SOME PROPERTIES OF RECURRENT RANDOM WALK[1]

BY

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1. Introduction

Consider the stochastic process

$$S_n = S_0 + X_1 + X_2 + \cdots + X_n, \qquad n \geqq 1.$$

S_0 is an arbitrary integer, and the X_i are independent, identically distributed, integer-valued random variables. It is assumed that the state space of this process is the set of all integers, and that every point is visited infinitely often with probability one, for every starting point S_0. Formally, this means

$$(1.1) \qquad P(\textstyle\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} [S_k = b] \mid S_0 = a) \equiv 1 \qquad \text{for all } a, b = 0, \pm 1, \pm 2, \cdots.$$

In terms of the characteristic function

$$\phi(\theta) = \textstyle\sum_{k=-\infty}^{\infty} P(X_1 = k) e^{ik\theta} = E(e^{i\theta X_1}), \qquad -\infty < \theta < \infty,$$

equation (1.1) is equivalent to

$$(1.2) \qquad \phi(\theta) \neq 1 \quad \text{for } 0 < |\theta| \leqq \pi, \qquad \lim_{t \to 1^-} \int_{-\pi}^{\pi} \frac{d\theta}{1 - t\phi(\theta)} = +\infty.$$

This is so because S_n, according to (1.1), is an indecomposable recurrent Markov chain on the set of all integers, and the first condition in (1.2) is necessary and sufficient for indecomposability, while the second is necessary and sufficient for recurrence.

Let x_1 and x_2 be two distinct integers, $|x_1 - x_2| = k > 0$. Consider the imbedded Markov chain induced by the set $S = \{x_1, x_2\}$ of these two points. This is the Markov chain whose transition matrix $P(S) = (P_{ij}(S))$, $i, j = 1, 2$, is defined by

$$(1.3) \qquad P_{ij}(S) = \textstyle\sum_{n=1}^{\infty} P(S_\nu \notin S \text{ for } \nu = 1, 2, \cdots, n-1; S_n = x_j \mid S_0 = x_i), \qquad i, j = 1, 2,$$

i.e., the imbedded Markov chain is the original process, observed only when it assumes a value in S.

The central result of this paper (Theorem 1) asserts that

$$(1.4) \qquad P(S) = \begin{bmatrix} 1 - p_k & p_k \\ p_k & 1 - p_k \end{bmatrix},$$

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$$(1.5)\qquad \begin{aligned}(p_k)^{-1} &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1-\cos k\theta}{1-\phi(\theta)}\,d\theta \\ &= \textstyle\sum_{n=0}^{\infty} [2P(S_n = 0) - P(S_n = k) - P(S_n = -k)] < \infty.\end{aligned}$$

(In the case of unconditional probabilities, as in (1.5), the condition $S_0 = 0$ is understood.)

The proof of (1.4) and (1.5) begins with Fourier analytical estimates, based on a technique of Chung and Erdös [1] in §2. In §3 we prove (1.4) and (1.5), but curiously the proof seems to require an extension of the investigation to the imbedded Markov chain induced by a set of three instead of two states.[2] These extended results are summarized in Theorem 2. The necessary combinatorial work, involving identities between generating functions, is rendered simple by using an elegant technique due to P. Frank [3].

Identities such as (1.5) for p_k are well adapted to the study of asymptotic behavior for large k (which evidently depends on the behavior of $\phi(\theta)$ near $\theta = 0$). §4 is therefore devoted to certain new limit theorems valid for certain $\phi(\theta)$ in the domain of attraction of symmetric stable laws of index $1 \leqq \alpha \leqq 2$. The comparison, in Theorem 6, between absorption problems involving single points and intervals was made possible by H. Kesten who kindly made recent results [4] available before publication.

In [6] new methods are developed which lead to an explicit formula for the transition matrix of the imbedded Markov chain corresponding to an arbitrary finite set of states, under the restriction that the process is symmetric, i.e., that $\phi(\theta) = \phi(-\theta)$. The methods and results in [6] take the form of a discrete analogue of classical potential theory.

2. Some Fourier analysis

Let $\mathcal{C}$ denote the class of trigonometric polynomials $f(\theta) = \sum a_k e^{ik\theta}$, where all but a finite number of the coefficients a_k, $k = 0, \pm 1, \pm 2, \cdots$, are zero and the remaining ones real, and such that $f(0) = f'(0) = 0$. It follows that for each $f \in \mathcal{C}$ there is a constant c such that $|f(\theta)| \leqq c\theta^2$ for $|\theta| \leqq \pi$. Examples of functions in $\mathcal{C}$ which we shall use are $1 - \cos k\theta$ and trigonometric polynomials of the form $\sum b_k(1 - e^{ik\theta})$, with $\sum kb_k = 0$, b_k real. In fact every f in $\mathcal{C}$ is of this latter type. The above-mentioned bound on $|f(\theta)|$ enables us to prove the following lemma (it is easy to see that the proof would go through if it were only assumed that for some $\varepsilon > 0$, $|f(\theta)| \leqq c|\theta|^{1+\varepsilon}$ for all $|\theta| \leqq \pi$.

LEMMA 1. *If $f(\theta) = \sum a_k e^{ik\theta} \in \mathcal{C}$, then the function $f(\theta)[1 - \phi(\theta)]^{-1}$ is integrable on $-\pi \leqq \theta \leqq \pi$, and*

$$(2.1)\qquad \lim_{t\to 1^-} \int_{-\pi}^{\pi} \frac{f(\theta)}{1 - t\phi(\theta)}\,d\theta = \int_{-\pi}^{\pi} \frac{f(\theta)}{1-\phi(\theta)}\,d\theta < \infty.$$

[2] This is no longer true; see footnote 3, page 239.

Proof. Since $\phi(\theta)$ is continuous, and satisfies (1.2), it suffices to prove that for some $h > 0$, $|f(\theta)|\,|1 - \phi(\theta)|^{-1} \in L_1(-h, h)$. Since $|f(\theta)| \leqq c\theta^2$, it suffices to show that $\theta^2|1 - \phi(\theta)|^{-1} \in L_1(-h, h)$. Using the method of Chung and Erdös [1], we obtain

$$(2.2)\qquad \begin{aligned} |\phi(\theta)|^2 &= 1 - 2\sum_{k=1}^{\infty} r_k \sin^2(k\theta/2),\\ r_k &= P(|X_1 - X_2| = k), \quad k = 0, 1, 2, \cdots, \end{aligned}$$

where X_1, X_2 are independent random variables, each with characteristic function $\phi(\theta)$. Since $|\sin x| \geqq |2x(\pi)^{-1}|$ when $|x| \leqq \pi/2$, we have, for every $\delta > 0$

$$2\sum_{k=1}^{\infty} r_k \sin^2(k\theta/2) \geqq 2\sum_{k=1}^{[\pi/\delta]} r_k \sin^2(k\theta/2) \geqq (2\theta^2/\pi^2)\sum_{k=1}^{[\pi/\delta]} k^2 r_k = \theta^2 A(\delta).$$

We choose δ sufficiently small so that $A(\delta) > 0$, and obtain

$$(2.3)\qquad \begin{aligned} |\phi(\theta)|^2 &\leqq 1 - \theta^2 A(\delta) \leqq e^{-\theta^2 A(\delta)} \quad \text{for} \quad |\theta| \leqq \delta,\\ A(\delta) &= (2/\pi^2)\sum_{k=1}^{[\pi/\delta]} k^2 r_k > 0. \end{aligned}$$

It follows that, for $|\theta| \leqq \delta$, except at $\theta = 0$,

$$\begin{aligned} \theta^2|1 - \phi(\theta)|^{-1} &\leqq \theta^2\sum_{n=0}^{\infty} |\phi(\theta)|^n \leqq 2\theta^2\sum_{n=0}^{\infty} |\phi(\theta)|^{2n}\\ &\leqq 2\theta^2[1 - e^{-\theta^2 A(\delta)}]^{-1} \in L_1(-\delta, \delta). \end{aligned}$$

Hence $|f(\theta)|\,|1 - \phi(\theta)|^{-1}$ is in $L_1(-\delta, \delta)$ and also in $L_1(-\pi, \pi)$.

We obtain equation (2.1) from the observation that $t|1 - z| \leqq |1 - tz|$ whenever $0 \leqq t \leqq 1$ and the complex number z is of absolute value $|z| \leqq 1$. Since $|\phi(\theta)| \leqq 1$, we have

$$\left|\frac{f(\theta)}{1 - t\phi(\theta)}\right| \leqq \frac{1}{t}\left|\frac{f(\theta)}{1 - \phi(\theta)}\right| \leqq 2\left|\frac{f(\theta)}{1 - \phi(\theta)}\right|, \qquad \frac{1}{2} \leqq t \leqq 1,$$

and the dominated convergence theorem completes the proof of Lemma 1.

LEMMA 2.
$$\lim_{k\to\infty} k^{-2}\left[\lim_{t\to1^-}\int_{-\pi}^{\pi} \frac{f_k(\theta)}{1 - t\phi(\theta)}\,d\theta\right] = 0$$
when $f_k(\theta) = 1 - \cos k\theta$, *and when* $f_k(\theta) = 1 + e^{-ik\theta} - e^{-2ik\theta} - e^{ik\theta}$.

Proof. Clearly both sequences of trigonometric polynomials $f_k(\theta)$ belong to the class $\mathcal{C}$, and for each sequence the proof is the same, starting with the observation that there is a positive constant c such that

$$|f_k(\theta)| \leqq ck^2\theta^2 \quad \text{for all} \quad |\theta| \leqq \pi, \quad k = 1, 2, 3, \cdots.$$

By Lemma 1,

$$\begin{aligned} b_k &= \left|\lim_{t\to1^-}\int_{-\pi}^{\pi} \frac{f_k(\theta)}{1 - t\phi(\theta)}\,d\theta\right|\\ &\leqq ck^2\int_{-\delta}^{\delta} \theta^2|1 - \phi(\theta)|^{-1}\,d\theta + 2\int_{\delta<|\theta|\leqq\pi} |1 - \phi(\theta)|^{-1}\,d\theta, \end{aligned}$$

for every $\delta > 0$. Hence, also for every $\delta > 0$,

$$0 \leqq \overline{\lim_{k\to\infty}}\, k^{-2} b_k \leqq c \int_{-\delta}^{\delta} \theta^2 \,|\, 1 - \phi(\theta) \,|^{-1}\, d\theta,$$

and for all sufficiently small $\delta > 0$ (for which $A(\delta)$ in (2.3) is positive)

$$0 \leqq \overline{\lim_{k\to\infty}}\, k^{-2} b_k \leqq 2c \int_{-\delta}^{\delta} \theta^2 [1 - e^{-\theta^2 A(\delta)}]^{-1}\, d\theta.$$

Suppose $0 < \delta_0 < \pi$, $A(\delta_0) = A_0$. Then $A(\delta) \geqq A_0$ for $0 < \delta < \delta_0$, and

$$0 \leqq \overline{\lim_{k\to\infty}}\, k^{-2} b_k \leqq 2c \int_{-\delta}^{\delta} \theta^2 [1 - e^{-\theta^2 A_0}]^{-1}\, d\theta, \qquad 0 < \delta < \delta_0,$$

and this integral tends to zero as $\delta \to 0$, so that $\lim_{k\to\infty} k^{-2} b_k = 0$.

LEMMA 3. *For* $f(\theta) = \sum a_k e^{ik\theta} \in \mathcal{C}$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(\theta)}{1 - \phi(\theta)}\, d\theta = \sum_{n=0}^{\infty} [\sum a_k P(S_n = -k)].$$

Proof.

$$\sum_{n=0}^{N} [\sum a_k P(S_n = -k)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)[1 + \phi(\theta) + \cdots + \phi^N(\theta)]\, d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(\theta)}{1 - \phi(\theta)}\, d\theta - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(\theta)\phi^{N+1}(\theta)}{1 - \phi(\theta)}\, d\theta.$$

Here we used Lemma 1. As $|\, \phi(\theta) \,| = 1$ at most at a finite number of points in the interval $[-\pi, \pi]$, the last integral tends to zero, and Lemma 3 is proved.

3. The imbedded Markov chain

Let S be the set $\{x_1, x_2, \cdots, x_n\}$ where the x_i are distinct integers. For $0 \leqq t < 1$ we define the n by n matrix $P_t(S) = (P_t(S)_{ij})$ by

$$\text{(3.1)} \quad \begin{aligned} P_t(S)_{ij} = \textstyle\sum_{n=1}^{\infty} t^n P(S_\nu \notin S \text{ for } \nu = 1, 2, \cdots, n-1;\\ S_n = x_j \mid S_0 = x_i), \qquad i, j = 1, 2, \cdots, n. \end{aligned}$$

Let T_m denote the time of the m^{th} visit to S, i.e., $T_m = k$ if exactly m of the random variables $S_1, S_2, \cdots, S_k$ have values in S and if in addition $S_k \in S$. It is then clear, by a simple renewal argument, that

$$[P_t^m(S)]_{ij} = \textstyle\sum_{n=1}^{\infty} t^n P(T_m = n; S_{T_m} = x_j \mid S_0 = x_i), \qquad m \geqq 1,$$

and

$$\text{(3.2)} \qquad \delta_{ij} + \textstyle\sum_{m=1}^{\infty} [P_t^m(S)]_{ij} = \sum_{n=0}^{\infty} t^n P(S_n = x_j \mid S_0 = x_i).$$

Since the row sums of $P_t(S)$ are all positive and less than one when $0 \leqq t < 1$, $I - P_t(S)$ has an inverse. It follows from equation (3.2) that

$$\text{(3.3)} \qquad I - P_t(S) = [\pi(t)]^{-1}, \qquad 0 \leqq t < 1,$$

where $\pi(t)$ is the n by n matrix whose elements are

$$\pi(t)_{pq} = \sum_{n=0}^{\infty} t^n P(S_n = x_q \mid S_0 = x_p) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-i\theta(x_q - x_p)}}{1 - t\phi(\theta)}\, d\theta,\ 0 \leqq t < 1, \tag{3.4}$$
$$p, q = 1, 2, \cdots, n.$$

Equation (3.3) is a special case of a result of P. Frank [3; Theorem VI].

At this point assume that the set $S = \{x_1, x_2\}$ with $x_2 = x_1 + k$, $k \neq 0$. Letting

$$v_k(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - e^{-ik\theta}}{1 - t\phi(\theta)}\, d\theta, \qquad s(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d\theta}{1 - t\phi(\theta)}, \quad 0 \leqq t < 1, \tag{3.5}$$

and performing the matrix inversion in (3.3), one obtains, for $0 \leqq t < 1$,

$$\begin{aligned} I - P_t(S) &= \left[v_k(t) + v_{-k}(t) - \frac{v_k(t)v_{-k}(t)}{s(t)} \right]^{-1} \\ &\qquad \cdot \begin{bmatrix} 1 & -1 + v_{-k}(t)/s(t) \\ -1 + v_k(t)/s(t) & 1 \end{bmatrix}. \end{aligned} \tag{3.6}$$

It is clear that $P_t(S) \to P(S)$ as $t \to 1^-$, with $P(S)$ as defined in equation (1.3). Hence the right-hand side in (3.6) has a limit as $t \to 1^-$. In addition the diagonal elements of $I - P(S)$ are equal, and $P(S)$ is a stochastic matrix, in view of (1.1). Therefore $P(S)$ has the form asserted in equation (1.4). Now

$$p_k = \lim_{t \to 1^-} \left[v_k(t) + v_{-k}(t) - \frac{v_k(t)v_{-k}(t)}{s(t)} \right]^{-1}.$$

Since $p_k = P(S)_{12}$ and the process is recurrent by (1.1), $p_k > 0$. Therefore

$$p_k^{-1} = \lim_{t \to 1^-} \left[v_k(t) + v_{-k}(t) - \frac{(v_k(t) + v_{-k}(t))^2}{4s(t)} + \frac{(v_k(t) - v_{-k}(t))^2}{4s(t)} \right] < \infty. \tag{3.7}$$

By Lemma 1

$$\lim_{t \to 1^-} [v_k(t) + v_{-k}(t)] = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1 - \cos k\theta}{1 - \phi(\theta)}\, d\theta.$$

As $s(t) \to +\infty$ when $t \to 1^-$, (3.7) becomes

$$p_k^{-1} = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1 - \cos k\theta}{1 - \phi(\theta)}\, d\theta + \lim_{t \to 1^-} \frac{(v_k(t) - v_{-k}(t))^2}{4s(t)} < \infty. \tag{3.8}$$

Note that the proof of (1.4) and (1.5) is finished now in the symmetric case, i.e., the case when $P(X_i = k) = P(X_i = -k)$, or $\phi(\theta) \equiv \phi(-\theta)$. For in this case $v_k(t) \equiv v_{-k}(t)$, so that (3.8) becomes (1.5). In the general case we begin by proving that

$$\lim_{t \to 1^-} v_1(t)[s(t)]^{-1/2} = V$$

exists and is finite, and that

$$\lim_{t \to 1^-} v_k(t)[s(t)]^{-1/2} = kV, \qquad k = \pm 1, \pm 2, \cdots . \tag{3.9}$$

Let $f(t) = [v_1(t) - v_{-1}(t)]\,[s(t)]^{-1/2}$. Equation (3.8) shows that $f^2(t)$ has a finite limit as $t \to 1^-$. As $f(t)$ is continuous for $0 \leqq t < 1$, $f(t)$ must also have a limit. By Lemma 1, $v_1(t) + v_{-1}(t)$ has a finite limit as $t \to 1^-$. Hence

$$\lim_{t \to 1^-} v_1(t)[s(t)]^{-1/2} = \lim_{t \to 1^-} f(t)/2 < \infty,$$

and we call this limit V. Again by Lemma 1, we know that $v_k(t) - kv_1(t)$ has a finite limit as $t \to 1^-$, and this yields equation (3.9).

Now equation (3.8) becomes

$$p_k^{-1} = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1 - \cos k\theta}{1 - \phi(\theta)}\, d\theta + k^2 V^2. \tag{3.10}$$

We want to prove equation (1.5), which follows from (3.10) and from Lemma 3 if $V = 0$. Unfortunately the proof that $V = 0$ takes an indirect route.[3]

We return to equations (3.3) and (3.4) and let T denote the set $\{x_1, x_2, x_3\}$ of three distinct integers. If $D(t)$ denotes the determinant of $\pi(t)$, $I - P_t(T)$ is of course $D(t)^{-1}$ times the matrix of cofactors of the transpose of $\pi(t)$, when $0 \leqq t < 1$. Since the stochastic process S_n has transition probabilities invariant under translation of the state space, it suffices to calculate

$$1 - P_t(T)_{11} = [D(t)]^{-1} s(t) \left[v_{x_2 - x_3}(t) + v_{x_3 - x_2}(t) - \frac{v_{x_2 - x_3}(t) v_{x_3 - x_2}(t)}{s(t)} \right],$$

$$-P_t(T)_{12} = [D(t)]^{-1} s(t) \left[v_{x_2 - x_1}(t) - v_{x_3 - x_1}(t) - v_{x_2 - x_3}(t) - \frac{v_{x_2 - x_3}(t) v_{x_3 - x_1}(t)}{s(t)} \right].$$

By equation (3.9), if we let $t \to 1^-$ and denote $\Delta = \lim_{t \to 1^-} D(t)^{-1} s(t)$,

$$1 - P(T)_{11} = \Delta \left[\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1 - \cos |x_2 - x_3|\,\theta}{1 - \phi(\theta)}\, d\theta + V^2 |x_2 - x_3|^2 \right], \tag{3.11}$$

$$\begin{aligned} P(T)_{12} = \Delta \Bigg[& \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 + e^{i\theta(x_1 - x_2)} - e^{i\theta(x_1 - x_3)} - e^{i\theta(x_3 - x_2)}}{1 - \phi(\theta)}\, d\theta \\ & + V^2 (x_3 - x_2)(x_3 - x_1) \Bigg]. \end{aligned} \tag{3.12}$$

(3.12) came from the fact that just like $v_{x_2 - x_3}(t) + v_{x_3 - x_2}(t)$,

$$v_{x_2 - x_1}(t) - v_{x_3 - x_1}(t) - v_{x_2 - x_3}(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{g(\theta)}{1 - t\phi(\theta)}\, d\theta$$

[3] This is unnecessary, as the referee has produced a simple proof that $V = 0$, based on the definition of V in the equation preceding (3.9) as the ratio of the integrals defining $v_1(t)$ and $s(t)$ and careful use of Schwarz's inequality. The present proof is retained as it will quickly lead to Theorem 2.

with $g(\theta) \in \mathfrak{C}$, so that Lemma 1 applies. Clearly $\Delta \geqq 0$ and finite. In fact $\Delta > 0$, for if $\Delta = 0$, $P(T)_{11} = 1$ which is impossible in a recurrent process.

To show that $V = 0$, consider the probability that the process, with $S_0 = x_1$ will assume the value x_2 before it assumes x_3, i.e.,

$$(3.13)\qquad {}_{x_3}f_{x_1,x_2} = \sum_{n=1}^{\infty} P(S_\gamma \notin \{x_2, x_3\} \text{ for } \gamma = 1, 2, \cdots, n-1,\ S_n = x_2 \mid S_0 = x_1).$$

One obtains

$$(3.14)\qquad {}_{x_3}f_{x_1,x_2} = P(T)_{12}[1 + P(T)_{11} + (P(T)_{11})^2 + \cdots] = P(T)_{12}/(1 - P(T)_{11}).$$

Specializing to $x_1 = 0$, $x_2 = k$, $x_3 = 2k$, $k > 0$,

$${}_{2k}f_{0,k} = \frac{(2\pi k^2)^{-1} \int_{-\pi}^{\pi} \dfrac{1 + e^{-ik\theta} - e^{-2ik\theta} - e^{ik\theta}}{1 - \phi(\theta)}\, d\theta + 2V^2}{(\pi k^2)^{-1} \int_{-\pi}^{\pi} \dfrac{1 - \cos k\theta}{1 - \phi(\theta)}\, d\theta + V^2}.$$

Lemma 2 implies that

$$\lim_{k\to\infty} {}_{2k}f_{0,k} = 2 \quad \text{if} \quad V \neq 0.$$

Since this is impossible, $V = 0$. That completes the proof of

THEOREM 1. *Equations* (1.4) *and* (1.5) *hold.*

Whereas the transition probabilities of the imbedded Markov chain on a set of more than two states are complicated (because the expression for Δ in in (3.11) and (3.12) is complicated) we saw that ${}_{x_3}f_{x_1,x_2}$ is independent of Δ. Another interesting quantity of this type is

$$(3.15)\qquad {}_{x_3}E_{x_1,x_2} = \sum_{n=1}^{\infty} P(S_\nu \neq x_3 \text{ for } \nu = 1, 2, \cdots, n-1;\ S_n = x_2 \mid S_0 = x_1).$$

When x_1, x_2, and x_3 are distinct, ${}_{x_3}E_{x_1,x_2}$ is the expected number of visits to x_2 before the first visit to x_3 when the process starts at $S_0 = x_1$. But we shall consider ${}_{x_3}E_{x_1,x_2}$ for arbitrary values of x_1, x_2, x_3. The result is

THEOREM 2. *When x_1, x_2, x_3 are distinct integers,*

$${}_{x_3}f_{x_1,x_2} = \frac{\sum_{n=0}^{\infty} [P(S_n=0)+P(S_n=x_2-x_1)-P(S_n=x_3-x_1)-P(S_n=x_2-x_3)]}{\sum_{n=0}^{\infty} [2P(S_n=0)-P(S_n=x_2-x_3)-P(S_n=x_3-x_2)]}.$$

When x_1, x_2, x_3 are arbitrary integers

$$(3.16)\qquad {}_{x_3}E_{x_1,x_2} = 1+\sum_{n=1}^{\infty} [P(S_n=0)+P(S_n=x_2-x_1) - P(S_n=x_3-x_1)-P(S_n=x_2-x_3)].$$

The proof of the first part follows from the identities (3.11) and (3.12) (with $V = 0$) applied to (3.14). Lemma 3 then converts the ratio of integrals into the ratio of the two series in Theorem 2.

When $x_1 = x_2 = x_3$, ${}_{x_3}E_{x_1,x_2} = 1$ and this agrees with (3.16). When $x_1 = x_2 \neq x_3$, we have

$$ {}_{x_3}E_{x_1,x_2} = {}_{x_3}E_{x_2,x_2} = {}_{x_3-x_2}E_{0,0} = {}_kE_{0,0}, \qquad k = x_3 - x_2 \neq 0. $$

Let S be the ordered set $\{0, k\}$, and $P(S)$ the corresponding two by two transition matrix. Then

$$ \begin{aligned} {}_{x_3}E_{x_2,x_2} &= \textstyle\sum_{n=1}^{\infty} n[P(S)_{11}]^n P(S)_{12} = [P(S)_{12}]^{-1} - 1 \\ &= -1 + \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1-\cos k\theta}{1-\phi(\theta)}\,d\theta \qquad (3.17) \\ &= -1 + \textstyle\sum_{n=0}^{\infty} [2P(S_n = 0) - P(S_n = k) - P(S_n = -k)] \end{aligned} $$

and this agrees with (3.16). (Lemma 3 was used to go from the integral in (3.17) to the series following it.)

When $x_1 \neq x_2 = x_3$, ${}_{x_3}E_{x_1,x_2} = 1$, and this agrees with (3.16).

When $x_1 = x_3 \neq x_2$, we omit a direct proof as (3.16) is then the theorem of Derman [2]: The expected number of visits to x_2 between successive returns to x_1 is the ratio of the invariant measures of the points x_2 and x_1. The (unique) invariant measure of the process S_n is constant. Hence ${}_{x_3}E_{x_1,x_2} = 1$ which agrees with (3.16).

Now only the case of distinct x_1, x_2, x_3 remains.

$$ {}_{x_3}E_{x_1,x_2} = {}_{x_3}f_{x_1,x_2}[1 + {}_{x_3}E_{x_2,x_2}]. $$

If S and T are the ordered sets $\{x_2, x_3\}$ and $\{x_1, x_2, x_3\}$, equations (3.14) and (3.17) give

$$ {}_{x_3}E_{x_1,x_2} = \frac{P(T)_{12}}{1 - P(T)_{11}} \cdot \frac{1}{P(S)_{12}}. $$

By Theorem 1 and equation (3.11) (with $V = 0$)

$$ [1 - P(T)_{11}]P(S)_{12} = \Delta, $$

and by equation (3.12) (with $V = 0$)

$$ {}_{x_3}E_{x_1,x_2} = \Delta^{-1}P(T)_{12} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 + e^{i\theta(x_1-x_2)} - e^{i\theta(x_1-x_3)} - e^{i\theta(x_3-x_2)}}{1-\phi(\theta)}\,d\theta, $$

which is equivalent to equation (3.16) by Lemma 3.

4. Limit theorems

From now on it is assumed that the characteristic function $\phi(\theta)$ satisfies equation (1.2) as well as the condition

$$ 0 < \lim_{\theta\to 0} \frac{1-\phi(\theta)}{|\theta|^{\alpha}} = Q < \infty \quad \text{for some } \alpha, \qquad 1 \leqq \alpha \leqq 2. \tag{4.1} $$

It is easily verified that (4.1), with $1 \leqq \alpha \leqq 2$, and only with $1 \leqq \alpha \leqq 2$, is compatible with (1.2). It is also clear that (4.1) holds with $\alpha = 2$ if and only if

$$E(X_1) = 0, \qquad 0 < E(X_1^2) = \sigma^2 < \infty.$$

In this case $Q = \frac{1}{2}\sigma^2$.

The asymptotic behavior of $p_n = P(S)_{12}$ when $S = \{0, n\}$ or, more generally, when $S = \{x_1, x_2\}$ with $|x_1 - x_2| = n$, is described by

THEOREM 3. *If* (4.1) *holds with*

$$1 < \alpha \leqq 2, \qquad \lim_{n\to\infty} n^{1-\alpha} p_n^{-1} = \frac{2}{\pi Q}\int_0^\infty \frac{1-\cos t}{t^\alpha}\,dt;$$

$$\textit{if } \alpha = 1, \qquad \lim_{n\to\infty} (\log n)^{-1} p_n^{-1} = \frac{2}{\pi Q}.$$

Remarks. When $\alpha = 2$, the integral in Theorem 3 has the value $\pi/2$, and one obtains, for arbitrary $\phi(\theta)$ satisfying (1.2), but not necessarily (4.1),

$$\lim\nolimits_{n\to\infty} np_n = \sigma^2/2 \leqq \infty. \tag{4.2}$$

One has the identity $np_n \equiv \sigma^2/2$ if and only if $\phi(\theta) = 1 - p + p\cos\theta$, $0 < p \leqq 1$. One can obtain more than (4.2) by assuming more about $\phi(\theta)$. For instance if $E \mid X_1^3 \mid < \infty$, one can show that

$$\lim_{n\to\infty}\left[\frac{1}{p_{n+1}} - \frac{1}{p_n}\right] = \frac{2}{\sigma^2}. \tag{4.3}$$

The proof of (4.3) depends on the identity

$$\frac{1}{p_{n+1}} - \frac{1}{p_n} = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}\,\mathrm{Re}\left[\frac{1-\cos\theta}{1-\phi(\theta)}\right]d\theta,$$

and on the properties of the Dirichlet kernel $\sin(n+\frac{1}{2})\theta\,(\sin\frac{1}{2}\theta)^{-1}$, which insure that (4.3) holds if $(1-\cos\theta)(1-\phi(\theta))^{-1}$ is sufficiently smooth at $\theta = 0$. The assumption that $E \mid X_1^3 \mid < \infty$ suffices to yield sufficient smoothness, whereas $\sigma^2 < \infty$ alone does not suffice.

To prove Theorem 3 let

$$\psi(\theta) = Q\cdot\mathrm{Re}\left[\frac{|\theta|^\alpha}{1-\phi(\theta)}\right], \quad \theta \neq 0, \qquad \psi(0) = 1.$$

By Theorem 1

$$Qp_n^{-1} = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1-\cos n\theta}{|\theta|^\alpha}\,\psi(\theta)\,d\theta.$$

Letting $\max_{|\theta|\leqq\pi}\psi(\theta) = M$, and given $\varepsilon > 0$, choosing $\delta > 0$ so that $|\psi(\theta) - 1| < \varepsilon$ for $|\theta| \leqq \delta$, we have

$$Qp_n^{-1} \leqq \frac{2}{\pi}(1+\varepsilon)\int_0^\delta \frac{1-\cos n\theta}{\theta^\alpha}\,d\theta + \frac{2M}{\pi}\int_\delta^\pi \frac{1-\cos n\theta}{\theta^\alpha}\,d\theta,$$

and a similar underestimate. Hence, for $\alpha > 1$

$$n^{1-\alpha} Qp_n^{-1} \leqq \frac{2}{\pi} (1 + \varepsilon) \int_0^{n\delta} \frac{1 - \cos t}{t^\alpha} dt + \frac{2M}{\pi} \int_{n\delta}^{n\pi} \frac{1 - \cos t}{t^\alpha} dt,$$

$$\frac{2}{\pi} (1 - \varepsilon) \int_0^\infty \frac{1 - \cos t}{t^\alpha} dt \leqq \varliminf_{n\to\infty} n^{1-\alpha} Qp_n^{-1} \leqq \varlimsup_{n\to\infty} n^{1-\alpha} Qp_n^{-1}$$

$$\leqq \frac{2}{\pi} (1 + \varepsilon) \int_0^\infty \frac{1 - \cos t}{t^\alpha} dt.$$

Since ε is arbitrary, Theorem 3 follows for $\alpha > 1$. For $\alpha = 1$ it follows from

$$(4.4) \qquad \lim_{n\to\infty} \frac{1}{\log n} \int_{n\delta}^{n\pi} \frac{1 - \cos t}{t} dt = 0, \qquad \lim_{n\to\infty} \frac{1}{\log n} \int_0^{n\delta} \frac{1 - \cos t}{t} dt = 1.$$

The parameter α plays an interesting role in the following limit theorem. Let $t_n = {}_{2n}f_{0,n}$. By (3.13) this is the probability that $S_\gamma = n$ before $S_\gamma = 2n$, if $S_0 = 0$.

THEOREM 4. *If* (4.1) *holds with some* $1 \leqq \alpha \leqq 2$,

$$\lim_{n\to\infty} t_n = 2^{\alpha-2}.$$

The proof consists of writing t_n as the ratio of two integrals, depending on n by use of Theorem 2 and Lemma 3. The asymptotic behavior of the denominator follows from Theorem 3, and that of the numerator by an analysis which imitates the proof of Theorem 3 and is therefore omitted.

That all the processes satisfying (4.1) possess a certain symmetry follows from the asymptotic behavior of $s_n = {}_{-n}f_{0,n}$, the probability that $S_\gamma = n$ before $S_\gamma = -n$, when $S_0 = 0$.

THEOREM 5. *If* (4.1) *holds with some* $1 \leqq \alpha \leqq 2$,

$$\lim_{n\to\infty} s_n = \tfrac{1}{2}.$$

From Theorem 2 it follows that

$$s_n - \tfrac{1}{2} = \tfrac{1}{2} \int_{-\pi}^{\pi} \frac{\sin 2n\theta - 2\sin n\theta}{1 - \phi(\theta)} d\theta \Big/ \int_{-\pi}^{\pi} \frac{1 - \cos 2n\theta}{1 - \phi(\theta)} d\theta.$$

The asymptotic behavior of the denominator is given by Theorem 3. Therefore it must be shown that, as $n \to \infty$

$$\text{for } \alpha > 1, \quad I_n(\alpha) = n^{1-\alpha} \int_{-\pi}^{\pi} \sin n\theta(1 - \cos n\theta) \cdot \operatorname{Im}\left[\frac{1}{1 - \phi(\theta)}\right] d\theta \to 0,$$

for $\alpha = 1$, the same with $n^{1-\alpha}$ replaced by $(\log n)^{-1}$.

Choosing δ such that $\operatorname{Im}[(1 - \phi(\theta))^{-1}] \leqq \varepsilon \mid \theta \mid^{-\alpha}$ for $\mid \theta \mid \leqq \delta$, we have

$$I_n(\alpha) \leqq 2\,\varepsilon \int_0^{\delta n} \frac{\mid \sin t \mid (1 - \cos t)}{t^\alpha} dt + 2M \int_{\delta n}^{\pi n} \frac{\mid \sin t \mid (1 - \cos t)}{t^\alpha} dt$$

when $\alpha > 1$, where $M = \max_{\delta \leqq \theta \leqq \pi} \operatorname{Im} [| \theta |^{\alpha}(1 - \phi(\theta))^{-1}]$. As there is a similar underestimate and ε is arbitrary, one finds that $I_n(\alpha) \to 0$. When $\alpha = 1$, the proof goes through employing equation (4.4).

Remark. This result can be extended to the probabilities ${}_0f_{[nt],n}$, where $[nt]$ is the greatest integer in nt, $0 < t < 1$. One finds

$$\lim_{n \to \infty} {}_0f_{[nt],n} = t \quad \text{if and only if} \quad \alpha = 2. \tag{4.5}$$

This is the same result as one obtains for the gambler's ruin problem, i.e., for the probability ${}_0g_{[nt],n}$ that $S_\nu \in [n, \infty)$ before $S_\nu \in (-\infty, 0]$ when $\alpha = 2$, and $S_0 = [nt]$. The solution of the gambler's ruin problem is not known when $1 < \alpha < 2$, whereas it is easy to calculate the limit in (4.5) also when $1 \leqq \alpha < 2$. It seems certain that[4]

$$\lim_{n \to \infty} {}_0f_{[nt],n} \neq \lim_{n \to \infty} {}_0g_{[nt],n} \qquad \text{when } 1 \leqq \alpha < 2.$$

These questions are related to the work in [4] and [5], and so are the following occupation time problems. Let $S_0 = 0$, and let the random variables

N_n = the number of visits to 0 before the first visit to n,

$N_{n,n}$ = the number of visits to 0 before the first visit to the set $\{-n, n\}$.

Equation (4.1) is assumed to hold and the discussion of $N_{n,n}$ is only valid under the additional assumption that $\phi(\theta) = \phi(-\theta)$. Both N_n and $N_{n,n}$ are defined so that $S_0 = 0$ counts as the first visit to 0. Equation (3.17) shows that

$$E[N_n] = p_n^{-1}. \tag{4.6}$$

A calculation, similar to (3.17) shows that, if $T = \{0, n, -n\}$,

$$E[N_{n,n}] = [1 - P(T)_{11}]^{-1}.$$

If $\phi(\theta) = \phi(-\theta)$, so that $v_k(t) = v_{-k}(t)$, equation (3.11) gives

$$E[N_{n,n}] = \Delta^{-1} p_{2n},$$

where

$$\Delta^{-1} = \lim_{t \to 1^-} v_{2n}(t)[4v_n(t) - v_{2n}(t)] = (1/2p_{2n})(2/p_n - 1/2p_{2n}),$$

$$E[N_{n,n}] = 1/p_n - 1/4p_{2n}. \tag{4.7}$$

Theorem 3 yields

$$\lim_{n \to \infty} E[N_n/n^{\alpha-1}] = \frac{2}{\pi Q} \int_0^\infty \frac{1 - \cos t}{t^\alpha} dt, \qquad 1 < \alpha \leqq 2,$$

$$\lim_{n \to \infty} E[N_{n,n}/n^{\alpha-1}] = (1 - 2^{\alpha-3}) \frac{2}{\pi Q} \int_0^\infty \frac{1 - \cos t}{t^\alpha} dt, \qquad 1 < \alpha \leqq 2, \tag{4.8}$$

$$\lim_{n \to \infty} E[N_n/\log n] = 2/\pi Q, \qquad \alpha = 1,$$

[4] This is now known, as the gambler's ruin problem has been solved by R. Getoor for the symmetric stable processes of index $1 < \alpha < 2$, and by H. Kesten who found the limit of ${}_0g_{[nt],n}$ as $n \to \infty$ for $1 \leqq \alpha < 2$.

$$\lim_{n\to\infty} E[N_{n,n}/\log n] = 3/2\pi Q, \qquad \alpha = 1.$$

By a trivial argument the above limit theorems are equivalent to the following theorem (N_n and $N_{n,n}$ have geometric distributions, which obviously yield exponential limit distributions).

THEOREM 6. *If* (4.1) *holds for some* $1 \leqq \alpha \leqq 2$, *and if* $\phi(\theta) = \phi(-\theta)$ *in the case of the random variable* $N_{n,n}$, *then* N_n *and* $N_{n,n}$, *normalized as in* (4.8), *have an exponential limit distribution as* $n \to \infty$. *The expected value of the limit distribution is the appropriate limit in* (4.8).

Remark. Let N_n^* be the number of visits to 0 before the first visit to $[n, \infty)$, and let $N_{n,n}^*$ be the number of visits to 0 before the first visit to $(-\infty, -n] \cup [n, \infty)$, when $S_0 = 0$. The analogue of equation (4.8) and of Theorem 6 for N_n^* and $N_{n,n}^*$ are investigated for $\alpha = 2$ in [5] and for $1 \leqq \alpha \leqq 2$ in [4]. It turns out that

$$\lim_{n\to\infty} E[N_n/n^{\alpha-1}] = \lim_{n\to\infty} E[N_n^*/n^{\alpha-1}],$$

as well as

$$\lim_{n\to\infty} E[N_{n,n}/n^{\alpha-1}] = \lim_{n\to\infty} E[N_{n,n}^*/n^{\alpha-1}],$$

if and only if $\alpha = 2$. H. Kesten (in a letter) described this phenomenon as follows: *Single points have the same absorbing power as semi-infinite intervals if the variance is finite.*

Added in proof. Recent work by J. G. Kemeny and J. L. Snell (*Potentials for denumerable Markov chains*, to appear in Journal of Mathematical Analysis and Applications) has provided methods which not only give Theorem 1 much more easily and naturally, but also generalize it and many results in [6] to a large class of recurrent Markov chains. For example, their work shows easily that the partial sums of the series $\sum_{n=0}^{\infty} [P(S_n = 0) - P(S_n = k)]$ are bounded. By a more delicate argument, based on Kemeny's and Snell's potential theory, one can show that the above series always converges when (1.1) holds.

REFERENCES

1. K. L. CHUNG AND P. ERDÖS, *Probability limit theorems assuming only the first moment I*, Mem. Amer. Math. Soc., no. 6, 1951.
2. C. DERMAN, *A solution to a set of fundamental equations in Markov chains*, Proc. Amer. Math. Soc., vol. 5 (1954), pp. 332–334.
3. P. FRANK, *Taboo generating functions and other topics in Markov chains*, Ph.D. Dissertation, Columbia University, ONR Technical report No. 2(N), NONR 266(55), 1959.
4. H. KESTEN, *On a theorem of Spitzer and Stone and random walks with absorbing barriers*, Illinois J. Math., vol. 5 (1961), pp. 246–266.
5. F. SPITZER AND C. STONE, *A class of Toeplitz forms and their application to probability theory*, Illinois J. Math., vol. 4 (1960), pp. 253–277.
6. F. SPITZER, *Recurrent random walk and logarithmic potential*, Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, to appear.

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