A DECOMPOSITION THEOREM FOR SUPERMARTINGALES

BY

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It is a well known fact (see [1], p. 296) that any discrete-parameter supermartingale $\{X_n\}_{n \in \mathbf{N}}$ can be represented as a sum:

$$X_n = Y_n + Z_n ,$$

$\{Y_n\}$ being a martingale, and $\{Z_n\}$ a process with decreasing sample functions, such that $Z_0 = 0$. Moreover, if the supermartingale $\{X_n\}$ is uniformly integrable, the same is true for $\{Y_n\}$ and $\{Z_n\}$. Doob has raised the problem of the existence of such a decomposition for continuous-parameter supermartingales. We shall solve this problem here, although the necessary and sufficient condition we give is not very easy to handle. Our proof has been adapted from that of a theorem in potential theory, concerning the representation of excessive functions as potentials of additive functionals ([3], pp. 75–83). A reader with some knowledge of Hunt's potential theory for Markov processes, and the theory of additive functionals, will easily recognize here some kind of a coarse potential theory, with supermartingales replacing excessive functions. Our terminology has been chosen in accordance with this idea.

We shall use freely the results contained in Chapter VII (martingale theory) of Doob's book. A number of definitions will be recalled, for the reader's convenience.

1. Let Ω be a set, $\mathfrak{F}$ a Borel field of subsets of Ω, $\mathbf{P}$ a probability measure defined on $(\Omega, \mathfrak{F})$. We are given a family $\{\mathfrak{F}_t\}_{t \in \mathbf{R}_+}$ of Borel subfields of $\mathfrak{F}$, such that

$$\mathfrak{F}_s \subset \mathfrak{F}_t , \qquad s < t.$$

We may, and do, suppose that the Borel field $\mathfrak{F}$ has been completed with respect to $\mathbf{P}$, and that each $\mathfrak{F}_t$ contains all $\mathfrak{F}$ sets of measure zero. A measurable stochastic process $\{X_t\}_{t \in \mathbf{R}_+}$ is *well adapted* to the $\mathfrak{F}_t$ family if, for each t, X_t is $\mathfrak{F}_t$-measurable. Let $\mathfrak{F}_{t_+}$ denote the intersection $\bigcap_{s > t} \mathfrak{F}_t$; any process which is well adapted to the $\mathfrak{F}_t$ family is well adapted to the $\mathfrak{F}_{t_+}$ family.

A *supermartingale* (relative to the $\mathfrak{F}_t$ family) is a real valued process $\{X_t\}$, well adapted to the $\mathfrak{F}_t$ family, such that

(i) $\forall_t , \quad \mathbf{E}[| X_t |] < \infty$,

(ii) $\forall_s , \quad \forall_t , \quad \mathbf{E}[X_{s+t} \mid \mathfrak{F}_s] \leqq X_s$ a.s.

If equality holds a.s. in (ii), the process is a *martingale*. We shall be concerned here only with sample right continuous supermartingales. If $\{X_t\}$ is such a supermartingale, then (ii) holds with $\mathfrak{F}_t$ replaced by $\mathfrak{F}_{t_+}$. Let indeed A be an event in $\mathfrak{F}_{s_+}$, and let s_n be a decreasing sequence which converges to s.

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For $s_n < s + t$

$$\int_A X_{s+t}\, d\mathbf{P} \leqq \int_A X_{s_n}\, d\mathbf{P}.$$

Now $X_{s_n} \rightarrow_{n\to\infty} X_s$, from sample right continuity; as the family $\{X_{s_n}\}$ is uniformly integrable (Theorem 4.2s of [1]), we may integrate to the limit, and this yields the desired inequality.

Consequently, when we deal with a right continuous supermartingale, we may suppose that the $\mathfrak{F}_t$ family is such that $\mathfrak{F}_t = \mathfrak{F}_{t_+}$ for every t—if necessary, we just replace the $\mathfrak{F}_t$ family by the $\mathfrak{F}_{t_+}$ family. From now on, we shall assume that this condition is realized, and delete the mention of the $\mathfrak{F}_t$ family, all well adapted processes or supermartingales being understood as relative to it.

Let $\{X_t\}$ be a supermartingale whose expectation $\mathbf{E}[X_t]$ is a right continuous function of t; it is possible then to find a right continuous supermartingale $\{Y_t\}$ equivalent to $\{X_t\}$ (for each t, $\mathbf{P}[X_t \neq Y_t] = 0$). To prove it, we define

$$Y_t(\omega) = X_{t_+}(\omega) = \lim_{s \downarrow t,\, s \text{ rational}} X_s(\omega),$$

on the set of ω such that this limit exists for every t, and $Y_t(\omega) = 0$ on the complement of this set (which is a null set). As $\mathfrak{F}_t = \mathfrak{F}_{t_+}$, we get a well adapted process. Using once more Theorem 4.2s of [1], we find that

$$\int_A Y_t\, d\mathbf{P} = \lim_{s \downarrow t} \int_A X_s\, d\mathbf{P} \leqq \int_A X_t\, d\mathbf{P} \qquad (A \in \mathfrak{F}_t).$$

If we take for A the set $\{X_{t_+} > X_t\}$, this yields that $Y_t \leqq X_t$ a.s. for each t, and equality follows from the right continuity of $\mathbf{E}[X_t]$.

A *stopping time* is a positive, possibly infinite, random variable $T(\omega)$ such that, for every $a \geqq 0$, the event $\{T < a\}$ belongs to $\mathfrak{F}_a$. We shall denote by $\mathfrak{F}_T$ the Borel field of events A in $\mathfrak{F}$ with the property that, for every $a \geqq 0$, $A \cap \{T < a\} \in \mathfrak{F}_a$. If t is a constant, and T a stopping time greater than t, then $\mathfrak{F}_t \subset \mathfrak{F}_T$ (the same is true with two stopping times S and T such that $S \leqq T$, but we shall not need it). Let $\{X_t\}$ be a uniformly integrable right continuous supermartingale; if we consider a random variable X_∞, a.s. equal to $\lim_{t\to\infty} X_t(\omega)$, then the process $\{X_t\}_{0\leqq t\leqq\infty}$ is a supermartingale (a martingale if the original one is). For any stopping time T, we may set $X_T(\omega) = X_{T(\omega)}(\omega)$ if $T(\omega) < \infty$, $X_\infty(\omega)$ if $T(\omega) = \infty$; it is easily seen that X_T is a random variable. The basic results about stopping times are Doob's "optional sampling theorems", which will be used several times in the remainder of this paper.

We are now ready for the first definition.

Definition 1. Let a be a number, $0 \leqq a \leqq \infty$, and let $\{X_t\}$ be a right continuous supermartingale, uniformly integrable on the interval $[0, a]$. We shall say that it *belongs to the class* (D) on this interval, if all the random variables X_T are uniformly integrable, T being any stopping time bounded by a. If $\{X_t\}$ belongs to the class (D) on every interval $[0, a]$, $a < \infty$, it will be said to belong *locally* to the class (D).

We have never met a right continuous supermartingale, uniformly integrable on an interval $[0, a]$, and not belonging to the class (D) on it. No satisfactory condition is known, implying that sufficiently general classes of right continuous supermartingales are contained in the class (D). The best we know in that direction is the following:

PROPOSITION 1. *Any right continuous martingale $\{X_t\}$ belongs locally to the class (D).*

Any right continuous supermartingale $\{X_t\}$, which is bounded from above, belongs locally to the class (D).

Any right continuous supermartingale $\{X_t\}$, which belongs locally to the class (D), and is uniformly integrable, belongs to the class (D).

Proof. If $a < \infty$, and T is a stopping time, $T \leqq a$, then $X_T = \mathbf{E}[X_a \mid \mathfrak{F}_T]$ a.s. Hence

$$\int_{\{|X_T|>n\}} |X_T| \, d\mathbf{P} \leqq \int_{\{|X_T|>n\}} |X_a| \, d\mathbf{P}.$$

As $n \cdot \mathbf{P}[|X_T| > n] \leqq \mathbf{E}[|X_T|] \leqq \mathbf{E}[|X_a|]$, the second integral goes to 0 as $n \to \infty$ uniformly in T, and the first assertion is proved. To prove the second one, we may suppose that the right continuous supermartingale is negative. Then

$$\int_{\{X_T<-n\}} X_T \, d\mathbf{P} \geqq \int_{\{X_T<-n\}} X_a \, d\mathbf{P}.$$

And we conclude as above, using the inequality ([1], p. 353)

$$n \cdot \mathbf{P}[\inf X_t < -n] \leqq \mathbf{E}[|X_a|].$$

The proof of the last assertion will require some definitions that will be useful later. As $\{X_t\}$ is uniformly integrable, we may set

$$X_t = \mathbf{E}[X_\infty \mid \mathfrak{F}_t] + (X_t - \mathbf{E}[X_\infty \mid \mathfrak{F}_t]).$$

The first process is a martingale, equivalent to a right continuous one, and the same reasoning as above will show that it belongs to the class (D). The second process is a positive right continuous supermartingale $\{Y_t\}$, such that $\lim_{t\to\infty} Y_t(\omega) = 0$ a.s. To recall the analogy with the Riesz decomposition, we shall call such a right continuous supermartingale a *potential*. Here, $\{Y_t\}$ is uniformly integrable, and belongs locally to the class (D); let us prove that it actually belongs to the class (D). Since both $\inf(T, a)$ and $\sup(T, a)$ are stopping times,

$$\int_{\{Y_T>n\}} Y_T \, d\mathbf{P} \leqq \int_{\{T \leqq a, Y_T>n\}} Y_T \, d\mathbf{P} + \int_{\{T>a\}} Y_T \, d\mathbf{P}$$

$$\leqq \int_{\{T \leqq a, Y_T>n\}} Y_T \, d\mathbf{P} + \mathbf{E}[Y_a].$$

As $\mathbf{E}[Y_a] \rightarrow_{a\to\infty} 0$, from the uniform integrability, and $\{Y_t\}$ belongs locally to the class (D), it is easily seen that the first integral is small when n is large enough, independently of T.

2. The aim of this section, and the following one, will be the solution of Doob's decomposition problem for a right continuous supermartingale which belongs to the class (D). The "Riesz decomposition" we have considered a few lines above shows that it is sufficient to solve the problem for a potential.

DEFINITION 2. A *right continuous increasing process* is a well adapted stochastic process $\{A_t\}$ such that: (i) $A_0 = 0$ a.s. (ii) For almost every ω, the function $t \rightarrow A_t(\omega)$ is positive, increasing (in the wide sense), and right continuous. Let $A_\infty(\omega)$ be $\lim_{t\to\infty} A_t(\omega)$; we shall say that the right continuous increasing process is *integrable* if $\mathbf{E}[A_\infty] < \infty$.

The right continuous increasing processes will play here the role played by additive functionals in the theory of Markov processes. It is obvious that the process $\{-A_t\}$ is a negative supermartingale, that it belongs to the class (D) locally, and globally if $\mathbf{E}[A_\infty] < \infty$; if $\{A_t\}$ is an integrable right continuous increasing process, any right continuous version of the supermartingale $\{\mathbf{E}[A_\infty \mid \mathfrak{F}_t] - A_t\}$ is therefore a potential of the class (D), which we shall call *the potential generated by* $\{A_t\}$. Such a potential can, from its very definition, be written in Doob's decomposed form, so that our result for the class (D) will be implied by the following, more precise statement.

THEOREM 1. *A potential $\{X_t\}$ belongs to the class (D) if, and only if, it is generated by some integrable right continuous increasing process.*

The sufficiency is obvious from what we have just said. The key to the proof of the necessity is the following lemma:

LEMMA 1. *Let $\{X_t\}$ be a right continuous supermartingale, and $\{X_t^n\}$ a sequence of decomposed right continuous supermartingales:*

$$X_t^n = M_t^n - A_t^n ,$$

the $\{M_t^n\}$ being martingales, and the $\{A_t^n\}$ right continuous increasing processes. Suppose that, for each t, the X_t^n converge to X_t in the L^1 topology, and the A_t^n are uniformly integrable in n. Then the decomposition problem is solvable for the supermartingale $\{X_t\}$; more precisely, there are a right continuous increasing process $\{A_t\}$, and a martingale $\{M_t\}$, such that $X_t = M_t - A_t$.

Proof. We denote by w the weak topology $w(L^1, L^\infty)$; a sequence of integrable random variables f_n converges to a random variable f in the w-topology, if and only if f is integrable, and $\mathbf{E}[f_n \cdot g] \rightarrow \mathbf{E}[f \cdot g]$ for any bounded random variable g. A fundamental theorem in functional analysis (see for instance [2], p. 294) asserts that any uniformly integrable sequence of random variables contains a w-convergent subsequence. Using a diagonal procedure, we may then find a sequence n_k of integers, such that the $A_t^{n_k}$ converge in the w-topology to random variables A'_t, for all rational values of t. To simplify the notations, we shall suppose that the whole sequence is so convergent. Obviously, $A_0 = 0$. As, $\mathfrak{F}_t$ being complete and containing all sets of measure 0, an integrable random variable f is $\mathfrak{F}_t$-measurable if and only if it is orthogonal to all bounded random variables g such that $\mathbf{E}[g \mid \mathfrak{F}_t] = 0$ a.s., it follows that A'_t is $\mathfrak{F}_t$-measurable (one can use the Hahn-Banach theorem, if the above assertion does not seem obvious enough). For $s < t$, s and t rational, $A'_t - A'_s$ is obviously a.s. positive, the inequality

$$\int_B (A_t^n - A_s^n)\, d\mathbf{P} \geqq 0,$$

where B denotes any $\mathfrak{F}$ set, behaving well as $n \to \infty$. As the $X_t^n \to X_t$ in a stronger topology than w (w-topology itself would be sufficient), the M_t^n converge to random variables M'_t for t rational, and the process $\{M'_t\}$ is easily seen to be a martingale; there is therefore a right continuous martingale $\{M_t\}$, defined for all values of t, such that $\mathbf{P}[M_t \neq M'_t] = 0$ for each rational t. If we define now $A_t = X_t + M_t$, $\{A_t\}$ is a right continuous increasing process, or at least becomes so after a modification on a set of measure zero. The lemma is then proved.

3. This section will be spent in a construction of the X_t^n, A_t^n for a potential $\{X_t\}$ of the class (D). We shall use the following device, which replaces in our theory the operators of a Markov semigroup: let $\{X_t\}$ be a right continuous supermartingale, and k a positive number; define $Y_t = \mathbf{E}[X_{t+k} \mid \mathfrak{F}_t]$. As

$$\begin{aligned}\mathbf{E}[Y_{t+s} \mid \mathfrak{F}_t] &= \mathbf{E}[\mathbf{E}[X_{t+s+k} \mid \mathfrak{F}_{t+s}] \mid \mathfrak{F}_t] = \mathbf{E}[X_{t+s+k} \mid \mathfrak{F}_t] \\ &= \mathbf{E}[\mathbf{E}[X_{t+s+k} \mid \mathfrak{F}_{t+k}] \mid \mathfrak{F}_t] \leqq \mathbf{E}[X_{t+k} \mid \mathfrak{F}_t] = Y_t,\end{aligned}$$

the Y_t process is a supermartingale smaller than $\{X_t\}$; its expectation $\mathbf{E}[Y_t] = \mathbf{E}[X_{t+k}]$ is a right continuous function of t, so that we may denote by $\{p_k X_t\}$ a right continuous version of $\{Y_t\}$. If $\{X_t\}$ is a potential, so is $\{p_k X_t\}$.

LEMMA 2. *Let $\{X_t\}$ be a potential and belong to the class (D). We consider the measurable, positive, well adapted processes $H = \{H_t\}$ with the property that the right continuous increasing processes*

$$A(H) = \{A_t(H, \omega)\} = \left\{\int_0^t H_s(\omega)\, ds\right\}$$

are integrable, and the potentials $Y(H) = \{Y_t(H, \omega)\}$ they generate are majorized by $\{X_t\}$. Then, for each t, the random variables $A_t(H)$ of all such processes $A(H)$ are uniformly integrable.

Proof. It is obviously sufficient to prove that the $A_\infty(H)$ are uniformly integrable.

(1) If the X_t process is bounded by some positive constant c, then $\mathbf{E}[A^2_\infty(H)] \leqq 2c^2$, and the uniform integrability follows.

$$A^2_\infty(H, \omega) = 2 \int_0^\infty [A_\infty(H, \omega) - A_u(H, \omega)] \, dA_u(H, \omega)$$

$$= 2 \int_0^\infty [A_\infty(H, \omega) - A_u(H, \omega)] H_u(\omega) \, du.$$

Hence, by using Fubini's theorem

$$\mathbf{E}[A^2_\infty(H)] = 2 \int_0^\infty \mathbf{E}[H_u \cdot (A_\infty(H) - A_u(H))] \, du.$$

We replace now the expectation $\mathbf{E}[\cdots]$ by $\mathbf{E}[\mathbf{E}[\cdots \mid \mathfrak{F}_u]]$, and use the fact that $Y_u(H) = \mathbf{E}[A_\infty(H) - A_u(H) \mid \mathfrak{F}_u]$, and H_u is $\mathfrak{F}_u$-measurable:

$$\mathbf{E}[A^2_\infty(H)] = 2 \int_0^\infty \mathbf{E}[Y_u(H) \cdot H_u] \, du$$

$$\leqq 2c \int_0^\infty \mathbf{E}[H_u] \, du = 2c\mathbf{E}[Y_0(H)] \leqq 2c^2.$$

(2) To reach the general case, it will be enough to prove now that any H process such that $Y(H)$ is majorized by $\{X_t\}$ is equal to a sum $H^c + H_c$ of two such processes, where $A(H^c)$ generates a potential bounded by c, and where $\mathbf{E}[A_\infty(H_c)]$ is smaller than some number ε_c, independent of H, such that $\varepsilon_c \rightarrow_{c\to\infty} 0$. Let G^c denote the indicator of the interval $[0, c]$. Define

$$H^c_t(\omega) = H_t(\omega) \cdot G^c \circ X_t(\omega); H_{ct} = H_t - H^c_t.$$

Let also $T^c(\omega)$ be the infimum of all t such that $X_t(\omega) \geqq c$; as c goes to infinity, $T^c(\omega) \rightarrow \infty$ a.s., therefore $X_{T^c} \rightarrow 0$ a.s., and the class (D) property implies that $\mathbf{E}[X_{T^c}] \rightarrow 0$. T^c is a stopping time, and $G^c \circ X_t = 1$ before time T^c. Hence

$$\mathbf{E}[A_\infty(H_c)] = \mathbf{E}\left[\int_0^\infty H_u(1 - G^c \circ X_u) \, du\right] \leqq \mathbf{E}\left[\int_{T^c}^\infty H_u \, du\right];$$

the last integral is also equal to

$$\mathbf{E}[A_\infty(H) - A_{T^c}(H)] = \mathbf{E}[\mathbf{E}[A_\infty(H) - A_{T^c}(H) \mid \mathfrak{F}_{T^c}]]$$

$$= \mathbf{E}[Y_{T^c}(H)] \leqq \mathbf{E}[X_{T^c}],$$

and the assertion relative to the H_c process is proved. We prove now that $Y(H^c)$ is bounded by c:

$$Y_t(H^c) = \mathbf{E}[A_\infty(H^c) - A_t(H^c) \mid \mathfrak{F}_t] \quad \text{a.s.}$$

$$= \mathbf{E}\left[\int_t^\infty H_u \cdot G^c \circ X_u \, du \mid \mathfrak{F}_t\right].$$

Let $S^c(\omega)$ be the infimum of all $s \geqq t$ such that $X_s(\omega) \leqq c$; the right sample continuity implies that $X_{S^c}(\omega) \leqq c$. As

$$\int_t^{S^c(\omega)} H_s(\omega) \cdot G^c \circ X_s(\omega)\, ds = 0,$$

we get

$$Y_t(H^c) \leqq \mathbf{E}\left[\int_{S^c}^{\infty} H_u\, du \mid \mathfrak{F}_t\right] = \mathbf{E}\left[\mathbf{E}\left[\int_{S^c}^{\infty} H_u\, du \mid \mathfrak{F}_{S^c}\right] \mid \mathfrak{F}_t\right]$$
$$= \mathbf{E}[Y_{S^c} \mid \mathfrak{F}_t] \leqq c \quad \text{a.s.}$$

This holds for each t, therefore a.s. for every rational t, and a.s. for every t in consideration of the right continuity.

Lemmas 1 and 2 will now imply Theorem 1, if we prove that we can find a sequence of processes H_n, verifying the assumptions of Lemma 2, and such that $Y_t(H_n)$ tends to X_t in the L^1 topology as $n \to \infty$. This is done in the following lemma.

LEMMA 3. *The notations are the same as in Lemma 2. Let k be a positive number, and $H_{t,k}(\omega) = (X_t(\omega) - p_k X_t(\omega))/k$. The processes $H_k = \{H_{t,k}\}$ verify the assumptions of Lemma 2, and their potentials increase to $\{X_t\}$ as $k \to 0$.*

Proof. Remark that, if $t < u$,

$$\mathbf{E}\left[\frac{1}{k}\left(\int_0^u [X_s - p_k X_s]\, ds - \int_0^t [X_s - p_k X_s]\, ds\right) \mid \mathfrak{F}_t\right]$$
$$= \mathbf{E}\left[\frac{1}{k}\int_t^u [X_s - p_k X_s]\, ds \mid \mathfrak{F}_t\right];$$

now, for $s \geqq t$, $\mathbf{E}[p_k X_s \mid \mathfrak{F}_t] = \mathbf{E}[\mathbf{E}[X_{s+k} \mid \mathfrak{F}_s] \mid \mathfrak{F}_t] = \mathbf{E}[X_{s+k} \mid \mathfrak{F}_t]$. The first member is therefore equal to

$$\mathbf{E}\left[\frac{1}{k}\int_t^u [X_s - X_{s+k}]\, ds \mid \mathfrak{F}_t\right]$$
$$= \mathbf{E}\left[\frac{1}{k}\int_t^{t+k} X_s\, ds \mid \mathfrak{F}_t\right] - \mathbf{E}\left[\frac{1}{k}\int_u^{u+k} X_s\, ds \mid \mathfrak{F}_t\right].$$

The last integral is positive, and its expectation is smaller than $\mathbf{E}[X_u]$; it tends to 0 as $u \to \infty$ in the L^1 norm, and there remains

$$Y_t(H_k) = \mathbf{E}\left[\frac{1}{k}\int_t^{\infty} [X_s - p_k X_s]\, ds \mid \mathfrak{F}_t\right] = \mathbf{E}\left[\frac{1}{k}\int_t^{t+k} X_s\, ds \mid \mathfrak{F}_t\right] \quad \text{a.s.,}$$

which is easily seen to increase to X_t as $k \to 0$.

4. We shall now extend Theorem 1 to right continuous supermartingales that belong *locally* to the class (D).

THEOREM 2. *Doob's decomposition problem is solvable for a right continuous supermartingale $\{X_t\}$ if and only if it belongs to the class (D) on every finite* *interval. More precisely,* $\{X_t\}$ *is then equal to the difference of a martingale and a right continuous increasing process.*

Proof. The necessity is obvious. To prove the sufficiency, we choose a positive number a and define

$$X'_t(\omega) = X_t(\omega) \quad (0 \leqq t \leqq a), \qquad X'_t(\omega) = X_a(\omega) \quad (a < t);$$

the X'_t process is a right continuous supermartingale of the class (D). Using Theorem 1, we write: $X'_t = M'_t - A'_t$, $\{M'_t\}$ and $\{A'_t\}$ being respectively a martingale and a right continuous increasing process. We let now a tend to infinity; it is easily seen in the above expression of the $Y_t(H_k)$ that the A'_t depend only on the values of $\{X'_t\}$ on intervals $[0, t + \varepsilon]$, with ε arbitrarily small. When $a \to \infty$, they do not vary any more once a has reached values greater than t; we use again Lemma 1 (or rather the trivial part of it), and Theorem 2 follows.

The most interesting consequence of Theorem 2 seems to be the following: Doob's decomposition problem is always solvable for positive *submartingales* (semimartingales). This was not known even for the absolute value of Brownian motion and might, according to Doob, simplify the proof of some rather difficult theorems of Lévy's in Brownian motion theory.

All the results we have found till now are invariant through a monotonic transformation of the real line; they are therefore true for supermartingales defined only on an interval.

5. Among the right continuous increasing processes, the continuous ones may be considered as particularly interesting, and it is natural to wonder whether a representation involving a continuous increasing process is possible for some classes of supermartingales. The answer to this question is given in Theorem 3 below. Our proof is nothing more than an adaptation to martingale theory of a proof given by Shur in [4]; we shall be therefore as brief as possible, the reader being referred to [4] for details.

DEFINITION 3. Let $\{X_t\}$ be a right continuous supermartingale and belong to the class (D). We shall say $\{X_t\}$ is *regular* if, for any increasing sequence of stopping times T_n, with the limit T, the equality $\mathbf{E}[X_T] = \lim_n \mathbf{E}[X_{T_n}]$ holds.

For instance, any uniformly integrable martingale is regular. A locally regular right continuous supermartingale can be defined as in Definition 2; the extension of the following results to locally regular supermartingales is left to the reader.

THEOREM 3. *Let* $\{X_t\}$ *be a potential and belong to the class* (D). $\{X_t\}$ *is generated by some integrable continuous increasing process if, and only if, it is regular.*

The proof will consist of several propositions or lemmas. Let us remark first that the necessity is obvious since, if $\{X_t\}$ is generated by some continuous increasing process $\{A_t\}$, $\mathbf{E}[X_{T_n}] = \mathbf{E}[\mathbf{E}[A_\infty - A_{T_n} \mid \mathfrak{F}_{T_n}]] = \mathbf{E}[A_\infty] - \mathbf{E}[A_{T_n}]$, and the regularity follows from Lebesgue's convergence theorem.

Proposition 2. *Let $\{X_t\}$ be a potential and belong to the class (D). There exists a series of bounded potentials $\{X_t^n\}$ whose sum is $\{X_t\}$. If $\{X_t\}$ is regular, so is each $\{X_t^n\}$.*

Proof. As the proof is nearly the same as that of an assertion in Lemma 2, we shall just sketch it: Let $\{A_t\}$ be an integrable right continuous increasing process which generates $\{X_t\}$ (Theorem 2), and let G_n be the indicator of the interval $[n, n+1[$. We denote by $\{X_t^n\}$ the potential generated by the right continuous increasing process

$$A_t^n(\omega) = \int_{[0,t]} G_n \circ X_s(\omega)\, dA_s(\omega).$$

It is obvious that $\{A_t\} = \sum_n \{A_t^n\}$. As the regularity property is equivalent to the fact that (the notations being those of Definition 3) $\mathbf{E}[A_{T_n}] \to \mathbf{E}[A_T]$, and for the general right continuous increasing process $\lim_n \mathbf{E}[A_{T_n}] \leqq \mathbf{E}[A_T]$, if $\{X_t\}$ is regular, so are the $\{X_t^n\}$. We prove now that $\{X_t^n\}$ is bounded by $n+1$ as in Lemma 2: Let S_n be the first time X_s enters the interval $[0, n+1]$ after time t; then

$$X_t^n = \mathbf{E}[A_\infty^n - A_t^n \mid \mathfrak{F}_t] \leqq \mathbf{E}[A_\infty - A_{S_n} \mid \mathfrak{F}_t] = \mathbf{E}[X_{S_n} \mid \mathfrak{F}_t] \leqq n+1 \quad \text{a.s.}$$

Our problem is thus reduced to a problem concerning potentials bounded by a constant c; indeed, if we may prove that $\{X_t^n\}$ is the potential of some continuous increasing process $\{B_t^n\}$, the series of continuous increasing functions $t \to B_t^n(\omega)$ of t is uniformly convergent on the whole real line for every ω such that $\sum_n B_\infty^n(\omega) < \infty$, i.e., almost every ω; its sum is therefore a.s. continuous.

The following lemma is due to Shur, and is the key to his proof.

Lemma 4. *Let ε and k be two positive numbers, $\{X_t\}$ a regular potential, and $T_k(\omega)$ the infimum of all t such that $X_t(\omega) - p_k X_t(\omega) \geqq \varepsilon$. Then $T_k(\omega)$ increases a.s. to infinity as k tends to 0.*

Proof. As the T_k increase when k decreases, it is sufficient to prove that the property holds when $k \to 0$ through a sequence of values. Define T as the limit of T_k. If $k < h$,

$$X_t(\omega) - p_h X_t(\omega) \geqq X_t(\omega) - p_k X_t(\omega);$$

giving t the value $T(\omega)$, and remarking that (right continuity)

$$X_{T_k} - p_k X_{T_k} \geqq \varepsilon$$

on the set $\{T_k(\omega) < \infty\}$, we get on this set

$$X_{T_k} - p_h X_{T_k} \geqq \varepsilon.$$

Remark now that $\{p_h X_t\}$ is a regular potential, take expectations, and let $k \to \infty$. There remains

$$\mathbf{E}[X_T - p_h X_T] \geqq \varepsilon \cdot \mathbf{P}[T < \infty].$$

Let now $h \to 0$; the first member tends to 0, and the lemma is proved.

LEMMA 5. *Let $\{X_t\}$ be a potential bounded by c. We set*

$$\Delta_k X_t = (X_t - p_k X_t)/k, \qquad A_t^k = \int_0^t \Delta_k X_s \, ds.$$

Let A_t be the w-limit of the A_t^k as k tends to 0; the convergence to A_t holds in fact in the L^2 sense.

Proof. We shall show that $\mathbf{E}[(A_t^k - A_t^h)^2] \to_{k\to 0, h\to 0} 0$, and this will imply the truth of the lemma. Using the same method as in the first part of Lemma 2, we get

$$\mathbf{E}[(A_t^k - A_t^h)^2] = \mathbf{E}\left[\int_0^t ds(\Delta_k X_s - \Delta_h X_s) \int_s^t (\Delta_k X_u - \Delta_h X_u)\, du\right];$$

writing $\int_s^t$ as $\int_s^\infty - \int_t^\infty$, taking then the conditional expectation of this first integral with respect to $\mathfrak{F}_s$, of the second one with respect to $\mathfrak{F}_t$, and using the computation of $\mathbf{E}[\int_s^\infty (X_u - p_k X_u)\, du \mid \mathfrak{F}_s]$ found in Lemma 3, there remains for the first member

$$\mathbf{E}\left[\int_0^t ds(\Delta_k X_s - \Delta_h X_s) \cdot \mathbf{E}\left[\frac{1}{k}\int_s^{s+k} X_u\, du - \frac{1}{h}\int_s^{s+h} X_u\, du \mid \mathfrak{F}_s\right]\right]$$
$$- \mathbf{E}\left[\int_0^t ds(\Delta_k X_s - \Delta_h X_s) \cdot \mathbf{E}\left[\frac{1}{k}\int_t^{t+k} X_u\, du - \frac{1}{h}\int_t^{t+h} X_u\, du \mid \mathfrak{F}_t\right]\right].$$

Split now the domain of integration (supposing $h \leqq k$) in two: the set where $T_k(\omega)$ is greater than, say, $t + 1$, and its complement. On the first set, the absolute value of the conditional expectation inside the symbol $\int_0^t$ is smaller than 4ε, and the integrals $\int_0^t \Delta_k X_s\, ds$ have an expectation smaller than c (Lemma 3); the absolute value of the whole integral, taken over this set, is therefore majorized by $16\varepsilon c$. On the set $\{T_k(\omega) \leqq t + 1\}$, whose measure tends to 0 with k, we majorize the conditional expectations by $2c$, and find that the whole expression is smaller than

$$4c \int_{[T_k \leqq k+1]} d\mathbf{P} \int_0^t (\Delta_k X_s + \Delta_h X_s)\, ds.$$

As the expectations $\mathbf{E}[(\int_0^t \Delta_k X_s\, ds)^2]$ have been seen to be bounded by $2c^2$, these random variables are uniformly integrable, and their integrals, on sets whose measures tend to 0, tend to 0; the integral goes therefore to 0 together with k, and the lemma is proved.

Let now $[0, t]$ be an interval, and $0 = t_1 < t_2 < \cdots < t_n = t$ be a subdivision of $[0, t]$; we consider a sequence of subdivisions whose "step" (the step of the t_i-subdivision is the number $\sup_i (t_{i+1} - t_i)$) decreases to 0. We have then the following lemma, the proof of which is similar to that of the preceding one, and is left to the reader:

LEMMA 6. *The sums*

$$\sum_i \mathbf{E}\left[\left(\int_{t_i}^{t_{i+1}} \Delta_k X_u \, du\right)^2\right]$$

tend to 0 *uniformly in* k, *as the step of the subdivision* $(t_i)_{i=0,\dots,n}$ *tends to* 0.

We can now conclude, after Shur: Fatou's lemma implies that the sum $\sum_i \mathbf{E}[(A_{t_{i+1}} - A_{t_i})^2]$ tends to 0 together with the step of the subdivision, and this implies, as the functions $A_t(\omega)$ are increasing, that they are a.s. continuous.

6. We shall now apply the results we have obtained to a generalization of the stochastic integral defined in Chapter IX of Doob's book. The following proposition will be useful for that purpose.

PROPOSITION 3. *Let* $\{Y_t\}$ *be a potential; it belongs to the class* (D) *if and only if, for any increasing sequence* T_n *of stopping times which increases a.s. to infinity,* $\lim_n \mathbf{E}[Y_{T_n}] = 0$.

Proof. The necessity is obvious. Let S_n be, conversely, the first time Y_t enters the interval $[n, \infty]$; the sample functions of Y_t being a.s. bounded on finite intervals, S_n increases a.s. to infinity together with the integer n, and our hypothesis implies that $\mathbf{E}[Y_{S_n}] \to \infty$. We consider now a stopping time T, and prove that $\int_{\{Y_T > n\}} Y_T \, d\mathbf{P}$ tends to 0 as n goes to infinity, independently from T. Let $R_n(\omega)$ be equal to ∞ if $T(\omega) < S_n(\omega)$, to $T(\omega)$ if $T(\omega) \geqq S_n(\omega)$: our integral is majorized by $\mathbf{E}[Y_{R_n}]$ and, R_n being a stopping time greater than S_n, by $\mathbf{E}[Y_{S_n}]$. The proposition is proved.

THEOREM 4. *Let* $\{X_t\}$ *be a square integrable martingale. There exists a right continuous increasing process* $\{A_t\}$ *such that, for any* $s < t$

$$\mathbf{E}[(X_t - X_s)^2 \mid \mathfrak{F}_s] = \mathbf{E}[A_t - A_s \mid \mathfrak{F}_s].$$

Proof. A reasoning on an interval $[0, a]$, with $a < \infty$, will be sufficient, as it will be possible to obtain such a right continuous increasing process on each interval $[na, (n+1)a]$, and the sum of the increasing processes A_t^n obtained in this manner, and taken as equal to 0 on the interval $[0, na]$, to $A_{(n+1)a}^n$ on $[(n+1)a, \infty[$, works well for the whole process. A monotonic change in the scale of time transforms the interval $[0, a]$ into $[0, \infty]$, and we are brought back to the same problem, with the new feature that $\sup_t \mathbf{E}[X_t^2] < \infty$.

We define now

$$Y_t = \mathbf{E}[(X_\infty - X_t)^2 \mid \mathfrak{F}_t].$$

Let s and t be two numbers, $s < t$; orthogonality of increments implies that

$$\mathbf{E}[(X_\infty - X_s)^2 \mid \mathfrak{F}_t] = \mathbf{E}[(X_\infty - X_t)^2 \mid \mathfrak{F}_t] + \mathbf{E}[(X_t - X_s)^2 \mid \mathfrak{F}_t],$$

and therefore, taking expectations with respect to $\mathfrak{F}_s$, the same equality with $\mathfrak{F}_t$ replaced by $\mathfrak{F}_s$ (or any smaller Borel field). Thus

$$Y_s \geqq \mathbf{E}[(X_\infty - X_t)^2 \mid \mathfrak{F}_s] = \mathbf{E}[Y_t \mid \mathfrak{F}_s].$$

The Y_t process is therefore a supermartingale. Since the X_t process is right continuous in the L^2 sense (Theorem 4.2, p. 328, of [1]), $\mathbf{E}[Y_t]$ is a right continuous function, and we may choose a right continuous version of the Y_t process. Let T_n be a sequence of stopping times which increases a.s. to infinity: X_{T_n} tends to 0 in the L^2 sense (Theorem 4.1 (iii) of [1], p. 319), and it follows that $\mathbf{E}[Y_{T_n}] = \mathbf{E}[(X_\infty - X_{T_n})^2] \to 0$. Using Proposition 3, we find that $\{Y_T\}$ is a potential and belongs to the class (D). By our Theorem 2, we may find a right continuous increasing process $\{A_t\}$ which generates $\{Y_t\}$:

$$\mathbf{E}[(X_\infty - X_s)^2 \mid \mathfrak{F}_s] = \mathbf{E}[A_\infty - A_s \mid \mathfrak{F}_s],$$

and, by the same orthogonality argument as above,

$$\mathbf{E}[(X_t - X_s)^2 \mid \mathfrak{F}_s] = \mathbf{E}[A_t - A_s \mid \mathfrak{F}_s].$$

7. We come now to the stochastic integral: Our aim is to define an integral

$$\int_0^\infty Y_t(\omega)\, dX_t(\omega)$$

for well adapted processes $\{Y_t\}$; we begin with step processes defined in the following manner: $0 = t_0 < t_1 < \cdots < t_n$ is a sequence of numbers, the $Y_{t_i}(\omega)$ are $\mathfrak{F}_{t_i}$-measurable square integrable random variables, and $Y_{t_n} = 0$. We set $Y_t(\omega) = Y_{t_i}(\omega)$ on the interval $[t_i, t_i + 1[$ $(t_{n+1} = \infty)$. The integral $\int_0^\infty Y_t\, dX_t$ is defined as the sum

$$\textstyle\sum_i Y_{t_i}(\omega)[X_{t_{i+1}}^-(\omega) - X_{t_i}(\omega)],$$

$X_{t_{i+1}}^-$ being the left limit of X_t at time t_{i+1}. We compute its L^2 norm, using the orthogonality of the increments, and we get

$$\begin{aligned}\mathbf{E}\left[\left(\int_0^\infty Y_t\, dX_t\right)^2\right] &= \mathbf{E}\left[\sum_i Y_{t_i}^2 \cdot \mathbf{E}[X_{t_{i+1}}^- - X_{t_i})^2 \mid \mathfrak{F}_{t_i}]\right] \\ &= \mathbf{E}\left[\sum_i Y_{t_i}^2 \cdot \mathbf{E}[A_{t_{i+1}}^- - A_{t_i} \mid \mathfrak{F}_{t_i}]\right] \\ &= \mathbf{E}\left[\int_0^\infty Y_t^2\, dA_t\right].\end{aligned}$$

Define the "norm" of a well adapted step process as this last integral. The preceding relation allows the extension of the stochastic integral, by standard Hilbert-space methods, to all well adapted processes which have a finite "norm", and are equal to the limit of a convergent sequence (in the sense of the "norm" just defined) of well adapted step processes. Among them are all right continuous well adapted processes, but it seems hard to show (though it is certainly true) that the full class of well adapted processes whose "norm" is finite has been attained by this procedure.

Note added in proof. Some results concerning the uniqueness of the representation of a supermartingale as a potential of a right continuous increasing process will appear later in this journal.

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