $T(k, x)$.*

We conclude this short section by noting, without giving the easy proof, that the interesting polynomials θ_j satisfy the following identity for all k, x, and y.

(9.5) THEOREM.

$$\theta(k, x + y) = \theta(k, x)\theta(0, y) + \theta(k - 1, x)\theta(1, y) + \cdots + \theta(1, x)\theta(k - 1, y) + \theta(0, x)\theta(k, y).$$

10. Some identities concerning $T(k, x)$

The following sequence of identities culminating in one that we shall soon need are easily verified in order. (Primes over functions denote differentiation.)

$$\theta'(j + 1, x) = \theta(j, x + 1). \tag{10.1}$$

Let $d(j, x) = \theta(j, x)e^{-(x+j)}$. Then,

$$d'(j + 1, x) = d(j, x + 1) - d(j + 1, x). \tag{10.2}$$

Since

$$T(k + 1, x) = 1 - \textstyle\sum_{j=0}^{k} d(j, x), \tag{10.3}$$

one obtains

$$T'(j + 1, x) = T(j, x + 1) - T(j + 1, x). \tag{10.4}$$

Replace j by $j + 1$, and differentiate again getting

$$T''(j + 2, x) = T'(j + 1, x + 1) - T'(j + 2, x). \tag{10.5}$$

Replace the right side of (10.5) by the values indicated by (10.4), and obtain

$$T''(j + 2, x) = T(j, x + 2) - 2T(j + 1, x + 1) + T(j + 2, x). \tag{10.6}$$

11. Main result

(11.1) THEOREM. *Let f_n, $n = 0, 1, 2, \cdots$, be a nonnegative subfair process that begins at the positive real number x. Then, for each $y > 0$ and all integers $k \geqq 1$, the probability that the process experiences k or more rises of size at least y is strictly less than $T(k, x/y)$. Moreover, this bound is best possible.*

Of course, this theorem implies the pointwise convergence theorem for nonnegative lower semimartingales proved in [3]. Theorem (11.1) gives, in a sense, precise bounds to the rate of convergence.

That the bound cannot be improved, and a little more, follows from (8.3).

That $T(k, x/y)$ is a bound will be established by induction on k. We find, however, that in order to effect the induction we must also prove simultaneously by induction the following sets of inequalities:

$$(11.2)\quad T(k, z) \geqq \frac{s}{s+g} T(k-1, z+g) + \frac{g}{s+g} T(k, z-s) \qquad \text{for all } k \geqq 1, 0 < s < z < z+1 \leqq z+g.$$

$$(11.3)\quad T(k, z) > \frac{s}{s+g} T(k-1, z+g) + \frac{g}{s+g} T(k+1, z-s) \qquad \text{for all } k \geqq 1, 0 < s < z < z+1 \leqq z+g.$$

$$(11.4)\quad T(k, z) > \tfrac{1}{2}T(k-1, z+1) + \tfrac{1}{2}T(k+1, z-1) \qquad \text{for all } k \geqq 1, z \geqq 1.$$

(11.5) $T(k+1, z)$ is strictly concave in z for $z \geqq 0$.

(11.6) $T'(k, z) < T(k, z) - T(k+1, z-1)$ for all $k \geqq 1$, $z \geqq 1$.

In view of (9.4), *these inequalities yield quantitative information about the Poisson process.*

(11.7) LEMMA. *For any fixed* $k \geqq 1$, *the following implications are valid*:

$$(11.1) \Rightarrow (11.2) \Rightarrow (11.3) \Rightarrow (11.4) \Rightarrow (11.5) \Rightarrow (11.6).$$

Proof. Assume (11.1), and consider a gambler whose initial fortune is z. Suppose that he selects a fair gamble that either increases his fortune to $z + g$ or decreases it to $z - s$. Suppose that, for either outcome, he continues to gamble according to the scheme in Section 2. According to (8.3), the probability that he achieves k rises of size 1 is arbitrarily close to the right side of (11.2). But according to (11.1) this is dominated by the left side of (11.2).

That (11.2) implies (11.3) is immediate from the fact that $T(k, x) > T(k+1, x)$ for all $x > 0$.

Letting $s = g = 1$ in (11.3) yields (11.4).

To see that (11.4) implies (11.5), let $j + 1 = k$, and $z + 1 = x$ in (10.6), obtaining $T''(k+1, z-1) < 0$ for all $z \geqq 1$.

Finally, assume (11.5). Then (10.6) implies (11.4). That is,

$$T(k-1, z+1) - T(k, z) < T(k, z) - T(k+1, z-1).$$

Hence by (10.4), $T'(k, z) < T(k, z) - T(k+1, z-1)$. This completes the proof of (11.7).

Proof of (11.1). We may suppose that $y = 1$, for this is a mere change of scale. The case $k = 1$ is the content of (1.1). We proceed by induction, assuming that (11.1) holds for all positive integers less than or equal to k and proceed to consider $k + 1$, with $k \geqq 1$.

Let A be the set of all finite sequences $(f_0, \cdots, f_n)$ of nonnegative real numbers such that $k + 1$ or more rises of size 1 occur in $(f_0, \cdots, f_n)$. If $\{f_n, n = 0, 1, \cdots\}$ is a stochastic process that begins at x, and if, for all n, the probability that $f_n > f_0$ is 0, then the probability P that some $(f_0, \cdots, f_n) \in A$ is 0, which is strictly less than $T(k + 1, x)$. Thus, we may assume that $\{f_n\}$ is advancing. Clearly, A is extensionally closed. The program now is to define an appropriate Q and then apply (6.4).

Temporarily, $f_0, \cdots, f_n$ designates nonnegative reals. $Q(f_0, \cdots, f_n)$ is intended to correspond to the probability that $k + 1$ or more rises of size 1 occur in the infinite sequence $f_0, \cdots, f_n, f_{n+1}, \cdots$ given that $f_0, \cdots, f_n$ are the initial values, and that f_{n+1} and all succeeding fortunes arise as a consequence of gambling according to the schemes in Sections 2 and 5.

Thus, define

$$Q(f_0) = T(k + 1, f_0). \tag{11.8}$$

For $n \geqq 1$, let $z = z(f_0, \cdots, f_{n-1})$ be the number of rises of size 1 in the sequence $f_0, \cdots, f_{n-1}$. If $z \geqq k + 1$, define

$$Q(f_0, \cdots, f_{n-1}, f_n) = 1. \tag{11.9}$$

If $z = z(f_0, \cdots, f_{n-1}) < k + 1$, let

$$i = i(f_0, \cdots, f_{n-1}) = k + 1 - z(f_0, \cdots, f_{n-1}).$$

Let $l = l(f_0, \cdots, f_{n-1})$ be the time, or subscript, at which the last rise of size 1 in $(f_0, \cdots, f_{n-1})$ is completed. If no such rise has occurred, let $l = 0$. Let $m = m(f_0, \cdots, f_{n-1})$ be the minimum of the numbers $f_l, \cdots, f_{n-1}$. (Of course $l = l(f_0, \cdots, f_{n-1})$.)

If $z = z(f_0, \cdots, f_{n-1}) < k + 1$, and hence $i = i(f_0, \cdots, f_{n-1}) \geqq 1$, define

$$\begin{aligned} Q(f_0, \cdots, f_n) &= T(i, f_n) && \text{for } f_n \leqq m, \\ &= (f_n - m)T(i - 1, m + 1) \\ &\quad + (1 - (f_n - m))T(i, m) && \text{for } m < f_n < m + 1, \\ &= T(i - 1, f_n) && \text{for } m + 1 \leqq f_n, \end{aligned} \tag{11.10}$$

where $m = m(f_0, \cdots, f_{n-1})$ and $i = i(f_0, \cdots, f_{n-1})$.

Clearly, (11.8), (11.9), and (11.10) define Q for all finite sequences $(f_0, \cdots, f_n)$ of nonnegative reals. Obviously, Q is nonnegative. Also, in view of (11.9), $Q(f_0, \cdots, f_n) \geqq 1$ whenever $(f_0, \cdots, f_n) \in A$. It is also easy to see that Q is nondecreasing and patient. Therefore, to apply (6.4) it is only necessary to prove that Q is strongly coordinatewise concave. To see that it is coordinatewise concave it is necessary to verify that (11.8), (11.9), and (11.10) are concave functions of their last argument. That is, it is necessary to prove that $T(k+1, f)$ is concave in f and that

$$(11.11)\quad \begin{aligned} q(i, m, f) &= T(i, f) && \text{for } f \leqq m, \\ &= (f-m)T(i-1, m+1) \\ &\quad +(1-(f-n))T(i, m) && \text{for } m < f < m+1, \\ &= T(i-1, f) && \text{for } m+1 \leqq f \end{aligned}$$

is concave in f for all m and all i satisfying $1 \leqq i \leqq k+1$.

That $T(k+1, f)$ is concave, and even strictly concave, follows from (11.7), that is, (11.1) implies (11.5).

To prove that $q(i, m, f)$ is concave in f, notice that it is linear in f for $m < f < m+1$ with slope equal to $T(i-1, m+1) - T(i, m)$. To prove it concave, it is necessary and sufficient to prove that $T(i, f)$ and $T(i-1, f)$ are concave, and that

$$T'(i, m) \geqq T(i-1, m+1) - T(i, m) \geqq T'(i-1, m+1)$$

for $1 \leqq i \leqq k+1$. But in view of (10.4), the inductive assumption, and (11.7), all these facts are true. Thus, Q is coordinatewise concave.

To check that Q is strongly coordinatewise concave it is necessary to see that $Q(f_0, \cdots, f_{n-1}, f_n)$ is strictly concave in f_n for $f_n < f_0$ when $f_j = f_0$ for $0 \leqq j \leqq n-1$. That is, it is necessary to see that $q(i, m, f)$ is strictly concave in f for $f < m$ when $i = k+1$. The question is: Is $T(k+1, f)$ strictly concave in f? A look at (11.5) completes the proof that Q is strongly coordinatewise concave. Now apply (6.4) to complete the induction and therewith the proof of (11.1).

(11.12) COROLLARY. *$T(k, x)$ is strictly concave in $x \geqq 0$ for all $k \geqq 1$.*

12. Upcrossings for unfair binomial processes

Let r, s, and x be positive real numbers with $r < s$. Suppose that a gambler's initial fortune is x. Suppose first that $x = r$. Let the gambler select a fair gamble that either increases his fortune to s, or decreases it to 0. Since the gamble is fair, his fortune increases to s with probability $W = r/s$, and decreases to 0 with probability $L = 1 - W = (s-r)/s$. If the gambler loses, and has his fortune decreased to 0, play is terminated. If, however, his fortune increases to s, let him discard $s - r$ units of his fortune, or, in probabilistic terms, let him select a gamble that loses $s - r$ with probability 1, so that, with probability 1, his fortune is again equal to r. Let the gambler then repeat this process indefinitely. With probability 1, his fortune will, after a finite number of gambles, become and remain zero. Let B be the number of times that the gambler's fortune increases from r to s. It is elementary to verify that for each integer $k \geqq 0$, the probability that B is greater than or equal to k is $W^k = r^k s^{-k}$.

Suppose next that $0 < x < r$. Let the gambler first choose a fair gamble that either increases his fortune to s or decreases it to 0, the former event occurring, of course, with probability (x/s). If his fortune does increase to s, let him continue gambling as above. Now, for each $k \geqq 1$, the probability that $B \geqq k$ is $(x/s)W^{k-1}$.

Lastly, suppose that $x > r$. This time the gambler begins by discarding $x - r$ units of his fortune and then continuing as above. Thus for this case the probability that $B \geqq k$ is equal to W^k, for $k \geqq 0$.

Thus we have the following simple theorem.

(12.1) THEOREM. *For each triple of positive real numbers, r, s, and x with $r < s$, there is a nonnegative expectation-decreasing process f_n, $n = 0, 1, \cdots$, that begins at x such that for each integer $k \geqq 1$, the probability that the process experiences k or more upcrossings of the interval $[r, s]$ is equal to*

$$\min\,[(x/s), (r/s)](r/s)^{k-1}.$$

(For a definition of upcrossings see [3, page 315].)

13. An extremal distribution for upcrossings

(13.1) THEOREM. *Let f_n, $n = 0, 1, 2, \cdots$, be a nonnegative expectation-decreasing process that begins at the positive real number x. Then, for each r and s with $0 \leqq r < s$, and for each $k \geqq 1$, the probability that the process f_n upcrosses the interval $[r, s]$ at least k times is less than or equal to* $\min\,(r, x)r^{k-1}s^{-k}$. *This bound is best possible.*

Proof. The preceding section showed that this bound cannot be improved. To see that it is a bound, let r, s, and k be fixed, and let A be the set of all finite sequences of nonnegative real numbers in which at least k upcrossings of $[r, s]$ occur.

The program will now be to define an appropriate Q and then apply (4.5). The value of $Q(f_0, \cdots, f_n)$ is intended to correspond to the probability that k or more upcrossings occur in the infinite sequence $f_0, \cdots, f_n, f_{n+1}, \cdots$, given that $f_0, \cdots, f_n$ are the initial fortunes, and that f_{n+1} and all succeeding fortunes arise as a consequence of gambling according to the scheme suggested in the preceding section. Thus, define

$$Q(f_0) = \min\,(r, f_0)s^{-1}W^{k-1}. \tag{13.2}$$

Let $z = z(f_0, \cdots, f_{n-1})$ be the number of upcrossings of $[r, s]$ that have been completed by the sequence $(f_0, \cdots, f_{n-1})$.

If $z(f_0, \cdots, f_{n-1}) \geqq k$, define

$$Q(f_0, \cdots, f_n) = 1. \tag{13.3}$$

Lastly, if $z(f_0, \cdots, f_{n-1}) < k$, let

$$i = i(f_0, \cdots, f_{n-1}) = k - z(f_0, \cdots, f_{n-1}).$$

Let $l = l(f_0, \cdots, f_{n-1})$ be the time at which the z^{th} upcrossing is completed.

Here l is to be taken as 0 if $z = 0$. Of course, $l(f_0, \cdots, f_{n-1}) \leqq n - 1$. Define $\alpha = \alpha(f_0, \cdots, f_{n-1})$ to be 1 or 0 according as the minimum of $f_l, f_{l+1}, \cdots, f_{n-1}$ is or is not less than or equal to r. Here $l = l(f_0, \cdots, f_{n-1})$. Of course, if $\alpha = 0$, then $r < f_{n-1}$. If $\alpha = 1$, *the process is ready for another upcrossing at time n.*

If $z(f_0, \cdots, f_{n-1}) < k$, and $\alpha = 1$, define

$$Q(f_0, \cdots, f_{n-1}, f_n) = (f_n/s)W^{i-1} \quad \text{if } f_n < s, \qquad = W^{i-1} \quad \text{if } f_n \geqq s, \tag{13.4}$$

where $i = i(f_0, \cdots, f_{n-1})$.

Finally, if $z(f_0, \cdots, f_{n-1}) < k$, and $\alpha = 0$, define

$$Q(f_0, \cdots, f_{n-1}, f_n) = W^i. \tag{13.5}$$

It is simple to verify that Q and A satisfy the hypotheses of (4.5). Therefore, $P \leqq Q(E(f_0)) = Q(x)$. This completes the proof.

The following result of Doob [4] was established by him by making use of potential-theoretic ideas.

(13.6) Corollary (Doob). *Let f_n, $n = 0, 1, 2, \cdots$, be a nonnegative expectation-decreasing process that begins at $x > 0$. Then, for each r and s with $0 < r < s$, the expected number of upcrossings of the interval $[r, s]$ does not exceed* $\min(x, r)(s - r)^{-1}$.

Proof. The expected number of upcrossing equals the sum over $k \geqq 1$, of the probability that the process experiences k or more upcrossings. According to (13.1) this does not exceed

$$\min(r, x)s^{-1}\textstyle\sum_{i=0}^{\infty} W^i = \min(r, x)(s - r)^{-1}.$$

This completes the proof.

14. Downcrossings for unfair binomial processes

(14.1) Theorem. *For each triple of positive real numbers, r, s, and x with $r < s$, there is a nonnegative expectation-decreasing process f_n, $n = 0, 1, \cdots$, that begins at x such that for each integer $k \geqq 1$ the probability that the process experiences k or more downcrossings of the interval $[r, s]$ is equal to* $\min((x/s), 1)(r/s)^{k-1}$.

Proof. Let a gambler with an initial fortune of x select a fair or unfair gamble that maximizes the probability that his fortune is s at the end of the gamble. If $x < s$, this probability is (x/s). If $x \geqq s$, this probability is 1. Suppose that then the gambler discards $s - r$ units of his fortune. Thus with probability $\min((x/s), 1)$ there is at least one downcrossing of $[r, s]$. Let the process then evolve according to the scheme in Section 12. There are $k - 1$ additional downcrossings if and only if there are $k - 1$ additional upcrossings. Thus with probability W^{k-1}, there are at least $k - 1$ more downcrossings. Since $W = (r/s)$, this completes the argument.

15. An extremal distribution for downcrossings

(15.1) THEOREM. *Let f_n, $n = 0, 1, \cdots$, be a nonnegative expectation-decreasing process that begins at the positive real number x. Then for each r and s with $0 < r < s$, and for each $k \geqq 1$, the probability that the process $\{f_n\}$ downcrosses the interval $[r, s]$ at least k times is less than or equal to*

$$\min((x/s), 1)(r/s)^{k-1}.$$

The proof is the same as for (13.1).

We recall a discovery of Hunt [10].

(15.2) COROLLARY (Hunt). *The expected number of downcrossings does not exceed* $\min(x, s)(s - r)^{-1}$.

In view of (14.1), *Hunt's bound is sharp.*

Of course, if one does not assume that f_0 is a constant, then the upper bound in (15.1) is $E(\min((f_0/s), 1))(r/s)^{k-1}$, and the upper bound in (15.2) is $E(\min(f_0, s))(s - r)^{-1}$.

Remark. Interesting results akin to (1.1), (11.1), (13.1), and (15.1) when the processes are not assumed to be nonnegative have yet to be found.

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