IMAGINARY QUADRATIC FIELDS WITH UNIQUE FACTORIZATION

Dedicated to Hans Rademacher
on the occasion of his seventieth birthday

BY

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1. Introduction

Nine imaginary quadratic fields are known in which the ring of integers has unique factorization, namely the fields with discriminants

$$-4,\ -8,\ -3,\ -7,\ -11,\ -19,\ -43,\ -67,\ -163.$$

Heilbronn and Linfoot [3] proved that there can exist at most one more such field. Dickson [2] showed that if this tenth field actually exists, then its discriminant must be numerically greater than 1 500 000, while Lehmer [5] improved this bound to 5 000 000 000.

It is easy to prove (see the last footnote on p. 294 of [3]) that if an imaginary quadratic field other than those with discriminants -4 and -8 has unique factorization, then its discriminant must be of the form $-p$, where p is a prime congruent to 3 modulo 4. We shall use $h(-p)$ to denote the number of classes of ideals in the ring of integers of the imaginary quadratic field with discriminant $-p$, and $L_p(s)$ to denote the Dirichlet L-function formed from the unique real nonprincipal residue-character modulo p. The latter is given by the formulas

$$L_p(s) = \sum_{n=1}^{\infty} \left(\frac{-p}{n}\right)\frac{1}{n^s} = \sum_{n=1}^{\infty} \left(\frac{n}{p}\right)\frac{1}{n^s} \qquad (s > 0)$$

in terms of the Kronecker and Legendre symbols respectively.

There are various results showing that if $h(-p) = 1$ for some prime p greater than 163, then $L_p(s)$ must have a real zero rather close to 1. For example, S. Chowla and A. Selberg [1] showed that if $h(-p) = 1$ for some prime p greater than 163, then $L_p(\frac{1}{2}) < 0$ and so $L_p(s)$ has a real zero between $\frac{1}{2}$ and 1 (since $L_p(1)$ is positive).

A more specific result follows from an inequality of Hecke, which is proved in [4]. If $0 < a \leqq 2$ and $L_p(s)$ has no real zeros greater than $1 - a/\log p$, Hecke showed that

$$h(-p) > \frac{a}{11000}\frac{p^{1/2}}{\log p}.$$

(This is trivial if $p < 10^{10}$, and otherwise follows from the inequality at the

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top of page 290 of [4].) This shows in particular that if $h(-p) = 1$ for some prime p greater than $2 \cdot 10^{10}$, then $L_p(s)$ must have a real zero between $1 - 2/\log p$ and 1.

In this paper we show that one of the lemmas in Heilbronn and Linfoot's paper [3] implies the following sharper result.

THEOREM. *If $p > 163$ and if the ring of integers of the imaginary quadratic field with discriminant $-p$ has unique factorization, then $L_p(s)$ has a real zero greater than*

$$1 - \frac{6}{\pi p^{1/2}}\left(1 + \frac{6 \log (p/4)}{\pi p^{1/2}}\right).$$

If we combine this with the result of Lehmer's calculation, we see that if $p > 163$ and $h(-p) = 1$, then $L_p(s)$ has a real zero greater than

$$1 - \frac{6}{\pi p^{1/2}}\left(1 + \frac{1}{16000}\right).$$

It is interesting to compare this with an unpublished result of Rosser that for any p there are no zeros of $L_p(s)$ greater than

$$1 - \frac{6}{\pi p^{1/2}}.$$

If Rosser's result could be improved very slightly, then we would be able to infer the nonexistence of the elusive tenth imaginary quadratic field whose ring of integers has unique factorization.

2. Preliminary lemmas

The first lemma is the cornerstone of our argument.

LEMMA 1. (Heilbronn and Linfoot). *If p is a prime such that $h(-p) = 1$ and if $\frac{1}{2} < s < 1$, then*

$$\zeta(s)L_p(s) \geqq \zeta(2s)(1 - 4^s p^{-s}) + 2^{2s-1} p^{1/2-s} \zeta(2s - 1) \int_{-\infty}^{\infty} (u^2 + 1)^{-s}\, du. \tag{1}$$

Proof. This is Lemma 2 of [3]. The proof is not very difficult.

Remark. If the expression on the right-hand side of (1) is positive when $s = s_0$, where $\frac{1}{2} < s_0 < 1$, then $L_p(s_0)$ must be negative, and so $L_p(s_1) = 0$ for some s_1 between s_0 and 1. For given p with $h(-p) = 1$ we wish to prove the existence of a zero of $L_p(s)$ as close to 1 as possible, and thus we shall try to prove the positivity of the right-hand side of (1) when $s = 1 - \delta$, where δ is as small as possible. We shall need some lemmas about the functions occurring on the right-hand side of (1).

LEMMA 2. *If $0 < \delta < \frac{1}{2}$, then*

$$\zeta(2 - 2\delta) > \tfrac{1}{6}\pi^2(1 + 1.1\,\delta).$$

Proof. By Taylor's theorem with remainder there is a number θ between 0 and 1 such that

$$\zeta(2-2\delta) = \zeta(2) - 2\delta\zeta'(2) + 2\delta^2\zeta''(2-2\theta\delta).$$

Since $\zeta''(2-2\theta\delta) > 0$ and $\zeta'(2)/\zeta(2) < -0.55$, the result follows.

LEMMA 3. *If* $0 < \delta < \frac{1}{2}$, *then*

$$\zeta(1-2\delta) > -(1-\delta)/(2\delta).$$

Proof. We have

$$\zeta(1-2\delta) = -\frac{1}{2\delta} + \frac{1}{2} - (1-2\delta)\int_1^\infty \left(u - [u] - \frac{1}{2}\right) u^{-2+2\delta}\, du.$$

Since the integral here is negative, the result follows.

LEMMA 4. *If* $0 < \delta < \frac{1}{2}$, *then*

$$\int_{-\infty}^{\infty} (u^2+1)^{-1+\delta}\, du < \pi(1-0.6\,\delta)/(1-2\delta).$$

Proof. For nonnegative integral n put

$$I_n = \int_0^\infty (u^2+1)^{-1} \log^n (u^2+1)\, du.$$

Using the substitutions $u^2 + 1 = t$ and $\log t = v$ in turn, we obtain

$$\begin{aligned} I_n &= \frac{1}{2}\int_1^\infty \frac{\log^n t}{t(t-1)^{1/2}}\, dt \\ &= \frac{1}{2}\int_0^\infty v^n e^{-v/2}(1-e^{-v})^{-1/2}\, dv \\ &= \frac{1}{2}\int_0^\infty v^n e^{-v/2}\left\{\sum_{k=0}^\infty \binom{k-\frac{1}{2}}{k} e^{-kv}\right\} dv \\ &= \frac{1}{2}\sum_{k=0}^\infty \binom{k-\frac{1}{2}}{k}\int_0^\infty v^n e^{-(k+1/2)v}\, dv \\ &= \frac{1}{2}n!\sum_{k=0}^\infty \binom{k-\frac{1}{2}}{k}\left(k+\frac{1}{2}\right)^{-n-1} \\ &= 2^n n!\sum_{k=0}^\infty \binom{k-\frac{1}{2}}{k}(2k+1)^{-n-1}. \end{aligned}$$

Now $I_0 = \pi/2$, while for $n > 0$ we have

$$\begin{aligned} I_n &\leqq 2^n n!\sum_{k=0}^\infty \binom{k-\frac{1}{2}}{k}(2k+1)^{-2} \\ &< 2^n n!\left\{1 + \frac{1}{18} + \frac{3}{200} + \frac{5}{16}\sum_{k=3}^\infty (2k+1)^{-2}\right\} \\ &< (1.097)2^n n! < 0.7(\pi/2)2^n n!. \end{aligned}$$

Hence

$$\begin{aligned}\int_0^\infty (u^2+1)^{-1+\delta}\,du &= \int_0^\infty (u^2+1)^{-1}\left\{\sum_{n=0}^\infty \frac{\delta^n \log^n (u^2+1)}{n!}\right\} du \\ &= \sum_{n=0}^\infty I_n\,\delta^n/n! \\ &< \frac{1}{2}\pi\left\{1+0.7\sum_{n=1}^\infty (2\delta)^n\right\} \\ &= \frac{1}{2}\pi(1-0.6\,\delta)(1-2\delta)^{-1}.\end{aligned}$$

LEMMA 5. *If p is a prime number such that $h(-p) = 1$ and if $0 < \delta < \frac{1}{2}$, then*

$$\zeta(1-\delta)L_p(1-\delta) \geqq \frac{\pi^2}{6}\{1+1.1\,\delta\}\left\{1-\left(\frac{4}{p}\right)^{1-\delta}\right\} - \frac{\pi}{\delta p^{1/2}}\,\frac{(1-\delta)(1-0.6\,\delta)}{1-2\delta}\left(\frac{p}{4}\right)^{\delta}.$$

Proof. The result is merely a combination of the previous lemmas.

LEMMA 6. *If p is a prime number such that $h(-p) = 1$ and if $0 < \delta < \frac{1}{20}$, then*

$$\zeta(1-\delta)L_p(1-\delta) \geqq \frac{\pi}{\delta p^{1/2}}\,\frac{(1-\delta)(1-0.6\,\delta)}{1-2\delta}\,F(\delta),$$

where

$$F(\delta) = \tfrac{1}{6}\pi\{1+0.6\,\delta\}\{1-(4/p)^{1-\delta}\}\delta p^{1/2} - e^{\delta\log(p/4)}.$$

Proof. Since

$$(1+1.1\,\delta)(1-2\delta) > (1-\delta)(1-0.6\,\delta)(1+0.6\,\delta)$$

for $0 < \delta < \frac{1}{20}$, this follows from Lemma 5.

3. Proof of the theorem

Suppose now that p is a prime greater than 163 such that $h(-p) = 1$. In view of the calculations of Dickson and Lehmer we may (and shall) assume that $p > 1\,500\,000$.

It is easy to see that the quantity $F(\delta)$ of Lemma 6 is negative if $\delta \leqq 6/(\pi p^{1/2})$, but becomes positive when δ is a little bit larger than $6/(\pi p^{1/2})$. Thus we take

$$\delta = \frac{6}{\pi p^{1/2}}(1+\eta), \qquad 0 < \eta < \frac{1}{10},$$

where η will later be chosen just large enough to make $F(\delta) > 0$. In particular, $\delta < \frac{1}{20}$, so that Lemma 6 is applicable.

Since $(4/p)^{19/20} < 0.01\, p^{-1/2} < 0.01\,\delta$, we have

$$\begin{aligned}\tfrac{1}{6}\pi\{1 + 0.6\,\delta\}\{1 - (4/p)^{1-\delta}\}\delta p^{1/2} &> \tfrac{1}{6}\pi\delta p^{1/2}(1 + 0.5\,\delta)\\ &= (1 + \eta)(1 + 0.5\,\delta)\\ &> 1 + 0.5\,\delta + \eta.\end{aligned}\tag{2}$$

On the other hand, since $\delta \log (p/4) < \delta \log^2 (p/4) < \frac{1}{3}$, we have

$$\begin{aligned}e^{\delta \log(p/4)} &< 1 + \delta \log (p/4) + \tfrac{1}{2}\{\delta \log (p/4)\}^2\{1 - \tfrac{1}{2}\delta \log (p/4)\}^{-1}\\ &< 1 + \delta \log (p/4) + 0.6\,\{\delta \log (p/4)\}^2\\ &< 1 + \delta \log (p/4) + 0.2\,\delta.\end{aligned}\tag{3}$$

If we now take

$$\eta = \frac{6}{\pi p^{1/2}} \log \frac{p}{4}$$

and combine (2) and (3), we have in the notation of Lemma 6

$$\begin{aligned}F(\delta) &> \eta - \delta \log (p/4) + 0.3\,\delta\\ &= -\left(\frac{6}{\pi p^{1/2}} \log \frac{p}{4}\right)^2 + 0.3\,\frac{6}{\pi p^{1/2}}\,(1 + \eta)\\ &> \frac{6}{\pi p^{1/2}}\left\{0.3 - \frac{6}{\pi p^{1/2}}\left(\log \frac{p}{4}\right)^2\right\}\\ &> 0.\end{aligned}$$

In view of Lemma 6 and our choice of δ and η, the theorem is proved. (Cf. the remark after Lemma 1.)

Addendum, December 1, 1961. The proof given by Heilbronn and Linfoot for our Lemma 1 actually shows that if a, b, c are real numbers with $a > 0$ and $d = b^2 - 4ac < 0$, and if $Z(s)$ is the analytic continuation of

$$\tfrac{1}{2} \sum_{(m,n) \neq (0,0)} (am^2 + bmn + cn^2)^{-s},$$

then for $\frac{1}{2} < s < 1$ we have

$$\left| a^s Z(s) - \zeta(2s) - (4a^2/|d|)^{s-1/2}\, \zeta(2s - 1) \int_{-\infty}^{+\infty} (u^2 + 1)^{-s}\, du \right| \leqq (4a^2/|d|)^s\, \zeta(2s).$$

In fact their proof incidentally establishes the continuation of $Z(s)$ from the domain Re $s > 1$ into the domain Re $s > \frac{1}{2}$, $s \neq 1$. The basic idea of the preceding inequality goes back to M. Deuring (*Imaginäre quadratische Zahlkörper mit der Klassenzahl 1*, Math. Zeitschrift, vol. 37 (1933), pp. 405–415). To see that Lemma 1 is contained in the above result, we need only note that if $h(-p) = 1$ for some prime p congruent to 3 modulo 4, $p > 3$, then

$$\zeta(s)L_p(s) = \frac{1}{2} \sum_{(m,n)\neq(0,0)} \{m^2 + mn + \tfrac{1}{4}(p+1)n^2\}^{-s}$$

for Re $s > 1$.

Accordingly, the argument of the present paper shows that if $d/a^2 < -2\cdot 10^6$, then $Z(s)$ has a real zero between

$$1 - \frac{3}{\pi}\left(\frac{4a^2}{|d|}\right)^{1/2}\left\{1 + \frac{3}{\pi}\left(\frac{4a^2}{|d|}\right)^{1/2}\log\left(\frac{|d|}{4a^2}\right)\right\} \quad \text{and} \quad 1 - \frac{3}{\pi}\left(\frac{4a^2}{|d|}\right)^{1/2}.$$

A less specific form of this assertion was obtained by M. Deuring (*Zetafunktionen quadratischer Formen*, J. Reine Angew. Math., vol. 172 (1935), pp. 226–252) and also by S. Chowla (*The class-number of binary quadratic forms*, Quart. J. Math. (Oxford), vol. 5 (1934), pp. 302–303, and *On an unsuspected real zero of Epstein's zeta function*, Proc. Nat. Inst. Sci. India, vol. 13, no. 4, 1 p. (1947)). Note, however, that Chowla uses $-4d$ in place of our d.

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