SEMIPROJECTIVE COMPLETIONS OF ABSTRACT CURVES[1]

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Introduction

Every embedding of a variety V can be essentially accomplished by adjoining new representatives to V. When an embedding of V is obtained by adjoining representatives of some projective variety, we call such an embedding semiprojective. In this paper we prove the following result: Given a variety V and a curve U which is a subvariety of V and has a representative on every representative of V, V can be semiprojectively embedded in a variety V' in such a way that the image of U is complete.

Our notation is that of Weil. In addition, we shall call a birational correspondence T between varieties V and V' pointwise biregular if T is biregular at every point P of V which corresponds to a point of V'. Also, if T is a correspondence between V and V' and U corresponds to U' under T, we shall write $T(U) = U'$.

1. The nonbiregular and pseudopoint loci

PROPOSITION 1.1. *Let T be a birational correspondence between the varieties V and V'. Then there exists a unique closed subset $\mathfrak{N}_T'$ of V' such that*

(i) *every component of $\mathfrak{N}_T'$ corresponds under T^{-1} to a subvariety of V,*

(ii) *if P' in V' corresponds nonbiregularly under T^{-1} to a point P in V, P' is in $\mathfrak{N}_T'$,*

(iii) *if P' is in $\mathfrak{N}_T'$ and P' corresponds to a point P in V, P' corresponds nonbiregularly.*

Moreover, if V, V', and T are defined over k, $\mathfrak{N}_T'$ is k-closed.

Proof. If V, V', and T are defined over k, by Weil [3], p. 514, Lemma 1, the set of points of V' where T^{-1} is not biregular is k-closed. Call this set $\mathfrak{A}'$, and let $\mathfrak{N}_T'$ be the (algebraic) projection of $(V \times \mathfrak{A}') \cap T$ on V'. Then $\mathfrak{N}_T'$ clearly has the stated properties.

If T is a birational correspondence between V and V', the closed subset of V' given by Proposition 1.1 will be called the *nonbiregular locus* of T on V' and will be denoted by $\mathfrak{N}_T'$.

We now make explicit the concept of adjoining representatives to a variety.

DEFINITION 1.1. Let $V = [V_\alpha ; \mathfrak{F}_\alpha ; T_{\beta\alpha}]$ and $V' = [V_\gamma'; \mathfrak{F}_\gamma'; T_{\delta\gamma}']$, $1 \leqq \alpha \leqq h$, $1 \leqq \gamma \leqq l$, be varieties, and T a birational correspondence between V and V' having representative $T_{\alpha\gamma}''$ on $V_\alpha \times V_\gamma'$. We shall say a variety

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$V^* = [V_\zeta^*; \mathfrak{F}_\zeta^*; T_{\eta\zeta}^*]$, $1 \leqq \zeta \leqq h + l$ is a *T-extension* of V and V' provided we can renumber the representatives of V^* so that

$$V_i^* = V_i, \quad \mathfrak{F}_i^* = \mathfrak{F}_i, \quad T_{ij}^* = T_{ij} \quad \text{for } 1 \leqq i, j \leqq h,$$

$$V_{h+i}^* = V_i', \quad \mathfrak{F}_{h+i}^* = \mathfrak{F}_i', \quad T_{h+i,h+j}^* = T_{ij}' \quad \text{for } 1 \leqq i, j \leqq l,$$

and

$$T_{i,h+j}^* = T_{ij}'' \quad \text{for } 1 \leqq i \leqq h, \quad 1 \leqq j \leqq l.$$

Moreover, we shall say V^* is an *extension* of V if there exist varieties V' and T such that V^* is a T-extension of V and V'.

It follows immediately that if T is a birational correspondence between V and V', there exists a T-extension of V and V' if and only if T is pointwise biregular. It is also easy to see (as we have done in [2]) that if V and V' are defined over k, V' is k-isomorphic to an extension of V if and only if there exists a dense k-embedding of V in V'.

In particular, if V, V', and T are defined over k and T is a birational correspondence between V and V', then there exists a T-extension of V and $V' - \mathfrak{N}_T'$, which is defined over k and which we shall denote by $(V, V' - \mathfrak{N}_T')$.[2]

PROPOSITION 1.2. *Let T be a correspondence between the varieties V and V', and let k be a field of definition for T, V, and V'. Then the set of all points P' of V' such that T is not complete over P' is a k-closed subset of V'.*

Proof. Let V_α, $1 \leqq \alpha \leqq h$, be the representatives of V, and let $\bar{V}_\alpha$ be that projective variety whose part at finite distance is V_α. Let T^* be the graph of T (considered as a mapping) on $\bar{V}_1 \times \cdots \times \bar{V}_h \times V' = V^*$. If $\mathfrak{F}_\alpha$ is the frontier on V_α, $V_\alpha - \mathfrak{F}_\alpha$ is a k-open subset of $\bar{V}_\alpha$, and $\bar{V}_\alpha - (V_\alpha - \mathfrak{F}_\alpha) = \bar{\mathfrak{F}}_\alpha$ is a k-closed subset of $\bar{V}_\alpha$. Then $\bar{\mathfrak{F}}_1 \times \cdots \times \bar{\mathfrak{F}}_h \times V' = \mathfrak{F}^*$ is a k-closed subset of V^*, so $T^* \cap \mathfrak{F}^*$ is also k-closed on V^*. Then the (algebraic) projection $\mathcal{P}'$ of $T^* \cap \mathfrak{F}^*$ on V' is k-closed.

Since $\bar{V}_1 \times \cdots \times \bar{V}_h$ is complete, the set-projection of $T^* \cap \mathfrak{F}^*$ on V' coincides with $\mathcal{P}'$; so a point P' of V' is in $\mathcal{P}'$ if and only if there exists a point $(P_1, \cdots, P_h, P')$ of $T^* \cap \mathfrak{F}^*$ lying over P'. But this is equivalent to saying T is not complete over P'.

The closed subset of V' given by Proposition 1.2 will be called the pseudo-point locus of T on V' and will be denoted by $\mathcal{P}_T'$.

DEFINITION 1.2. Given varieties U, V, and V' with U a subvariety of V, we shall say U can be *completed* (k-completed) by embedding V in V' pro-

[2] We are identifying T with the naturally induced correspondence between V and $V' - \mathfrak{N}_T'$, where here $V' - \mathfrak{N}_T'$ is the abstract variety defined by Weil on p. 179 of [4]. Where no confusion can result, we shall use the notation $V' - \mathfrak{N}_T'$ also to denote the set-complement of $\mathfrak{N}_T'$ in V'.

vided there exists an embedding (k-embedding) T of V in V' such that U corresponds under T to a complete subvariety of V'. We shall refer to V' as a *completion* (k-completion) of U under T, and to T as a *completing* (k-completing) of U in V'.

THEOREM 1.1. *Let V be a variety defined over a field k, and let U be a subvariety of V. U can be k-completed by embedding V if (and only if) there exist a variety V' defined over k and a birational correspondence T between V and V' and defined over k such that U corresponds biregularly to a complete subvariety U' of V' under T and $\mathfrak{N}_T' \cap \mathcal{P}_T' \cap U' = \emptyset$. Moreover, when there exist such a V' and T, the injection map of V into $(V, V' - \mathfrak{N}_T')$ is a k-completing of U.*

Proof. Suppose T and V' are defined over k, where T is a birational correspondence between V and V' with U corresponding biregularly to a complete subvariety U' of V' and $\mathfrak{N}_T' \cap \mathcal{P}_T' \cap U' = \emptyset$. If I is the injection map of V into $(V, V' - \mathfrak{N}_T')$ and I' the injection map of $V' - \mathfrak{N}_T'$ into $(V, V' - \mathfrak{N}_T')$, we have $I(U) = I'(U') = U^*$. Therefore, if K is a field of definition for U^*, U, U' containing k, and if P is a generic point of U over K and P' a corresponding generic point of U' over K, then there exists a generic point P^* of U^* over K such that $I(P) = I'(P') = P^*$. P^* then has the property that it agrees with P on any representative of U and with P' on any representative of U'. Moreover, $P \times P'$ is a generic point over K of the birational correspondence T^* between U and U' obtained by restricting T. U^* is complete. For if not, there exists a specialization $P^* \xrightarrow{K} Q^*$ where Q^* is the pseudopoint of U^*. But associated with this there is a specialization $(P, P') \xrightarrow{K} (Q, Q')$ where Q^* agrees with Q on any representative of V and with Q' on any representative of V'. Hence Q is the pseudopoint of U; and since U' is complete, Q' must be in $\mathfrak{N}_T' \cap U'$. But then T^* is not complete over Q', so Q' is in $\mathcal{P}_{T^*}'$; and therefore Q' is in $\mathcal{P}_{T^*}' \cap \mathfrak{N}_T' \cap U'$. But $\mathcal{P}_{T^*}' \subseteq \mathcal{P}_T' \cap U'$, so Q' is in $\mathcal{P}_T' \cap \mathfrak{N}_T' \cap U'$. This is a contradiction to the hypothesis that $\mathcal{P}_T' \cap \mathfrak{N}_T' \cap U' = \emptyset$. Thus, U^* is complete.

2. Semiprojective completions

We shall say a variety V is a *semiprojective* variety provided there exists a projective variety which is isomorphic to an extension of V. An extension V^* of V will be called a *semiprojective extension* provided V^* is an extension of the form $(V, V' - \mathfrak{N}_T')$ where V' is semiprojective. If a subvariety U of a variety V can be completed (k-completed) by embedding V in a semiprojective extension V^*, we shall say U can be *semiprojectively completed* (semiprojectively k-completed) by embedding V in V^*.

Any embedding, then, of a variety V in a variety $(V, \bar{V} - \bar{\mathfrak{N}}_T)$, where $\bar{V}$ is the projective join of the projectively embedded representatives of V, is a semiprojective embedding. In particular, it is easily seen that any surface with only a finite number of singularities can be semiprojectively completed by such an embedding.[3]

We now prove our main theorem.

THEOREM 2.1. *Let U^r be a subvariety of a variety V, let k be a field of definition for U and V, let $\bar{V}$ be the projective join of the projectively embedded representatives of V, and let $\bar{T}$ be the natural correspondence between V and $\bar{V}$. If U corresponds biregularly under $\bar{T}$ to a subvariety $\bar{U}$ of $\bar{V}$, then there exist a semiprojective variety V' and a birational correspondence T between V and V' such that*

(i) *both T and V' are defined over k,*

(ii) *U corresponds biregularly under T to a variety U' which is k-isomorphic to the projective variety $\bar{U}$,*

(iii) *$\mathfrak{N}_T' \cap \mathcal{P}_T' \cap U'$ is either empty or has dimension $\leqq r - 2$.*

Proof. Let $\bar{\mathfrak{N}}$ be the nonbiregular locus of $\bar{T}$ on $\bar{V}$, and let $f_1(x), \cdots, f_p(x)$ be a basis of forms for $\mathcal{I}(\bar{U})$ in $k[x_0, \cdots, x_n]$. There exists a form $g(x)$ in $\mathcal{I}(\bar{\mathfrak{N}})$ and not in $\mathcal{I}(\bar{U})$ in $k[x_0, \cdots, x_n]$; for if not, $\mathcal{I}(\bar{\mathfrak{N}}) \subseteq \mathcal{I}(\bar{U})$, and $\bar{U} \subseteq \bar{\mathfrak{N}}$. But this means U corresponds nonbiregularly to $\bar{U}$ under $\bar{T}$, a contradiction. If now $r_i(x) = f_i^{\rho}/g^{\gamma}$, where $\rho = \gamma\delta/\gamma_i$ and $\gamma_i = \deg f_i$, $\gamma = \text{l.c.m.}\, \gamma_i$, and $\delta = \deg g$, then the r_i are quotients of homogeneous polynomials of the same degree. Since $g(x)$ is not in $\mathcal{I}(\bar{U}) \supseteq \mathcal{I}(\bar{V})$, if $\bar{P}$ is a generic point of $\bar{V}$ over k, $g(\bar{P}) \neq 0$; so $r_i(\bar{P})$ is a function on $\bar{V}$. Then $r(\bar{P}) = (r_1(\bar{P}), \cdots, r_p(\bar{P}))$ is a point of the affine space S^p; so $(\bar{P}, r(\bar{P}))$ has a locus V' over k in $\bar{V} \times S^p$. If $\overline{S^p}$ is the projective variety having S^p as its part at finite distance, $\bar{V} \times \overline{S^p}$ is an extension of $\bar{V} \times S^p$; and $\bar{V} \times \overline{S^p}$ is isomorphic to the projective join of $\bar{V}$ and $\overline{S^p}$. Hence $\bar{V} \times S^p$ is semiprojective, and therefore V' is semiprojective too.

Let now P be a generic point of V over k, and $\bar{P}$ the corresponding generic point of $\bar{V}$, so that $(P, \bar{P})$ is a generic point of $\bar{T}$ over k. There is a natural birational correspondence between V and V', namely the locus T of $(P, \bar{P}, r(\bar{P}))$ over k. If Q is a generic point of U over k and $\bar{Q}$ the generic point of $\bar{U}$ over k corresponding to Q under $\bar{T}$, $r_i(\bar{Q}) = 0$ since $f_i(\bar{Q}) = 0$ and $g(\bar{Q}) \neq 0$. Therefore U corresponds under T to the subvariety U' of V' having generic point $(\bar{Q}, 0)$ over k. Then the projection of U' on $\bar{U}$ is clearly a k-isomorphism, so U' is k-isomorphic to the projective variety $\bar{U}$.

Suppose N' is a component of the nonbiregular locus $\mathfrak{N}_T'$ on V', and let $(\bar{P}_1, r)$ be a generic point of N' over $\bar{k}$. By definition of $\mathfrak{N}_T'$ there exists a point P_1 in V such that $(P_1, \bar{P}_1, r)$ is in T. Assume that $g(\bar{P}_1) \neq 0$. Then $\bar{P}_1$ is not in $\bar{\mathfrak{N}}$, so P_1 corresponds biregularly to $\bar{P}_1$ under $\bar{T}$. But also each $r_i(\bar{P})$ is in the specialization ring of $\bar{P}_1$ in $k(\bar{P})$ when $g(\bar{P}_1) \neq 0$, so each of

[3] For, if V is a surface with no singular curves, every component of $\bar{\mathfrak{N}}_T$ contracts to a point of V under T^{-1}. But then every point of $\bar{\mathfrak{N}}_T$ corresponds to a point of V, and hence T is complete over every point of $\bar{\mathfrak{N}}_T$. Therefore, $\bar{\mathfrak{N}}_T \cap \bar{\mathcal{P}}_T = \emptyset$. In [1] Nagata has made this observation for the case that V is normal.

the functions $r_i(\bar{P})$ is defined at $\bar{P}_1$ and $r_i = r_i(\bar{P}_1)$. Since P_1 corresponds biregularly to $\bar{P}_1$ and each $r_i(\bar{P})$ is in the specialization ring of $\bar{P}_1$ in $k(\bar{P})$, P_1 also corresponds biregularly to $(\bar{P}_1, r)$ under T; but this means N' corresponds biregularly under T, a contradiction. Therefore, $g(\bar{P}_1) = 0$. Then $f_i(\bar{P}_1) = 0$ for $i = 1, \cdots, p$ also; for otherwise, if $r = (r_1, \cdots, r_p)$ and $f_i(\bar{P}_1) \neq 0$, $r_i = r_i(\bar{P}_1) = \infty$ and $(\bar{P}_1, r)$ would not be a point. Hence, $\bar{P}_1$ is in $\bar{U}$, and $(\bar{P}_1, 0)$ is in U'.

If N^* is the locus of $(\bar{P}_1, 0)$ over $\bar{k}$, N^* is a proper subvariety of U' since its projection on $\bar{U}$ is different from $\bar{U}$ due to the fact $g(\bar{P}_1) = 0$. Moreover, since P_1 corresponds to $\bar{P}_1$ under $\bar{T}$, the projection from $\bar{T}$ to $\bar{V}$ is regular at $\bar{P}_1$. But then the specialization $\bar{P} \xrightarrow{k} \bar{P}_1$ extends only to the specialization $(P, \bar{P}) \xrightarrow{k} (P_1, \bar{P}_1)$, so a fortiori the specialization $(\bar{P}, r(\bar{P})) \xrightarrow{k} (\bar{P}_1, 0)$ extends only to the specialization $(P, \bar{P}, r(\bar{P})) \xrightarrow{k} (P_1, \bar{P}_1, 0)$; so N^* corresponds under T^{-1} to a subvariety of V (namely the locus of P_1 over $\bar{k}$), and T is complete over N^*.

Finally, observe that $N' \cap U' \subseteq N^*$. Let then $\mathfrak{N}^*$ be the union of all such N^* obtained from components of $\mathfrak{N}_T'$. Then $\mathfrak{N}_T' \cap U' \subseteq \mathfrak{N}^*$, and since $\mathfrak{N}^*$ is a proper closed subset of U' and therefore has dimension at most $r - 1$, any $(r - 1)$-dimensional component of $\mathfrak{N}_T' \cap U'$ must also be a component of $\mathfrak{N}^*$. But we have seen T is complete over every component of $\mathfrak{N}^*$, so no component of $\mathfrak{N}^*$ is $\subseteq \mathcal{P}_T'$, and therefore no $(r - 1)$-dimensional component of $\mathfrak{N}_T' \cap U'$ is $\subseteq \mathcal{P}_T'$. Thus, $(\mathfrak{N}_T' \cap U') \cap \mathcal{P}_T'$ has dimension at most $r - 2$.

COROLLARY 2.1. *Let U be a curve which is a subvariety of a variety V and has a representative on every representative of V, and suppose U and V are defined over a field k. Then there exist a semiprojective variety V' and a birational correspondence T between V and V' such that the injection map of V is a k-completing of U in $(V, V' - \mathfrak{N}_T')$.*

Proof. Apply Theorems 2.1 and 1.1.

Remarks. (i) The requirement that U have a representative on every representative of V in Corollary 2.1 may be removed if U is a normal curve on a surface V, since then U corresponds biregularly to $\bar{U}$ on $\bar{V}$ and Theorem 2.1 applies. Question: Is the "fully represented" condition necessary when V is, for instance, a nonsingular variety of dimension > 2?

(ii) In Theorem 2.1 the properties of $\bar{V}$ that are used are that $\bar{V}$ is projective, and that the projection from $\bar{T}$ to $\bar{V}$ is regular at every point of $\bar{V}$ which corresponds to a point of V. We could therefore have replaced $\bar{V}$ by any other variety with these properties.

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