A CONDITION FOR THE SOLVABILITY OF A FINITE GROUP

BY
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In [8] H. Wielandt introduced the concept of subinvariant subgroup and proved that the set of all subinvariant subgroups of a finite group G form a lattice, $\mathcal{S}(G)$, under the usual compositions of intersection and subgroup union. (Cf. [2], Chapter 8.) Clearly the definition of solvability for G requires $\mathcal{S}(G)$ to form a rather substantial skeleton for $\mathcal{L}(G)$, the lattice of all subgroups of G under the same compositions, and suggests that there exist relations between $\mathcal{S}(G)$ and $\mathcal{L}(G)$ which insure the solvability of G. One such relationship was given by Wielandt in [8]: A finite group G is nilpotent if and only if $\mathcal{L}(G) = \mathcal{S}(G)$.

Now a direct extension of a portion of this result based only on the ratio of the number of elements in $\mathcal{L}(G)$ to the number in $\mathcal{S}(G)$ is impossible as the direct product of a simple nonabelian group of small order with a nilpotent group of large order indicates. Thus the distribution of the elements of $\mathcal{S}(G)$ in $\mathcal{L}(G)$ must be considered. We prove here that, stated roughly, if the elements of $\mathcal{S}(G)$ comprise over 20% of $\mathcal{L}(G)$ and are rather uniformly distributed throughout $\mathcal{L}(G)$, then G is a solvable group.

1. On maximal subgroups

Two intermediate results essential to the proof of the theorem mentioned above are proved in this section. Both are results concerning maximal subgroups and are of some interest in themselves.

THEOREM 1. *If the finite group G contains a maximal subgroup M which is nilpotent of class less than* 3, *then G is solvable.*

This result is properly contained in a theorem of B. Huppert [4] *except* when M contains a 2-subgroup of class 2. However this rather special case attains a degree of importance from some work of N. Itô and of J. G. Thompson. In [6] Thompson announced that a finite group which contains as a maximal subgroup a nilpotent group of odd order is a solvable group, while in [5] Itô showed that certain nonsolvable linear fractional groups contain maximal subgroups which are nilpotent (and of even order).

Now if the theorem is not true, then among the nonsolvable groups which contain as maximal subgroups nilpotent groups of class less than 3 there is one (at least) of minimal order. Denote such a group by G. We shall show that G cannot exist and thereby prove the theorem.

First of all G contains no normal subgroups ($\neq(1)$) which lie entirely in M. For if $K \subseteq M$ is normal in G, then G/K satisfies the hypothesis of the theorem. Since the order of G/K, $o(G/K)$, is less than $o(G)$, G/K is solv-

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able. Also K is solvable since it is a subgroup of M. But the extension of a solvable group by a solvable group is solvable. This means, since G is non-solvable, that G has no such subgroups as K.

Next we shall show that G contains a normal subgroup H such that $G/H \cong M$, and we use the theory of the transfer to do this. Let p be a rational prime divisor of $o(M)$, and let P be the p-Sylow subgroup of M. Then P is also a Sylow subgroup of G, for suppose a p-subgroup S of G properly contains P. If $\mathfrak{z}(S)$, the center of S, lies in P, then $\mathfrak{z}(S) \neq (1)$ is normal in G since $G = M \cup S$. If $\mathfrak{z}(S)$ does not lie in P, then $\mathfrak{z}(P)$ is normal in G since $M \cup \mathfrak{z}(S) = G$ in this case. Thus the supposition that P is not a Sylow subgroup of G leads to the existence of a normal subgroup of G lying in M, a possibility that was ruled out above. So P is a Sylow subgroup of G.

Now consider the transfer of G into P. According to the First Theorem of Grün ([9], p. 140) the transferred group is isomorphic with P/P_1, where

$$P_1 = (P \cap \mathfrak{N}'(P)) \cup (P \cap x_1 P' x_n^{-1}) \cup \cdots \cup (P \cap x_n P' x_n^{-1}).$$

Here $\mathfrak{N}'(P)$ is the commutator subgroup of $\mathfrak{N}(P)$, the normalizer of P in G, P' is the commutator subgroup of P, and the x_i are the $n = o(G)$ elements of G. We must show that P_1 is a proper subgroup of P, and since P is of class 1 or 2, there are two cases to consider.

Case 1. P is abelian. Therefore $P' = (1)$ so that $P_1 = P \cap \mathfrak{N}'(P)$. Since M contains no subgroups normal in G, and since M is maximal in G, it follows that $\mathfrak{N}(P) = M$. Thus $\mathfrak{N}'(P) = M'$, and since M is the direct product of its Sylow subgroups, $P \cap M'$ is precisely P' which is (1) since P is abelian. So $P_1 = (1)$ in this case, and G is homomorphic with P.

Case 2. P is of class 2. Then $P \neq \mathfrak{z}(P) = Z$ but $\mathfrak{z}(P/Z) = P/Z$, and consequently $P' \subseteq Z$. Now let $x \in G$, $x \notin M$, and suppose $xZx^{-1} \cap M = T \neq (1)$. Then $T = xZx^{-1}$, for consider the normalizer of T in G. $\mathfrak{N}(T)$ contains xMx^{-1} since T lies in the center of xMx^{-1}. Also $\mathfrak{N}(T)$ contains $\mathfrak{z}(M)$ since T is in M. Since T cannot be normal in G, $\mathfrak{N}(T) = xMx^{-1}$, so that $\mathfrak{z}(M) \subseteq xMx^{-1}$. But this means that xZx^{-1} lies in the normalizer of $\mathfrak{z}(M)$, so that if $T \neq xZx^{-1}$, then $G = M \cup xZx^{-1}$ is the normalizer of $\mathfrak{z}(M)$, an impossibility. Moreover $Z \cap xZx^{-1} = (1)$ since $\mathfrak{C}(Z \cap xZx^{-1})$, the centralizer of $Z \cap xZx^{-1}$, equals $M \cup xMx^{-1}$ which is G.

Now form the subgroup $U = Z \cup xZx^{-1}$. U is normal in M for if $y \in M$ and $z_1 \in xZx^{-1}$, then $yz_1y^{-1} = z_1 z$ for some $z \in Z$ since P is of class 2. Similarly, U is normal in xMx^{-1}, and so, since $G = M \cup xMx^{-1}$, U is normal in G. As this possibility was ruled out earlier, the assumption $xZx^{-1} \cap M \neq (1)$ when $x \notin M$ cannot hold. Consequently,

$$(P \cap x_1 P' x_1^{-1}) \cup \cdots \cup (P \cap x_n P' x_n^{-1}) \subseteq Z.$$

Since $P \cap \mathfrak{N}'(P)$ still equals P' exactly as in Case 1, we see that $P_1 \subseteq Z$. Thus P_1 is a proper subgroup of P.

However we can carry the argument further in the second case and obtain the same conclusion as in the first, namely, that G is homomorphic with P. The above proves that G contains a normal subgroup K such that $G/K \cong P/P_1$. Therefore P_1 is a p-Sylow subgroup of K. Consider the transfer of K into P_1, and apply Grün's Theorem again. Then the transferred group is isomorphic with P_1/P_2, where

$$P_2 = (P_1 \cap \mathfrak{N}'_K(P_1)) \cup (P_1 \cap x_1 P'_1 x_1^{-1}) \cup \cdots \cup (P_1 \cap x_r P'_1 x_r^{-1}).$$

Here $\mathfrak{N}'_K(P_1)$ denotes the commutator subgroup of the normalizer of P_1 in K. By the Second Isomorphism Theorem ([9], p. 34) $P_1 = P \cap K$ and is normal in P, hence in M. Therefore $\mathfrak{N}_K(P_1) \supseteq M \cap K$. But if $\mathfrak{N}_K(P_1)$ contains an element not in M, then P_1 is normal in G. Thus $\mathfrak{N}_K(P_1) = M \cap K$. Since $M \cap K$ is nilpotent, $P_1 \cap \mathfrak{N}'_K(P_1) = P'_1$ which equals (1) since P_1 is abelian. Therefore $P_2 = (1)$ and K is homomorphic with P_1. This yields the desired result: G is homomorphic with P in both cases.

If the kernel of the above homomorphism is denoted by H_p, then $H = \bigcap H_p$ for all the distinct prime divisors of $o(M)$ is precisely a normal subgroup of G having the property that $G/H \cong M$. Moreover $G = HM$ and $H \cap M = (1)$. The elements of M induce automorphisms of H; in particular, an element x from $\mathfrak{z}(M)$ induces an automorphism of H which fixes no element of H except the identity. For if $xhx^{-1} = h \neq 1$ in H, then x generates a cyclic subgroup of M which is normal in G since $M \cup (h) = G$. Therefore H possesses a fixed-point-free automorphism of prime order, so that by a result of J. G. Thompson [7] H is a nilpotent group. (We note that a method of I. Herstein [3] can also be adapted to prove that H is nilpotent.)

This means, however, that the nonsolvable group G is an extension of a nilpotent group H by a nilpotent group M. Since such an extension is solvable, we see that G is both solvable and nonsolvable. Clearly this is impossible. So, G does not exist, and the theorem is proved.

The intersection of all maximal subgroups of a finite group G is known to be a nilpotent group, the Frattini subgroup of G. The nature of the generalized Frattini subgroups [1] of the next theorem will prove useful in the following section.

THEOREM 2. *The intersection $\mathfrak{F}_p(G)$ of all the maximal subgroups of the finite group G whose indices in G are not divisible by the prime p is a solvable group.*

Let P be a p-Sylow subgroup of $F = (\mathfrak{F}_p G)$. If $P = (1)$, then F is the Frattini subgroup of G, for otherwise G contains a maximal subgroup N such that $F \cup N = G$. However $G/F \cong N/F \cap N$, so that the index of N in G is not divisible by p. But that means $F \subseteq N$ and $F \cup N = N \neq G$. Therefore F is the Frattini subgroup of G and is certainly solvable.

Now suppose $P \neq (1)$. By the Sylow theorems G contains a Sylow subgroup S such that $P = F \cap S$. Therefore $S \subseteq \mathfrak{N}(P)$, the normalizer of P

in G, so that if $\mathfrak{N}(P) \neq G$, then $\mathfrak{N}(P)$ is contained in a maximal subgroup M of G whose index in G is not divisible by p. Suppose $y \in G$, $y \notin M$; then yPy^{-1} is also a Sylow subgroup of F. Therefore F contains an element x such that $x(yPy^{-1})x^{-1} = P$, which means that $xy \in \mathfrak{N}(P)$ and thus M. But $F \subseteq M$ so that $x^{-1} \in M$, and therefore $y \in M$. This impossibility means that $\mathfrak{N}(P) = G$.

Now consider $\mathfrak{F}_p(G/P)$; we again have the first case in that the p-Sylow subgroup of $\mathfrak{F}_p(G/P) \cong \mathfrak{F}_p(G)/P$ is (1). Therefore $\mathfrak{F}_p(G)/P$ is the Frattini subgroup of G/P, and so F is solvable since it is an extension of a nilpotent group by a nilpotent group.

Example. Let G be the group defined by the relations $a^5 = b^4 = 1$ and $a^2b = ba$. In this group of order 20 the Frattini subgroup is (1), while $\mathfrak{F}_5(G)$ is the subgroup of order 10 generated by a and b^2. $\mathfrak{F}_5$ is solvable but not nilpotent.

2. Subinvariance and variance of a group

A subgroup H of a group G is *subinvariant* in G if there exist subgroups $H_0 = G \supset \cdots \supset H_n = H$ such that H_i is a normal subgroup of H_{i-1} for $i = 1, 2, \cdots, n$. For some of the properties and applications of subinvariance see the aforementioned paper of Wielandt [8] and Chapter 8 of [2]. We shall need the following additional property.

LEMMA 1. *If the subinvariant subgroup H of the finite group G is contained in the nonnormal maximal subgroup K of G, and if no subgroup of K properly containing H is subinvariant in G, then H is normal in G.*

Let $y \in K$, $y \notin H$. Then yHy^{-1} is also subinvariant in G, for if

$$G = H_0 \supset H_1 \supset \cdots \supset H_n = H$$

with H_i normal in H_{i-1}, then $G = yH_0y^{-1} \supset yH_1y^{-1} \supset \cdots \supset yH_ny^{-1}$ with yH_iy^{-1} normal in $y(H_{i-1})y^{-1}$. A result of Wielandt states that $H \cup yHy^{-1}$ is also subinvariant in G, and since $H \subseteq H \cup yHy^{-1} \subseteq K$, it follows that $yHy^{-1} = H$. Thus $\mathfrak{N}(H) \supseteq K$. On the other hand $\mathfrak{N}(H) \supset H_{n-1}$ which is not a subgroup of K since it is subinvariant in G and contains $H_n = H$ properly. Therefore $\mathfrak{N}(H) = K \cup H_{n-1} = G$, and so H is normal in G.

A collection of subgroups $L_0, L_1, \cdots, L_n$ of G is called an *upper chain*, $\mathcal{C}_n$, of G of length n if $L_0 = G$ and if each L_i is maximal in L_{i-1}, $i = 1, 2, \cdots, n$. The *subinvariance* of $\mathcal{C}_n$, $s(\mathcal{C}_n)$, is defined to be the number of $L_i \neq L_0$ which are subinvariant in G. The *variance* of $\mathcal{C}_n$, $v(\mathcal{C}_n)$, is defined as $n/s(\mathcal{C}_n)$ if $s(\mathcal{C}_n) \neq 0$, and as n otherwise. Then $v(G)$, the *variance* of G, is the maximum of the $v(\mathcal{C}_n)$ for all $\mathcal{C}_n$ of G. (Only finite groups are being considered here.) This number $v(G)$ describes to some extent the distribution of the elements of $\mathcal{S}(G)$ in $\mathcal{L}(G)$. For example, $v(G) = 1$ if and only if $\mathcal{S}(G) = \mathcal{L}(G)$.

LEMMA 2. *If H is a nonnormal maximal subgroup of the finite group G, then $v(H) < v(G)$.*

Let $\mathcal{C}'_n:H = K_0 \supset K_1 \supset \cdots \supset K_n$ be an upper chain of H; then $\mathcal{C}_{n+1}:G = L_0 \supset L_1 = K_0 \supset L_2 = K_1 \supset \cdots \supset L_{n+1} = K_n$ is an upper chain of G. Since a subgroup of H is subinvariant in G only if it is subinvariant in H, we see that $s(\mathcal{C}_{n+1}) \leqq s(\mathcal{C}'_n)$. If $s(\mathcal{C}_{n+1}) > 0$, then

$$v(\mathcal{C}_{n+1}) = (n + 1)/s(\mathcal{C}_{n+1}) \geqq (n + 1)/s(\mathcal{C}'_n) > n/s(\mathcal{C}'_n) = v(\mathcal{C}'_n).$$

If $s(\mathcal{C}_{n+1}) = 0$, then $v(\mathcal{C}_{n+1}) = n + 1 > n \geqq v(\mathcal{C}'_n)$, which is n if $s(\mathcal{C}'_n) = 0$ and $n/s(\mathcal{C}'_n)$ otherwise. Since G is finite, the strict inequality is preserved when maxima are considered, and hence $v(G) > v(H)$.

A simple consequence of this result is the following characterization of nilpotent groups:

A finite group G is nilpotent if and only if $v(G) = v(H)$ for every proper subgroup H of G.

For if $v(G) = v(H)$ when H is maximal in G, then H must be normal, and a group all of whose maximal subgroups are normal is certainly nilpotent. Conversely if G is nilpotent, then $v(G) = 1$, and since a subgroup H of G is also nilpotent, $v(H) = 1$. Therefore $v(G) = v(H)$.

LEMMA 3. *If $v(G) = r$ and H is a normal subgroup of G, then $v(G/H) \leqq r$.*

Let $v(G/H) = m$. Then G/H contains an upper chain

$$\mathcal{C}^*_n:G/H = L^*_0 \supset L^*_1 \supset \cdots \supset L^*_n,$$

such that $v(\mathcal{C}^*_n) = m$. Therefore G contains the upper chain

$$\mathcal{C}_n:G = L_0 \supset L_1 \supset \cdots \supset L_n,$$

where L_i contains H and $L_i/H = L^*_i$, $i = 0, 1, 2, \cdots, n$. Clearly L_i is subinvariant in G if and only if L^*_i is subinvariant in G/H, so that $v(\mathcal{C}_n) = r$. Since $v(G) \geqq v(\mathcal{C}_n)$, it follows that $v(G/H) \leqq r$.

LEMMA 4. *If H is a nonnormal maximal subgroup of the finite group G, if H is solvable but contains no subgroups $\neq (1)$ subinvariant in G, and if $v(G) < n$, then $o(H)$ is the product of at most $n - 2$ not necessarily distinct primes.*

Let $H = H_0 \supset H_1 \supset \cdots \supset H_r = (1)$ be a composition series for H; therefore $o(H)$ is a product of r primes. Then

$$\mathcal{C}_{r+1}:G = L_0 \supset L_1 = H \supset L_2 = H_1 \supset \cdots \supset L_{r+1} = (1)$$

is an upper chain of G, and $s(\mathcal{C}_{r+1}) = 1$ since L_{r+1} is the only element of the chain which is subinvariant in G. Hence $n > v(G) \geqq (r + 1)/1 = r + 1$, so that $r < n - 1$ and $r \leqq n - 2$.

Now we are in position to prove our main result.

THEOREM 3. *The finite group G is solvable if* (i) $v(G) < 5$ *and* $(o(G), 3) = 1$, *or* (ii) $v(G) < 4$.

Since there are actually two theorems here, we shall prove the truth of the conclusion for hypothesis (i) and then indicate the few minor changes which yield a proof in the other case.

If the conclusion is not true, then there is a nonsolvable group G having $v(G) < 5$ and $(o(G), 3) = 1$ such that any group of smaller order with these properties is solvable. We shall prove the theorem by showing that such a group does not exist, and we will do this by considering the maximal subgroups of G.

First of all, G must contain nonnormal maximal subgroups since a group having all its maximal subgroups normal is nilpotent, hence solvable. So let H be a nonnormal maximal subgroup of G. Then since $v(H) < v(G)$, by Lemma 2, and since $o(H)$ divides $o(G)$, we see that H satisfies (i). Since G is a nonsolvable group of minimal order satisfying (i), we see that H must be solvable. Hence all of the nonnormal maximal subgroups of G are solvable groups.

We can obtain still more information about these subgroups. Suppose H contains a subgroup $\neq (1)$ which is subinvariant in G; then it contains a subgroup K maximal in this property. Therefore, by Lemma 1, K is actually normal in G. Now consider G/K. By Lemma 3, $v(G/K) \leqq v(G)$, and since $o(G/K)$ divides $o(G)$, it follows that G/K satisfies (i). Again the minimality of $o(G)$ among the orders of the nonsolvable groups satisfying (i) implies that G/K is solvable. But K is also a solvable group since it is a subgroup of a nonnormal maximal subgroup H of G. This means that G is an extension of a solvable group by a solvable group and hence is solvable itself. Thus the supposition that H contains a subgroup $\neq (1)$ which is subinvariant in G cannot hold. Now we can apply Lemma 4 to the nonnormal maximal subgroups of G. Thus we have the following information about some of the maximal subgroups of G.

(*) *G has nonnormal maximal subgroups each of which is a solvable group whose order is a product of at most three primes.*

Now consider the normal maximal subgroups of G. By a well-known theorem ([9], p. 139) the order of a nonsolvable group is divisible by either 12 or the cube of the smallest prime dividing the order of the group. Since G is nonsolvable, and since $(o(G), 3) = 1$, $o(G)$ has a prime divisor p such that $p^3 \mid o(G)$. Let P be a p-Sylow subgroup of G. If G contains no normal subgroups, then P is contained in a nonnormal maximal subgroup M of G. But by (*), $o(M)$ is a product of at most three primes, so that $M = P$ and $o(M) = p^3$. Since a group of order p^3 is nilpotent of class $\leqq 2$, this implies, by Theorem 1, that G is solvable. Thus the supposition that G contains no normal subgroups leads to a contradiction of the nonsolvability of G. If $L \neq (1)$ is a normal subgroup of G, then G/L is solvable due to the minimality of $o(G)$. Therefore we conclude that G must contain a maximal subgroup which is normal. We note further that the above argument also shows that each p-Sylow subgroup of G, for any prime p such that $p^3 \mid o(G)$, is contained only in a maximal subgroup which is normal. Thus we have the following additional information about the maximal subgroups of G:

(**) *G contains at least one maximal subgroup which is normal. Moreover, if p is the smallest prime dividing $o(G)$, then the p-Sylow subgroups of G are contained only in normal maximal subgroups of G.*

Now we are in a position to prove that there exists no nonsolvable group satisfying both (*) and (**). For suppose G_1 is a group with properties (*) and (**). Then if p is the smallest prime dividing $o(G_1)$, $\mathfrak{F}_p(G_1)$, the intersection of all maximal subgroups of G_1 which have indices relatively prime to p, is the intersection of certain of the normal maximal subgroups of G. Therefore $G_1/\mathfrak{F}_p(G_1)$ is abelian since it is the direct product of images of G modulo those normal maximal subgroups. Theorem 2, however, states that $\mathfrak{F}_p(G_1)$ is a solvable group, so that G_1, as an extension of a solvable group by an abelian group, is a solvable group. Thus there does not exist a nonsolvable group with properties (*) and (**), and so the proof (for hypothesis (i)) is complete.

The proof for hypothesis (ii) runs along the same lines, with one simplification. In the above it was necessary to have the cube of a prime dividing $o(G)$ since this, together with the order of a nonnormal maximal subgroup being the product of at most three primes, enabled us to prove that G had a normal subgroup. However, if $v(G) < 4$, the nonnormal maximal subgroups of a nonsolvable G with minimal $o(G)$ will have orders which are products of two primes. This means it will only be necessary to have the square of some prime divide $o(G)$, something which is true for every nonsolvable group.

We note that the variance of A_5, the alternating group of order 60, is 4. Hence the condition on $o(G)$ in (i) is necessary.

References

1. W. E. Deskins, *On maximal subgroups*, Proceedings of Symposia in Pure Mathematics, vol. 1 (Finite Groups), Amer. Math. Soc., 1959, pp. 100–104.
2. M. Hall, *The theory of groups*, New York, Macmillan, 1959.
3. I. Herstein, *A remark on finite groups*, Proc. Amer. Math. Soc., vol. 9 (1958), pp. 255–257.
4. B. Huppert, *Normalteiler und maximale Untergruppen endlicher Gruppen*, Math. Zeitschrift, vol. 60 (1954), pp. 409–434.
5. N. Itô, *On the factorizations of the linear fractional group $LF(2, p^n)$*, Acta Sci. Math. Szeged, vol. 15 (1953), pp. 79–84.

6. J. G. THOMPSON, *Finite groups with normal p-complements*, Proceedings of Symposia in Pure Mathematics, vol. 1 (Finite Groups), Amer. Math. Soc., 1959, pp. 1–4.
7. ———, *Finite groups with fixed-point-free automorphisms of prime order*, Math. Zeitschrift, to appear.
8. H. WIELANDT, *Eine Verallgemeinerung der invarianten Untergruppen*, Math. Zeitschrift, vol. 45 (1939), pp. 209–244.
9. H. ZASSENHAUS, *The theory of groups* (translation), New York, Chelsea, 1949.

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