COMPLETIONS OF GROUPS OF CLASS 2[1]

BY
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1. Introduction

Let G be a group with center Z and commutator subgroup C, and suppose that $G \supseteq Z \supseteq C$. If H is a complete ($nH = H$ for all $n > 0$) nilpotent group of class 2 that contains G and no proper complete subgroup of H contains G, then we say that H is a *completion* of G. We prove (Theorem 3.2) that there exists a completion of G if and only if $\{g \in G : ng \in C \text{ for some } n > 0\} \subseteq Z$. If C is torsion free, then there exists a completion of K of G such that the commutator subgroup of K is torsion free and the center of K is the abelian completion of Z. Moreover, any other such completion of G is isomorphic to K (Theorem 3.3). These results generalize the corresponding results of Baer for abelian groups, and also Vinogradov's result for torsion free G.

The author originally had a long transfinite proof of Theorem 2.1, and all other results were restricted by the hypothesis that G contains no elements of order 2. This hypothesis on G has been removed, and the author wishes to thank Reinhold Baer for suggesting the elegant proof of Theorem 2.1.

Notation. N and Δ will always denote additive abelian groups with elements $0, a, b, \cdots$ and $\theta, \alpha, \beta, \gamma, \cdots$ respectively. F will denote the group of all *factor mappings* of $\Delta \times \Delta$ into N. Thus $f \in F$ if and only if $f: \Delta \times \Delta \to N$ and for all $\alpha, \beta \in \Delta$

$$f(\alpha, \theta) = f(\theta, \beta) = 0,$$

and

$$f(\alpha, \beta + \gamma) + f(\beta, \gamma) = f(\alpha + \beta, \gamma) + f(\alpha, \beta).$$

Each $f \in F$ determines a central extension G of N by Δ, where $G = \Delta \times N$ and, for all (α, a) and (β, b) in G,

$$(\alpha, a) + (\beta, b) = (\alpha + \beta, f(\alpha, \beta) + a + b).$$

The mappings of $f, g \in F$ are *equivalent* if there exists $t: \Delta \to N$ such that for all $\alpha, \beta \in \Delta$,

$$f(\alpha, \beta) = g(\alpha, \beta) - t(\alpha + \beta) + t(\alpha) + t(\beta).$$

In this case the mapping $(\alpha, a) \in G(\Delta, N, f)$ upon $(\alpha, a + t(\alpha))$ in $G(\Delta, N, g)$ is an isomorphism.

G will always denote an additive group with commutator group C and center Z, and we shall always assume that $G \supseteq Z \supseteq C$. Suppose that N is a subgroup of G between Z and C. Let $\Delta = G/N$, and let π be the natural homo-

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morphism of G onto Δ. Choose a mapping r of Δ into G such that $r(\theta) = 0$ and $r(\alpha)\pi = \alpha$ for all $\alpha \in \Delta$. For $\alpha, \beta \in \Delta$ define that

$$f(\alpha, \beta) = -r(\alpha + \beta) + r(\alpha) + r(\beta).$$

Then $f \in F$, and the mapping of $r(\alpha) + a$ upon (α, a), where $a \in N$, is an isomorphism of G onto $G(\Delta, N, f)$. We shall frequently make use of this representation of G.

2. The relationship between bilinear mapping and the commutator function

Let $G \supseteq Z \supseteq N \supseteq C$, $\Delta = G/N$, and let F be the group of all factor mappings of $\Delta \times \Delta$ into N. If $a, b \in G$, then the commutator function $[a, b] = -a - b + a + b$ is bilinear and skew-symmetric. Conversely, if the commutator function of a group H is bilinear, then H is nilpotent and of class 2 (see [4], I, p. 100 for proofs). For $A = a + N$ and $B = b + N$ in G/N, define that $g(A, B) = [a, b]$. Then since $N \subseteq Z$, g is a bilinear and skew-symmetric mapping of $\Delta \times \Delta$ into N. Let P (Q) be the set of all symmetric (skew-symmetric) mappings in F. Clearly P and Q are subgroups of F.

LEMMA 2.1. *If N contains no elements of order 2, then Q is the set of all bilinear skew-symmetric mappings of $\Delta \times \Delta$ into N.*

Proof. It is easy to verify that each bilinear mapping of $\Delta \times \Delta$ into N is a factor mapping. Thus it suffices to show that each f in Q is also bilinear. Let $G(\Delta, N, f)$ be the central extension of N by Δ that is determined by f. Then

$$\begin{aligned}[(\alpha, 0), (\beta, 0)] &= -((\beta, 0) + (\alpha, 0)) + (\alpha, 0) + (\beta, 0) \\ &= -((\beta + \alpha, 0) + (\theta, f(\beta, \alpha))) + (\alpha + \beta, 0) + (\theta, f(\alpha, \beta)) \\ &= (\theta, -f(\beta, \alpha) + f(\alpha, \beta)) = (\theta, 2f(\alpha, \beta)).\end{aligned}$$

Therefore $2f$ is bilinear, and hence

$$\begin{aligned}2f(\alpha, \beta + \gamma) &= 2f(\alpha, \beta) + 2f(\alpha, \gamma) \\ &= 2(f(\alpha, \beta) + f(\alpha, \gamma)).\end{aligned}$$

Thus, because N contains no elements of order 2,

$$f(\alpha, \beta + \gamma) = f(\alpha, \beta) + f(\alpha, \gamma).$$

We note the equivalence of the following properties of N. (a) $N = 2N$, and N contains no elements of order 2. (b) The mapping $a \rightarrow 2a$ is an automorphism of N. Suppose that N satisfies (a). For $f \in F$ and $\alpha, \beta \in \Delta$ define that

$$p(\alpha, \beta) = \tfrac{1}{2}(f(\alpha, \beta) + f(\beta, \alpha))$$

$$q(\alpha, \beta) = \tfrac{1}{2}(f(\alpha, \beta) - f(\beta, \alpha)).$$

It is easy to show that p and q belong to F. Clearly p is symmetric, q is skew-symmetric, and $f = p + q$. Also this representation of f as the sum of a symmetric and a skew-symmetric function is unique. Therefore $F = P \oplus Q$. All bilinear mappings of $\Delta \times \Delta$ into N belong to F, and each such f is equivalent to its skew-symmetric part ([3], Theorem 5.2).

THEOREM 2.1. *If the mapping $a \to 2a$ is an automorphism of N, then the following are equivalent*:

(i) *N is a direct summand of every abelian extension of N by Δ.*

(ii) *Each central extension of N by Δ is determined by a bilinear skew-symmetric factor mapping.*

Proof. Let $D = \{f \in F : f(\alpha, \beta) = -t(\alpha + \beta) + t(\alpha) + t(\beta)$ for all $\alpha, \beta \in \Delta$, where t is a mapping of Δ into $N\}$. Then F/D is the group of all nonequivalent central extensions of N by Δ. Clearly (i) is equivalent to $D = P$, and by Lemma 2.1, (ii) is satisfied if and only if each coset in F/D has a nonvoid intersection with Q. If $D = P$, then $F = P \oplus Q = D \oplus Q$, and hence Q is a set of representatives for F/D. Thus (i) implies (ii). Conversely if $D \neq P$, then P contains a nonzero $X \in F/D$, and hence $X \cap Q$ is the null set. Therefore (ii) implies (i).

Note that if N contains no elements of order 2, and if f and g are skew-symmetric mappings from the same coset in F/D, then $f - g$ is both symmetric and skew-symmetric, and hence $f = g$. Thus the bilinear skew-symmetric factor mappings mentioned in (ii) are uniquely determined by the particular central extensions of N by Δ. Hence if $a \to 2a$ is an automorphism of N and (i) is satisfied, then there exists a $1-1$ correspondence between Q and the set of all nonequivalent central extensions of N by Δ. If N is complete ($nN = N$ for all $n > 0$), then it is a direct summand of every abelian extension. Thus if N is complete and contains no elements of order 2, then (i) and (ii) are satisfied for any abelian group Δ.

Next suppose that Δ is a free abelian group, and let A be an abelian extension of N by Δ. There exists a homomorphism π of A onto Δ with kernel N. Let S be a free set of generators of Δ, and for each α in S let $r(\alpha)$ be an element in A such that $r(\alpha)\pi = \alpha$. There exists an extension of r to a homomorphism of Δ into A. Thus A splits over N, and hence N is a direct summand of A. Therefore if Δ is free abelian, then (i) and (ii) are satisfied for all abelian groups N such that $a \to 2a$ is an automorphism of N.

In Lemma 3.4 it is shown that if N is complete and Δ is torsion free, then (ii) is satisfied. Also if $2\Delta = \Delta$ and (ii) is satisfied, then (i) is satisfied. For let $f : \Delta \times \Delta \to N$ be a bilinear skew-symmetric factor mapping that determines an abelian extension of N by Δ; then $f(\alpha, \beta) = f(\beta, \alpha) = -f(\alpha, \beta)$, and hence $0 = 2f(\alpha, \beta) = f(2\alpha, \beta)$ for all α and β in N. Thus, since $2\Delta = \Delta$, $f(\alpha, \beta) \equiv 0$, and hence N is a direct summand of A.

3. Completions of nilpotent groups of class 2

Each abelian group A is contained in a complete abelian group, and each such extension of A contains at least one complete subgroup M that is minimal among those containing A. Between any two such minimal complete abelian groups containing A there exists an isomorphism that induces the identity on A (see [4], I for proofs). M is the *abelian completion* of A, and we shall denote it by A^*. As before let G be a central extension of N by Δ. A minimal complete nilpotent group of class 2 that contains G will be called a *completion* of G. Example I at the end of this paper shows that an abelian group may have a non-abelian completion.

LEMMA 3.1. *If S is a subgroup of G, then $d(S) = \{g \in G : ng \in S$ for some $n > 0\}$ is a subgroup of G that contains S. If G is complete, then so is $d(S)$, and if S is normal, then so is $d(S)$. We shall call $d(S)$ the d-hull of S in G.*

Proof. Clearly $d(S)$ is closed with respect to inverses. If $a, b \in d(S)$, then there exist positive integers m and n such that $ma, nb \in S$, and since the commutator function is bilinear, $mn[b, a] = [nb, ma] \in S \cap C \subseteq S \cap Z$. By a simple induction argument

$$u(a + b) = ua + ub + (u(u - 1)/2)[b, a]$$

for all positive integers u. In particular, for $u = 2mn$,

$$\begin{aligned} 2mn(a + b) &= 2mna + 2mnb + mn(2mn - 1)[b, a] \\ &= 2n(ma) + 2m(nb) + (2mn - 1)[nb, ma] \end{aligned}$$

which belongs to S. Thus $a + b \in d(S)$, and hence $d(S)$ is a subgroup of G.

If G is complete, and $ng = s$ for g in G and s in $d(S)$, then there exists $m > 0$ such that $mng = ms \in S$. Thus $g \in d(S)$ and it follows that $d(S)$ is complete. Suppose that S is normal and that $s \in d(S)$ and $g \in G$. There exists $n > 0$ such that $ns \in S$, and hence $n(-g + s + g) = -g + ns + g \in S$. Thus $-g + s + g \in d(S)$, and therefore $d(S)$ is normal.

COROLLARY I. *$T = \{g \in G : ng = 0$ for some $n > 0\}$ is a normal subgroup of G, called the torsion subgroup of G.*

COROLLARY II. *If H is a completion of G, then H is the d-hull of G in H.*

COROLLARY III. *If C is torsion free, then the d-hull $d(A)$ of any abelian subgroup A of G is abelian, and if $A \subseteq Z$, then $d(A) \subseteq Z$. In particular, $T \subseteq Z$, and G/Z is torsion free.*

Proof. If $ma + nb = nb + ma$ for a, b in G and positive integers m and n, then $a + b = b + a$. For $mn[a, b] = [ma, nb] = 0$, and thus $[a, b] = 0$. If $ng \in Z$ for $g \in G$ and $n > 0$, then $ng + a = a + ng$ for all $a \in G$. Thus $g + a = a + g$, and hence $g \in Z$. Therefore G/Z is torsion free, and the remainder of this corollary is now obvious.

LEMMA 3.2. *Let H be a complete nilpotent group of class 2.* (a) *The torsion subgroup of H belongs to the center $Z(H)$ of H.* (b) *The commutator subgroup $C(H)$ of H is complete.* (c) *If $G \subseteq H$, then the d-hull of C in G is contained in Z, where C in the commutator subgroup of G and Z is the center of G.*

Proof. (a) is a special case of a theorem by Černikov that the periodic part of a complete ZA-group is contained in the center (see [4], II, p. 234 for a proof). In particular, a finite nilpotent group of class 2 has a completion if and only if it is abelian. If $a, b \in H$, then for each $n > 0$ there exists an $a' \in H$ such that $na' = a$. Thus $n[a', b] = [na', b] = [a, b]$, and it follows that $C(H)$ is complete. Finally suppose that $a \in G \subseteq H$ and that $na \in C$ for some $n > 0$. $C \subseteq C(H) \subseteq Z(H)$, and $C(H)$ is complete. Thus $na = nc$ for some $c \in C(H)$, and hence $n(a - c) = 0$. Therefore $a - c$ and c belong to $Z(H)$, and hence $a \in Z(H) \cap G \subseteq Z$. Thus the d-hull of C is contained in Z. In Theorem 4.1 we show that $Z(H)$ is also complete.

A mapping g of $\Delta \times \Delta$ into N will be called a (*)-*mapping* if

(a) g is bilinear and skew-symmetric, and

(b) $g(\alpha, \alpha) = 0$ for all $\alpha \in \Delta$.

Note that if N contains no elements of order 2, then (a) implies (b). If G is a central extension of N by $\Delta = G/N$ and N contains no elements of order 2, then without loss of generality $G = G(\Delta, N, f) \subseteq H(\Delta, N^*, f)$, and N^* contains no elements of order 2. Thus by Theorem 2.1, H is determined by a (*)-mapping g of $\Delta \times \Delta$ into N^*. If g can be extended to a (*)-mapping k of $\Delta^* \times \Delta^*$ into N^*, then $K(\Delta^*, N^*, k)$ is a complete nilpotent group of class 2 that contains an isomorphic copy of G. This is the method that we use to obtain a completion of G. But we do not wish to impose any restrictions on N, and hence we cannot use Theorem 2.1.

LEMMA 3.3. *Suppose that Δ is torsion free, $N = N^*$ is complete, and g is a (*)-mapping of $\Delta \times \Delta$ into N.*

(a) *There exists an extension of g to a (*)-mapping g^* of $\Delta^* \times \Delta^*$ into N.*

(b) *If N is torsion free, then g^* is unique.*

Proof. (b) Suppose that h and k are bilinear mappings of $\Delta^* \times \Delta^*$ into N that induce g on $\Delta \times \Delta$. If $\alpha, \beta \in \Delta^*$, then there exists $n > 0$ such that $n\alpha, n\beta \in \Delta$, and hence $n^2h(\alpha, \beta) = h(n\alpha, n\beta) = g(n\alpha, n\beta) = k(n\alpha, n\beta) = n^2k(\alpha, \beta)$. Thus since N is torsion free, $h(\alpha, \beta) = k(\alpha, \beta)$.

(a) We may assume that $\Delta \neq \Delta^*$. Thus there exists a $\delta \in \Delta^*$, $\delta \notin \Delta$, such that $p\delta \in \Delta$ for some prime p and $n\delta \notin \Delta$ for $n = 1, 2, \cdots, p - 1$. Let Δ' be the subgroup of Δ^* that is generated by Δ and δ. It suffices to show that g can be extended to a (*)-mapping of $\Delta' \times \Delta'$ into N, for then an application of Zorn's lemma completes the proof. The elements α', β', γ' in Δ' have unique representations

$$\begin{aligned} \alpha' &= \alpha + m\delta, & 0 &\leqq m < p, & \alpha &\in \Delta, \\ \beta' &= \beta + n\delta, & 0 &\leqq n < p, & \beta &\in \Delta, \\ \gamma' &= \gamma + r\delta, & 0 &\leqq r < p, & \gamma &\in \Delta. \end{aligned}$$

For each $\alpha \in \Delta$ define $r(\alpha) = g(p\delta, \alpha)$. Then r is a homomorphism of Δ into N, and $r(p\delta) = g(p\delta, p\delta) = 0$. There exists an extension of r to a homomorphism r^* of Δ^* into N such that $r^*(\delta) = 0$. For the mapping of $\alpha' \in \Delta'$ upon $r(\alpha)$ is an extension of r to a homomorphism of Δ' into N, and since N is complete, this can be extended to a homomorphism of Δ^* into N. For each $\alpha \in \Delta^*$, define $\rho(\alpha) = (1/p)\alpha$. Since Δ^* is a rational vector space, ρ is an automorphism of Δ^*. Let $r' = r^*\rho$. Then r' is a homomorphism of Δ^* into N and $pr'(\alpha) = pr^*\rho(\alpha) = pr^*((1/p)\alpha) = r^*(\alpha)$ for all α in Δ^*. Now we extend g to a mapping g' of $\Delta' \times \Delta'$ into N by defining that for all $\alpha \in \Delta$ and all $\alpha', \beta' \in \Delta'$

$$g'(\delta, \alpha) = r'(\alpha) = -g'(\alpha, \delta),$$

and

$$g'(\alpha', \beta') = g'(\alpha + m\delta, \beta + n\delta) = g(\alpha, \beta) + mg'(\delta, \beta) + ng'(\alpha, \delta).$$

It follows that

$$\begin{aligned} g'(\beta', \alpha') &= g'(\beta + n\delta, \alpha + m\delta) \\ &= g(\beta, \alpha) + ng'(\delta, \alpha) + mg'(\beta, \delta) \\ &= -\, g(\alpha, \beta) - ng'(\alpha, \delta) - mg'(\delta, \beta) \\ &= -g'(\alpha', \beta'), \end{aligned}$$

and

$$\begin{aligned} g'(\alpha', \alpha') &= g'(\alpha + m\delta, \alpha + m\delta) \\ &= g(\alpha, \alpha) + mg'(\delta, \alpha) + mg'(\alpha, \delta) \\ &= 0 + mg'(\delta, \alpha) - mg'(\delta, \alpha) = 0. \end{aligned}$$

Thus it remains to be shown that g' is bilinear. For $\beta, \gamma \in \Delta$,

$$g'(\delta, \beta + \gamma) = r'(\beta + \gamma) = r'(\beta) + r'(\gamma) = g'(\delta, \beta) + g'(\delta, \gamma),$$

and for $\alpha', \beta', \gamma' \in \Delta'$,

$$g'(\alpha', \beta' + \gamma') = g'(\alpha + m\delta, \beta + \gamma + (n + r)\delta).$$

If $n + r < p$, then

$$\begin{aligned} g'(\alpha', \beta' + \gamma') &= g(\alpha, \beta + \gamma) + mg'(\delta, \beta + \gamma) + (n + r)g'(\alpha, \delta) \\ &= g(\alpha, \beta) + mg'(\delta, \beta) + ng'(\alpha, \delta) + g(\alpha, \gamma) \\ &\quad + mg'(\delta, \gamma) + rg'(\alpha, \delta) \\ &= g'(\alpha', \beta') + g'(\alpha', \gamma'). \end{aligned}$$

If $n + r = p + q$, then since

$$g'(\delta, p\delta) = r'(p\delta) = r^*\rho(p\delta) = r^*(\delta) = 0$$

and

$$g(\alpha, p\delta) = -g(p\delta, \alpha) = -r(\alpha) = -r^*(\alpha) = -pr'(\alpha) = pg'(\alpha, \delta),$$

we have

$$\begin{aligned} g'(\alpha', \beta' + \gamma') &= g'(\alpha + m\delta, \beta + \gamma + p\delta + q\delta) \\ &= g(\alpha, \beta + \gamma + p\delta) + mg'(\delta, \beta + \gamma + p\delta) + qg'(\alpha, \delta) \\ &= g(\alpha, \beta + \gamma) + mg'(\delta, \beta + \gamma) + mg'(\delta, p\delta) + g(\alpha, p\delta) \\ &\quad + qg'(\alpha, \delta) \\ &= g(\alpha, \beta + \gamma) + mg'(\delta, \beta + \gamma) + (p + q)g'(\alpha, \delta) \\ &= g'(\alpha', \beta') + g'(\alpha', \gamma'). \end{aligned}$$

By symmetry, $g'(\alpha' + \beta', \gamma') = g'(\alpha', \gamma') + g'(\beta', \gamma')$.

LEMMA 3.4. *Suppose that $G = G(\Delta, N, f)$, where Δ is torsion free and N is complete. Let C^* be a completion of C in N where $0 \times C$ is the commutator subgroup of G. Then there exists a $(*)$-mapping g of $\Delta \times \Delta$ into C^* such that g is equivalent to f, and $2g(\alpha, \beta) = f(\alpha, \beta) - f(\beta, \alpha)$ for all $\alpha, \beta \in \Delta$.*

Proof. For $\alpha, \beta \in \Delta$, let $h(\alpha, \beta) = f(\alpha, \beta) - f(\beta, \alpha)$. Since h is the commutator mapping of $\Delta = G/N$ into C, it follows that h is a $(*)$-mapping. By Lemma 3.3 there exists an extension of h to a $(*)$-mapping h^* of $\Delta^* \times \Delta^*$ into C^*. Define that $g(\alpha, \beta) = 2h^*(\alpha/2, \beta/2)$ for all α, β in Δ. By a simple computation it follows that g is a $(*)$-mapping of $\Delta \times \Delta$ into C^*.

$$\begin{aligned} 2h^*(\alpha/2, \beta/2) - 2h^*(\beta/2, \alpha/2) &= 4h^*(\alpha/2, \beta/2) \\ &= h^*(\alpha, \beta) = h(\alpha, \beta) = f(\alpha, \beta) - f(\beta, \alpha) \end{aligned}$$

for all $\alpha, \beta \in \Delta$. Thus

$$\begin{aligned} f(\alpha, \beta) - g(\alpha, \beta) &= f(\alpha, \beta) - 2h^*(\alpha/2, \beta/2) \\ &= f(\beta, \alpha) - 2h^*(\beta/2, \alpha/2) \\ &= f(\beta, \alpha) - g(\beta, \alpha). \end{aligned}$$

Therefore $f - g$ is symmetric, and hence determines an abelian extension of N by Δ. Since N is complete, it is a direct summand of any such extension. Thus $f(\alpha, \beta) - g(\alpha, \beta) = -t(\alpha + \beta) + t(\alpha) + t(\beta)$ for some $t: \Delta \to N$. Therefore g is equivalent to f.

THEOREM 3.1. *If $G \supseteq Z \supseteq N \supseteq C$ and $\Delta = G/N$ is torsion free, then there exists a completion K of G such that K is a central extension of N^* by Δ^* and $C(K) = C^*$.*

Proof. We may assume without loss of generality that

$$G = G(\Delta, N, f) \subseteq H(\Delta, N^*, f).$$

Let C^* be a completion of C in N^*. By Lemma 3.4, there exists a $(*)$-

mapping g of $\Delta \times \Delta$ into C^* such that g is equivalent to f. Thus there exists a mapping t of Δ into N such that for all α, β in Δ,

$$f(\alpha, \beta) = g(\alpha, \beta) - t(\alpha + \beta) + t(\alpha) + t(\beta).$$

The mapping σ of $(\alpha, a) \in H$ upon $(\alpha, a + t(\alpha))$ is an isomorphism of H onto $\tilde{H}(\Delta, N^*, g)$. In particular, $G\sigma = \tilde{G} = \{(\alpha, a + t(\alpha)) : a \in N \text{ and } \alpha \in \Delta\}$. By Lemma 3.3, there exists an extension of g to a $(*)$-mapping g^* of $\Delta^* \times \Delta^*$ into C^*.

$$\tilde{G} \subseteq \tilde{H}(\Delta, N^*, g) \subseteq K(\Delta^*, N^*, g^*).$$

If $(\alpha, a) \in K$, then it follows by induction that $n(\alpha, a) = (n\alpha, na)$ for all $n > 0$. Thus, since Δ^* and N^* are complete, K is a complete central extension of N^* by Δ^*.

Consider (α, a) and (β, b) in G. $[(\alpha, a), (\beta, b)] = (\theta, f(\alpha, \beta) - f(\beta, \alpha))$ and $[(\alpha, a), (\beta, b)]\sigma = [(\alpha, a + t(\alpha)), (\beta, b + t(\beta))] = (\theta, 2g(\alpha, \beta))$. By Lemma 3.4, $2g(\alpha, \beta) = f(\alpha, \beta) - f(\beta, \alpha)$, and hence it follows that $\theta \times C \subseteq C(K)$. Also if (α, a) and (β, b) belong to K, then

$$[(\alpha, a), (\beta, b)] = (\theta, 2g^*(\alpha, \beta))$$

which belong to $\theta \times C^*$. Thus $\theta \times C \subseteq C(K) \subseteq \theta \times C^*$, and by Lemma 3.2, $C(K)$ is complete. Therefore $C(K) = \theta \times C^*$.

Finally suppose that Q is a complete subgroup of K that contains $\tilde{G}$. $\theta \times N \subseteq \tilde{G} \subseteq Q$. If $(\theta, a) \in Q$ and n is a positive integer, then there exists a $(\beta, b) \in Q$ such that $(\theta, a) = n(\beta, b) = (n\beta, nb)$. Thus $n\beta = \theta$, and so $\beta = \theta$. It follows that $(\theta, b) \in Q$, and hence $\{a \in N^* : (\theta, a) \in Q\}$ is a complete subgroup of N^* that contains N. Therefore $\theta \times N^* \subseteq Q$. If $(\beta, b) \in Q$ and n is a positive integer, then there exists a $(\gamma, c) \in Q$ such that

$$(\beta, b) = n(\gamma, c) = (n\gamma, nc).$$

In particular, $n\gamma = \beta$. Thus $\{\beta \in \Delta^* : (\beta, b) \in Q \text{ for some } b \in N^*\}$ is a complete subgroup of Δ^* that contains Δ, and hence it must equal Δ^*. Since $\theta \times N^* \subseteq Q$, it follows that $\Delta^* \times 0 \subseteq Q$, and hence that $Q = K$.

Corollary. *If Z is its own d-hull in G, then there exists a completion K of G that is a central extension of Z^* by $(G/Z)^*$, $C(K) = C^*$, and $Z(K) = Z^*$.*

Proof. Since Z is its own d-hull, $\Delta = G/Z$ is torsion free. If we let $N = Z$ in the theorem, then we get a completion K of G such that K is a central extension of Z^* by Δ^* and $C(K) = C^*$. Now we must show that $Z(K) = Z^*$. As in the proof of the theorem, $\tilde{G} \subseteq \tilde{H}(\Delta, Z^*, g) \subseteq K(\Delta^*, Z^*, g^*)$, where g and g^* are $(*)$-mappings. If $(\alpha, a) \in Z(K)$, then $g^*(\alpha, \beta) = g^*(\beta, \alpha) = -g^*(\alpha, \beta)$ for all $\beta \in \Delta^*$, and hence $0 = 2g^*(\alpha, \beta) = g^*(\alpha, 2\beta)$ for all $\beta \in \Delta^*$. But since $2\Delta^* = \Delta^*$, $g^*(\alpha, \beta) = 0$ for all $\beta \in \Delta^*$. Now there exists $n > 0$ such that $n\alpha \in \Delta$, and hence $0 = ng^*(\alpha, \beta) = g^*(n\alpha, \beta) = g(n\alpha, \beta)$ for all $\beta \in \Delta$. But then $f(n\alpha, \beta) = - t(n\alpha + \beta) + t(n\alpha) + t(\beta) = f(\beta, n\alpha)$ for all $\beta \in \Delta$, and hence $(n\alpha, \theta)$ is in the center of $G(\Delta, Z, f)$, and so $n\alpha = \theta$. Thus since $\Delta = G/Z$ is torsion free, $\alpha = \theta$, and hence $(\alpha, a) = (\theta, a) \in \theta \times Z^*$. Therefore $Z(K) \subseteq \theta \times Z^*$, and clearly $Z(K) \supseteq \theta \times Z^*$.

THEOREM 3.2. *If $G \supseteq Z \supseteq C$, then there exists a completion of G if and only if the d-hull of C in G is contained in Z.*

Proof. If $Z \supseteq N$, where N is the d-hull of C in G, then G/N is torsion free, and hence by Theorem 3.2 there exists a completion of G. The converse is part (c) of Lemma 3.2.

THEOREM 3.3. *If C is torsion free, then there exists a completion K of G such that*

$$(**) \qquad Z(K) = Z^* \quad \textit{and} \quad C(K) \textit{ is torsion free.}$$

If H is any other such completion of G, then $H \cong K$.

Proof. By Corollary III of Lemma 3.1, Z is its own d-hull in G. Thus by the Corollary to Theorem 3.1 there exists a completion of G that satisfies (**). Suppose that H is a completion of G that satisfies (**). By Lemma 3.2, $C(H)$ is complete. If $a, b \in H$, then by Corollary II of Lemma 3.1, there exists an $n > 0$ such that $na, nb \in G$. Thus $n^2[a, b] = [na, nb] \in C$. It follows that $C(H) = C^*$, and then by Corollary III of Lemma 3.1, $H/Z(H)$ is torsion free. If $z \in Z$ and $h \in H$, then there exists an $n > 0$ such that $nh \in G$, and hence $n[h, z] = [nh, z] = 0$. Thus since $C(H)$ is torsion free, $h + z = z + h$. It follows that $Z(H) \cap G = Z$, and hence

$$\frac{H}{Z(H)} \supseteq \frac{G + Z(H)}{Z(H)} \cong \frac{G}{G \cap Z(H)} = \frac{G}{Z}.$$

Thus $G(\Delta, Z, f) \subseteq H'(\Delta, Z(H), f) \subseteq H(\Delta'', Z(H), h)$, where $\Delta'' = H/Z(H)$, $\Delta = (G + Z(H))/Z(H)$, and h induces f. Also $\Delta'' = \Delta^*$; for $S(\Delta^*, Z(H), h) \supseteq G$ and it is complete, where Δ^* is a completion of Δ in Δ''.

Now suppose that H and K are two completions of G that satisfy (**). Then

$$G(\Delta, Z, f) \begin{cases} \subseteq H'(\Delta, Z(H), f) \subseteq H(\Delta^*, Z(H), h) \\ \subseteq K'(\Delta, Z(K), f) \subseteq K(\Delta^*, Z(K), k) \end{cases}$$

where h and k induce f. By Lemma 3.4 there exists a (*)-mapping $\bar{h}$ of $\Delta^* \times \Delta^*$ into $C(H)$ that is equivalent to h and a (*)-mapping $\bar{k}$ of $\Delta^* \times \Delta^*$ into $C(K)$ that is equivalent to k. Moreover, $2\bar{h}(\alpha, \beta) = h(\alpha, \beta) - h(\beta, \alpha)$ and $2\bar{k}(\alpha, \beta) = k(\alpha, \beta) - k(\beta, \alpha)$ for all $\alpha, \beta \in \Delta^*$. For $\alpha, \beta \in \Delta^*$ there exists $n = 2m > 0$ such that $m\alpha, m\beta \in \Delta$. Thus

$$\begin{aligned} n^2\bar{h}(\alpha, \beta) &= 4\bar{h}(m\alpha, m\beta) = 2(h(m\alpha, m\beta) - h(m\beta, m\alpha)) \\ &= 2(f(m\alpha, m\beta) - f(m\beta, m\alpha)) = 2(k(m\alpha, m\beta) - k(m\beta, m\alpha)) \\ &= 4\bar{k}(m\alpha, m\beta) = n^2\bar{k}(\alpha, \beta). \end{aligned}$$

Now suppose that there exists an isomorphism σ of $Z(H)$ onto $Z(K)$ such that $C(H)\sigma = C(K)$ and $c\sigma = c$ for all $c \in C$. Then $n^2(\bar{h}(\alpha, \beta)\sigma) = (n^2\bar{h}(\alpha, \beta))\sigma = n^2\bar{h}(\alpha, \beta) = n^2\bar{k}(\alpha, \beta)$ because $n^2\bar{h}(\alpha, \beta) \in C$. Thus since $C(K)$ is torsion free, $\bar{h}(\alpha, \beta)\sigma = \bar{k}(\alpha, \beta)$ for all $\alpha, \beta \in \Delta^*$. Define that $(\alpha, a)\pi = (\alpha, a\sigma)$ for all $(\alpha, a) \in H$. Then π is an isomorphism of H onto K. Thus to complete the proof we need the following

LEMMA. *Suppose that C and Z are abelian groups, $C \subseteq Z$, C is torsion free, Z' and Z'' are abelian completions of Z, and C' and C'' are completions of C in Z' and Z'' respectively. Then there exists an isomorphism σ of Z' onto Z'' such that $C'\sigma = C''$ and $c\sigma = c$ for all $c \in C$.*

Proof. $Z' = C' \oplus T' \oplus D'$ and $Z'' = C'' \oplus T'' \oplus D''$, where T' and T'' are the torsion groups of Z' and Z'' respectively, because all these groups are complete and C' and C'' are torsion free. Since Z' and Z'' are completions of Z, they are isomorphic, and hence $T' \cong T''$. Since C' and C'' are completions of C, there exists an isomorphism of C' onto C'' that induces the identity on C. Thus it suffices to show that the rational vector spaces D' and D'' have the same dimension. Let B be the basis for D', and for each $b \in B$ pick a positive integer $n = n(b)$ such that $nb \in Z$. Then $nb = x + y + \bar{b}$, where $x \in C''$, $y \in T''$, and $\bar{b} \in D''$. Let $\bar{B}$ be the set of all such elements $\bar{b}$. Suppose (by way of contradiction) that $c_1, \cdots, c_k$ are distinct dependent elements of $\bar{B}$. Then there exist integers n_i, not all zero, such that $\sum n_i c_i = 0$. For each c_i there exist a $b_i \in B$ and an integer $m_i > 0$ such that $m_i b_i = x_i + y_i + c_i$, where $x_i \in C''$ and $y_i \in T''$. $\sum n_i m_i b_i = \sum n_i(x_i + y_i)$. Thus since the y_i are of finite order and $C'' = C^*$, there exists an integer $n > 0$ such that $\sum nn_i m_i b_i = \sum nn_i x_i \in C$. But since $D' \cap C = 0$, $\sum nn_i m_i b_i = 0$, a contradiction. It follows that the rank of $D' \leqq$ rank of D'', and by symmetry the rank of $D'' \leqq$ rank of D'.

COROLLARY (Vinogradov). *If G is torsion free, then there exists a unique (to within an isomorphism that induces the identity on G) torsion free completion of G.*

Proof. It follows from the theorem that there exists a completion K of G such that K is a central extension of Z^* by $(G/Z)^*$. But since Z^* and $(G/Z)^*$ are torsion free, K is torsion free. If H and K are torsion free completions of G, then $Z(H)$ and $Z(K)$ are completions of Z, and $C(H)$ and $C(K)$ are completions of C in $Z(H)$ and $Z(K)$. There exists a unique isomorphism σ of $Z(H)$ onto $Z(K)$ such that $z\sigma = z$ for all $z \in Z$. In particular, $c\sigma = c$ for all $c \in C$. Thus $C(H)\sigma = C(K)$ because $C(H)$ $(C(K))$ is the unique completion of C in $Z(H)$ $(Z(K))$. Therefore the isomorphism π of H onto K (in the proof of the theorem) induces the identity on G.

Remarks. Our results include the classical case where G is abelian. For in this case C is torsion free, and hence by Theorem 3.3 there exists a unique abelian completion of G. We pose the following questions. Is the completion in the Corollary of Theorem 3.1 unique? Can we impose further conditions on the completion in Theorem 3.1 so that it is unique?

4. Structure of complete nilpotent groups of class 2

As usual $G \supseteq Z \supseteq C$. The join of an ascending chain of subgroups of G each of which has C for its center is a subgroup of G with center C. Thus there exist maximal subgroups of G with center C.

LEMMA 4.1. *Suppose that $Z = C \oplus S$ for some complete subgroup S of Z. If G' is a maximal subgroup of G with center C, then $G = G' \oplus S$, and conversely.*

Proof. $G'/C \cap Z/C = C$. Thus since Z/C is complete, there exists a subgroup H of G such that$H \supseteq G'$ and $G/C = H/C \oplus Z/C$. S is normal in G because $S \subseteq Z$, and H is normal in G because $H \supseteq C$. $H \cap S = H \cap Z \cap S = C \cap S = 0$, and $H + S = H + C + S = H + Z = G$. Therefore $G = H \oplus S$. If a is in the center of H, then $a \in Z = C \oplus S$, and hence $a = c + s$ with $c \in C$ and $s \in S$. Thus $a - c = s \in H \cap S = 0$, and hence $a \in C$. Thus C is the center of H, and hence $H = G'$.

Conversely, if $G = G'' \oplus S$, then since S is abelian, $C \subseteq G''$, and hence $C \subseteq Z(G'')$. But $Z = Z(G'') \oplus S$, and so $Z(G'') = C$. If K is a subgroup of G that properly contains G'', then $K \cap S \neq 0$. Thus $Z(K)$ properly contains C, a contradiction. Therefore G'' is a maximal subgroup of G with center C.

THEOREM 4.1. *Suppose that G is complete, and let G' be a maximal subgroup of G with center C. Then C and Z are complete, $Z = C \oplus S$, G/Z is torsion free, $G = G' \oplus S$, and G' is complete. Moreover G'/C is torsion free and $G' \cong G'(\Delta', C, f)$, where $\Delta' = G'/C$ and f is a $(*)$-mapping of $\Delta' \times \Delta'$ into C such that $\{2f(\alpha, \beta) : \alpha, \beta \in \Delta'\}$ generates C. G' is torsion free if and only if C is torsion free.*

Proof. By Lemma 3.2, C is complete. Let D be the d-hull of C in G. By Lemma 3.1, D is complete, and by Lemma 3.2, $D \subseteq Z$. Thus by Lemma 3.4 we may assume that $G = G(\Delta, D, g)$, where $\Delta = G/D$ and g is a $(*)$-mapping of $\Delta \times \Delta$ into D. If $(\alpha, a) \in Z$, then $g(\alpha, \beta) = g(\beta, \alpha) = -g(\alpha, \beta)$, and hence $0 = 2g(\alpha, \beta) = g(\alpha, 2\beta)$ for all $\beta \in \Delta$. But since Δ is complete, $g(\alpha, \beta) = 0$ for all $\beta \in \Delta$. Thus

$$Z = \{(\alpha, a) \in G : g(\alpha, \beta) = 0 \text{ for all } \beta \in \Delta\}.$$

If $n(\alpha, a) = (n\alpha, na) \in Z$ for $(\alpha, a) \in G$ and $n > 0$, then $0 = g(n\alpha, \beta) = ng(\alpha, \beta) = g(\alpha, n\beta)$ for all $\beta \in \Delta$, and hence $g(\alpha, \beta) = 0$ for all $\beta \in \Delta$. Therefore $(\alpha, a) \in Z$, and it follows that G/Z is torsion free and that Z is complete. Since C is complete, $Z = C \oplus S$, and by Lemma 4.1, $G = G' \oplus S$. In particular, G' is complete. If $x \in G'$ and $nx \in C$ for some $n > 0$, then since C is complete, $nx = nc$ for some $c \in C$. Thus $n(x - c) = 0$, and hence by

Lemma 3.2, $x - c \in C$, and thus $x \in C$. Therefore $\Delta' = G'/C$ is torsion free, and the remainder of the theorem follows at once from Lemma 3.4.

Suppose that G is complete and torsion free. Then by the last theorem Z and C are complete. Let N be any complete subgroup of Z that contains C, and let $\Delta = G/N$. Then N and Δ are rational vector spaces. Thus if f is the (*)-mapping of $\Delta \times \Delta$ into N that determines G, then $sf(\alpha, \beta) = f(s\alpha, \beta) = f(\alpha, s\beta)$ for all rational numbers s. For $nf((m/n)\alpha, \beta) = f(n(m/n)\alpha, \beta) = f(m\alpha, \beta) = mf(\alpha, \beta)$, and hence $f((m/n)\alpha, \beta) = (m/n)f(\alpha, \beta)$.

Next let

$$\alpha = x_1 \alpha_{s_1} + \cdots + x_m \alpha_{s_m} \quad \text{and} \quad \beta = y_1 \alpha_{t_1} + \cdots + y_n \alpha_{t_n}$$

be elements of Δ, where $\alpha_1, \cdots, \alpha_\lambda, \cdots$ is a basis for Δ.

$$f(\alpha, \beta) = \sum x_i y_j f(\alpha_{s_i}, \alpha_{t_j}).$$

Thus f can be represented by a skew-symmetric matrix with entries $f(\alpha_i, \alpha_j)$ from N, and the dimension of this matrix is equal to the rank of Δ.

Now suppose that we are given complete torsion free abelian groups N and Δ. Fix a basis for Δ. Then by Theorem 2.1 and the above remarks there exists a $1-1$ correspondence between the set of all nonequivalent central extensions of N by Δ and the set of all skew-symmetric matrices over N with dimension equal to the rank of Δ.

Finally suppose that G is complete and that C is torsion free. Then by Theorem 4.1, $G = G' \oplus S$, where G' is a torsion free complete subgroup of G with center C and S is an abelian subgroup of G. Thus G' is determined by the rational vector spaces G'/C and C, and a skew-symmetric matrix M over C with dimension equal to the rank of G'/C. Since C is the center of G', it follows that M has no zero rows. Thus we have a complete structure theorem for G.

The most important property of a complete abelian group is that it is a direct summand of every containing abelian group. The following is a slight generalization of this result.

If G is complete and contained in a group H such that for some abelian subgroup Q of H, $H = G + Q$ and $[G, Q] = 0$, then G is a direct summand of H.

For it follows that $Z \subseteq Z(H)$, and by Theorem 4.1, Z is complete. Thus $Z(H) = Z \oplus D$ for some subgroup D of $Z(H)$. Clearly $D \cap G = 0$, and D and G are normal subgroups of H. Each h in H has a representation $h = g + q$, where $g \in G$ and $q \in Q$, and $q = z + d$, where $z \in Z$ and $d \in D$ because $Q \subseteq Z(H)$. Thus $h = (g + z) + d \in G + D$, and hence $H = G \oplus D$. It seems that a better result should be obtainable by using Theorem 2.1 and Theorem 4.1, but the author has been unsuccessful with this problem. We conclude by giving the following two examples.

Example I. *An abelian group with a non-abelian completion.* Let I be the group of integers, $\Delta = I \oplus I$, and $N = I/[3]$ = the group of integers modulo 3. For (a_1, a_2) and (b_1, b_2) in Δ define

$$g((a_1, a_2), (b_1, b_2)) = (a_1, a_2)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + [3] = a_1 b_2 - a_2 b_1 + [3].$$

It follows that g is a $(*)$-mapping of $\Delta \times \Delta$ into N, and hence $G(\Delta, N, g)$ is a central extension of $\theta \times N$ by Δ. It is easy to show that

$$Z = \{(\alpha, a) \in G : \alpha = (3a, 3b) \text{ for some } a, b \in I\},$$

and hence Z is not its own d-hull in G. By Lemma 3.3 there exists an extension of g to a $(*)$-mapping g^* of $\Delta^* \times \Delta^*$ into N^*.

$$G(\Delta, N, g) \subseteq H(\Delta, N^*, g) \subseteq K(\Delta^*, N^*, g^*),$$

K is a completion of G, and it is fairly easy to show that no proper complete subgroup of K contains Z.

Example II. *A group H such that $H \supseteqq Z(H) \supseteqq C(H)$, $Z(H) \supseteqq T(H)$, and the d-hull of $C(H)$ in H is not contained in $Z(H)$.* Let $H = I \times I \times G$, where G is the group in the last example. For (x_1, x_2) and (y_1, y_2) in $I \times I$ define that

$$f((x_1, x_2), (y_1, y_2)) = 3(x_1 y_2 - x_2 y_1).$$

Then f is a $(*)$-mapping of $(I \times I) \times (I \times I)$ into [3], and hence determines a mapping into the subgroup $[3] \times 0 \times 0$ of the center of G. Define that

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_5) &+ (y_1, y_2, y_3, y_4, y_5) \\ &= (x_1 + y_1, x_2 + y_2, x_3 + y_3 + 3(x_1 y_2 - x_2 y_1), \\ &\qquad x_4 + y_4, x_5 + y_5 + x_3 y_4 - x_4 y_3 + [3]).\end{aligned}$$

Then H is a group, $6(0, 0, 1, 0, 0) = (0, 0, 6, 0, 0) \in C(H)$ and $(0, 0, 1, 0, 0) \notin Z(H)$. Also H is nilpotent and of class 2. Thus by Corollary I of Theorem 3.1 there does not exist a completion of H.

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