GROUPS ON S^n WITH PRINCIPAL ORBITS OF DIMENSION $n-3$, II

BY

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1. Introduction

In a recent paper [1] under the same title the authors have studied the transformation groups (G, X) in which X is the n-sphere S^n with the usual differentiable structure, and G is a compact connected Lie group acting differentiably on X with principal orbits of dimension $n-3$ and with a stationary point α. It is proved that if α is an isolated singular orbit, then $n = 4$, G is effectively a circle group, and the orbit space X/G is a simply connected 3-manifold. If α is a singular orbit but not isolated, then the orbit space X/G is a simply connected 3-manifold with boundary such that the boundary is topologically a 2-sphere and that an orbit is singular or principal according as it is on the boundary or not. As a continuation of our previous study, we shall prove the following.

Theorem A. *Let G be a compact connected Lie group acting differentiably on the n-sphere with the usual differentiable structure. If principal orbits are $(n-3)$-dimensional, and if there is a stationary point, then there exists a second stationary point.*

In fact as a consequence of our method of proof we obtain the following sharper result.

Theorem B. *Under the hypothesis of Theorem* A, *the set of stationary points of G is a sphere of dimension zero, one, two, or three. In case it is a 2-sphere it is all of B, and principal orbits are $(n-3)$-spheres.*

In case $n = 3$, G acts trivially on the 3-sphere so that Theorems A and B become obvious.

In case $n = 4$, principal orbits are 1-dimensional and then are circles. Therefore G is effectively a circle group, and hence the existence of a second stationary point is assured [1]; also Theorem B follows.

Excluding these two rather simple cases, we shall assume $n > 4$. Throughout this paper, X denotes the n-sphere S^n with the usual differentiable structure, and G denotes a compact connected Lie group acting differentiably on X with principal orbits of dimension $n-3$ and with a stationary point α.

As usual, we let $X^* = X/G$ be the orbit space, and p the natural projection of X onto X^*. For every $x \in X$, we let G_x be the isotropy subgroup of G at x and G_x^* the identity component of G_x. Let U be the union of all principal

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orbits and B the union of all singular orbits. Then U and B are invariant and disjoined, and hence $U^* = p(U)$ and $B^* = p(B)$ are disjoined. Since $n > 4$, $X = U \cup B$ (that means, there is no exceptional orbit of dimension $n-3$), B^* is topologically a 2-sphere, and X^* is a simply connected 3-manifold with B^* as its boundary [2].

2. Singular orbits

Let $G(y)$ be a singular orbit, and consider G_x and G_x^* as functions of B into the space of compact subsets of G. If G_x is continuous at y, we call $G(y)$ a *regular singular orbit.* If G_x is not continuous at y but G_x^* is, we call $G(y)$ an *exceptional singular orbit.* If G_x^* is not continuous at y, we call $G(y)$ a *singular singular orbit.* Intuitively speaking all nearby singular orbits of a regular singular orbit are of the same type, all nearby singular orbits of an exceptional singular orbit are of the same dimension but not of the same type, and all nearby singular orbits of a singular singular orbit are not of the same dimension.

Let B_r be the union of all regular singular orbits, B_e the union of all exceptional singular orbits, and B_s the union of all singular singular orbits. Then

$$B = B_r \cup B_e \cup B_s ,$$

and B_r, B_e, B_s are invariant and mutually disjoint. Denote by B_r^*, B_e^*, B_s^* the respective images of B_r, B_e, B_s under p. We shall see later that B_e^* is empty (Lemma 1) and *that B_s^* either is empty or consists of a finite number of arcs joining two stationary points* (Lemmas 1 and 7). Various types of B_s^* may be seen from the following examples.

Let R^k denote the euclidean k-space, and let $S^k = R^k \cup \infty$ be the one-point-compactification of R^k. Then the rotation group $SO(k)$ has a natural action on S^k leaving ∞ fixed. Since the unitary 2-space may be regarded as R^4, the unitary group $U(2)$ has a natural action on $S^4 = R^4 \cup \infty$ leaving ∞ fixed.

(1) Let $G = SO(\xi)$ act on $X = S^{\xi+2}$ such that whenever $g \in G$, $x_1 \in R^\xi$, and $x_2 \in R^2$, $g(x_1, x_2) = (gx_1, x_2)$, where ξ is an integer $\geqq 2$. Then all singular orbits are stationary points so that B_s^* is empty.

(2) (due to Bredon) Let $G = SO(3)$ act on $X = S^6$ such that whenever $g \in G$, $x_1 \in R^3$, and $x_2 \in R^3$, $g(x_1, x_2) = (gx_1, gx_2)$. Then B_s^* consists of two stationary points.

(3) (due to Bredon) Let $G = U(2)$. Then the center C of G is a circle group, and G/C may be identified with $SO(3)$. Let G act on $X = S^7$ such that whenever $g \in G$, $x_1 \in R^4$, and $x_2 \in R^3$, $g(x_1, x_2) = (gx_1, (gC)x_2)$. Then B_s^* is an arc with two stationary points as its end points.

(4) Let $G = SO(\xi) \times SO(\eta)$ act on $X = S^{\xi+\eta+1}$ such that whenever $g_1 \in SO(\xi)$, $g_2 \in SO(\eta)$, $x_1 \in R^\xi$, $x_2 \in R^\eta$, and $x_3 \in R$, $(g_1, g_2)(x_1, x_2, x_3) = (g_1 x_1, g_2 x_2, x_3)$, where ξ and η are integers $\geqq 2$. Then B_s^* consists of two arcs with two stationary points as their common end points.

(5) Let $G = SO(\xi) \times SO(\eta) \times SO(\zeta)$ act on $X = S^{\xi+\eta+\zeta}$ such that whenever $g_1 \in SO(\xi)$, $g_2 \in SO(\eta)$, $g_3 \in SO(\zeta)$, $x_1 \in R^{\xi}$, $x_2 \in R^{\eta}$, and $x_3 \in R^{3}$, $(g_1, g_2, g_3)(x_1, x_2, x_3) = (g_1 x_1, g_2 x_2, g_3 x_3)$, where ξ, η, ζ are integers $\geqq 2$. Then B_s^* consists of three arcs with two stationary points as their common end points.

3. Lemmas on B^*

LEMMA 1. (i) *B_e^* is empty.*

(ii) *Every $y^* \in B_s^*$ has a compact neighborhood C^* in B^* which may be regarded as a circular disk of center y^* such that there are a finite number of radii $y^*z_1^*, \cdots, y^*z_k^*$ such that $B_s^* \cap C^* = y^*z_1^* \cup \cdots \cup y^*z_k^*$ ($= y^*$ if $k = 0$) and that for every $i = 1, \cdots, k$, all the orbits on $y^*z_i^* - y^*$ are of the same type. In case that y^* and z_1^* are of the same type, $k = 2$, and all the orbits on $B_s^* \cap C^* = y^*z_1^* \cup y^*z_2^*$ are of the same type.*

Proof. Let $y^* \in B^*$ and let $y \in p^{-1}(y^*)$. Let K be a closed cell which is a slice at y and on which G_y acts orthogonally, and let S be the boundary of K. Then $G(K)/G = K/G_y$ is topologically a cone of vertex y^* over $G(S)/G = S/G_y$, and for every $x^* \in G(S)/G$, all the orbits on the line segment y^*x^* except y^* are of the same type.

Denote by r the dimension of $G(y)$. Then S is an $(n-r-1)$-sphere and (G_y, S) is a transformation group with principal orbits of dimension $n-r-3$. Since B^* is topologically a 2-sphere [2], $B^* \cap G(S)/G$ is not empty, and every G_y-orbit in $B \cap S$ is of dimension $< n-r-3$. Using an argument similar to one in [4], we can easily show that S/G_y^* is a closed 2-cell and there is no exceptional G_y^*-orbit. Since there is no exceptional G-orbit of dimension $n-3$ [2], every G_y-orbit in K is actually a G_y^*-orbit. Hence for any $z \in K$, $G_z^* = G_y^*$ implies $G_z = G_y$, and consequently $y^* \notin B_e^*$. This proves that B_e^* is empty.

It is clear that $B^* \cap G(S)/G$ is the boundary of the closed 2-cell $G(S)/G = S/G_y^*$. Since $G(K)/G$ is topologically a cone of vertex y^* over $G(S)/G$ such that for every $x^* \in G(S)/G$, all the orbits on the line segment y^*x^* except y^* are of the same type, we may regard $C^* = B^* \cap G(K)/G$ as a circular disk of center y^* such that for every $z^* \in B^* \cap G(S)/G$, all the orbits on the radius y^*z^* except y^* are of the same type.

Suppose that all the orbits on $B^* \cap G(S)/G$ are of the same type. Then $B_s^* \cap G(K)/G \subset y^*$ and is equal to y^* if and only if y^* is of lower dimension than orbits in $B^* \cap G(S)/G$.

Suppose that not all the orbits on $B^* \cap G(S)/G$ are of the same type. Then $B_s^* \cap G(S)/G$ is not empty but finite. Let $B_s^* \cap G(S)/G = \{z_1, \cdots, z_k\}$. Then $B_s^* \cap C$ is equal to the union of the radii $y^*z_1^*, \cdots, y^*z_k^*$.

If y^* and z_1^* are of the same type, then $p^{-1}(z_1^*) \cap S$ is a stationary point of G_y, but not all the G_y-orbits on $B \cap S$ are stationary points. By the main theorem of [4], $B_s^* \cap G(S)/G$ consists of two elements, namely z_1^* and z_2^*, and $p^{-1}(z_2^*) \cap S$ is also a stationary point of G_y. Hence all the orbits on $B_s^* \cap C^* = y^*z_1^* \cup y^*z_2^*$ are of the same type.

As direct consequences of Lemma 1, we have

LEMMA 2. *B_2^* is a finite subpolyhedron of B^* of dimension $\leqq 1$.*

LEMMA 3. *If $\alpha \notin B_s$, then B_s is empty, $B = B_r$ consists of all the stationary points of G, and all principal orbits are $(n-3)$-spheres.*

From now on we shall assume that

$$\alpha \in B_s .$$

LEMMA 4. *There is no simple closed curve in B_s^* not containing $\alpha^* = p(\alpha)$.*

Proof. Suppose that there is a simple closed curve in B_s^* not containing α^*. Then there is a component Q^* of $B_r^* = B^* - B_s^*$ whose closure $\overline{Q^*}$ does not contain α^*. It is clear that all the orbits in Q^* are of the same type. Denote by t the dimension of orbits in Q^*. Then there is a $(t+2)$-cycle z mod 2 in $p^{-1}(\overline{Q^*})$ which is not bounding in B. (Notice that z is the fundamental cycle mod 2 of $(p^{-1}(\overline{Q^*}),\ p^{-1}(\overline{Q^*} - Q^*))$.) Hence there is an $(n-t-3)$-cycle z' mod 2 in $U = X - B$ linked with z.

Since the 3-manifold U^* has trivial homotopy groups [1], it is contractible so that the transformation group (G, U) has a cross-section M on which G_x is constant. M is homeomorphic to U^* and then is also contractible. Therefore for any $a \in M$ there is a map $h:M \times [0, 1] \to M$ such that whenever $x \in M$, $h(x, 0) = x$ and $h(x, 1) = a$. Hence the map $\bar{h}:U \times [0, 1] \to U$ defined by

$$\bar{h}(gx, t) = gh(x, t), \qquad g \in G,\ x \in M, \text{ and } t \in [0, 1],$$

deforms U into the orbit $G(a)$. Using this deformation if necessary, we may assume that z' is in $G(a)$.

Since G acts differentiably on X and leaves α fixed, there is an invariant compact neighborhood Y of α which is a closed n-cell contained in $X - p^{-1}(\overline{Q^*})$. Let $a \in Y$. Then z' is in Y and is bounding in

$$Y \subset X - p^{-1}(\overline{Q^*}),$$

contrary to our assumption that z' links with z. This proves Lemma 4.

Let Y be a closed n-cell which is a compact neighborhood of α and on which G acts orthogonally, and let Z be the boundary of Y. Let $Y^* = p(Y)$, $Z^* = p(Z)$, and $A^* = B^* - (Y^* - Z^*)$. Since $\alpha^* \in B_s^*$, it follows from Lemma 1 that $B^* \cap Y^*$ may be regarded as a circular disk of center α^* and boundary $B^* \cap Z^* = A^* \cap Z^*$.

LEMMA 5. *$p^{-1}(A^*)$ has trivial homology groups mod 2.*

Proof. Suppose that there is a nonbounding k-cycle z mod 2 in $p^{-1}(A^*)$, where z is reduced if $k = 0$. Then z is linked with an $(n-k-1)$-cycle z' mod 2 in $X - p^{-1}(A^*)$. Denote by $|z'|$ the support of z'. Then $p(|z'|)$ is a compact subset of $X^* - A^*$. Hence there is a triangulation K of X^* such that (i) no 3-simplex of K has all of its vertices in B^*, (ii) $B^* \cup Z^*$ is the polyhedron of some subcomplex of K, and (iii) the star Q^* of A^* in the barycentric subdivision K' of K does not meet $p(|z'|)$. Let P^* be the star of B^* in K'. It is not hard to see that $B^* \cap Q^*$ is a closed 2-cell containing A^* in its interior and that (P^*, Q^*) is topologically a cylinder over $(B^*, B^* \cap Q^*)$.

Let E^* be the closure of $X^* - P^*$. Since P^* is topologically a cylinder with B^* and $P^* \cap E^*$ as its bases, $P^* \cap E^*$ is a deformation retract of $P^* - B^*$ so that E^* is a deformation retract of U^*. It follows from the contractibility of U^* that E^* is contractible, and hence $p^{-1}(E^*)$ is topologically the product of E^* and a principal orbit.

Let F^* be the closure of $P^* - Q^*$. Then $E^* \cap F^*$ is topologically a closed 2-cell. Since both E^* and $E^* \cap F^*$ are contractible, $E^* \cap F^*$ is a deformation retract of E^*, so that $p^{-1}(E^* \cap F^*)$ is a deformation retract of $p^{-1}(E^*)$. Hence $p^{-1}(F^*)$ is a deformation retract of $p^{-1}(E^* \cup F^*)$.

Since z' is in $p^{-1}(E^* \cup F^*)$, we may as well assume that z' is in $p^{-1}(F^*)$. But $p^{-1}(F^*) \subset Y \subset X - p^{-1}(A^*)$ and z' is bounding in Y. It follows that z' is bounding in $X - p^{-1}(A^*)$, contrary to our assumption that z' links with z.

LEMMA 6. *$B_s^* \cap A^*$ contains no simple closed curve and is connected.*

Proof. The nonexistence of a simple closed curve in $B_s^* \cap A^*$ is a consequence of Lemma 4. By this result and Lemma 2 we may have a connected subpolyhedron T^* of A^* such that $\dim T^* \leqq 1$ and that T^* contains $B_s^* \cap A^*$ but does not contain any simple closed curve.

If $B_s^* \cap A^*$ is not connected, there is an arc a^*b^* in T^* which intersects B_s^* only at its end points a^* and b^*. The fundamental cycle z mod 2 of $(p^{-1}(a^*b^*), p^{-1}(a^* \cup b^*))$ is clearly a nonbounding cycle of $p^{-1}(T^*)$. Since $p^{-1}(T^*)$ is a deformation retract of $p^{-1}(A^*)$, z is not bounding in $p^{-1}(A^*)$, contrary to Lemma 5.

LEMMA 7. *There is a stationary point β in $p^{-1}(B_s^* \cap A^*)$ such that A^* may be regarded as a circular disk of center $\beta^* = p(\beta)$ such that for every $z^* \in A^* \cap z^*$, all the orbits on the radius $\beta^* z^*$ except β^* are of the same type. Hence $B_s^* \cap A^*$ is a finite union of radii of A^*.*

Proof. We first claim that $B_s^* \cap A^*$ is not empty. Suppose that $B_s^* \cap A^*$ is empty. If so, then all the orbits in A^* are of the same type. It follows from the contractibility of A^* that $p^{-1}(A^*)$ is topologically the product of A^* and an orbit in A^*. By Lemma 5, every orbit in A^* has trivial homology groups mod 2 and then is a stationary point. Hence, by Lemma 3 with an interior point of $p^{-1}(A^*)$ in place of α, B_s^* is empty, contrary to our assumption that $\alpha^* \in B_s^*$.

Since $B_s^* \cap A^*$ is a connected subpolyhedron of A^* of dimension $\leqq 1$ and it contains no simple closed curve (Lemmas 2 and 6), $p^{-1}(B_s^* \cap A^*)$ is a deformation retract of $p^{-1}(A^*)$. It follows from Lemma 5 that $p^{-1}(B_s^* \cap A^*)$ has trivial homology groups mod 2.

If all the orbits on $B_s^* \cap A^*$ are of the same type, then $p^{-1}(B_s^* \cap A^*)$ is topologically the product of $B_s^* \cap A$ and an orbit on $B_s^* \cap A^*$. Hence every orbit on $B_s^* \cap A^*$ has trivial homology groups mod 2 and thus is a stationary point. Since $B_s^* \cap A^*$ is not empty, it follows from Lemmas 1 and 6 that $B_s^* \cap A^*$ is either a single stationary point contained in the interior of the closed 2-cell A^* or an arc intersecting the boundary of A^* only at its end points. Let β be an interior point of $p^{-1}(B_s^* \cap A^*)$. Then the conclusion of Lemma 7 follows.

If not all the orbits on $B_s^* \cap A^*$ are of the same type, then there is some $\beta \in p^{-1}(B_s^* \cap A^*)$ such that $\beta^* = p(\beta)$ is in the interior of A^* and that nearby singular singular orbits of $G(\beta)$ are of higher dimension than $G(\beta)$. Since $p^{-1}(B_s^* \cap A^*)$ has trivial homology groups mod 2, on each component of $(B_s^* \cap A^*) - \beta^*$, all the orbits are of the same type. It follows from Lemma 1 that $B_s^* \cap A^*$ is a finite union of arcs such that each of the arcs has β^* and one point on the boundary of A^* as its end points and that any two of the arcs intersect only at β^*.

Now it is clear that $G(\beta)$ is a deformation retract of $p^{-1}(B_s^* \cap A^*)$ and then has trivial homology groups mod 2. Hence $\beta = G(\beta)$ is a stationary point.

4. Remarks

(1) Theorem A is an immediate consequence of Lemma 7.

(2) In the proof of Lemma 7 we have incidently proved Theorem B.

(3) If X^* is a closed 3-cell, we can easily prove that G acts linearly on X.

(4) Although Lemma 5 proves only that $p^{-1}(A^*)$ has trivial homology groups mod 2, we can see from Lemma 7 that $p^{-1}(A^*)$ is actually contractible.

(5) As mentioned earlier, our proofs also give

COROLLARY. *Under the hypothesis of Theorem* A, B_s^* *is either empty or consists of a pair of stationary points and a finite set of arcs joining them.*

BIBLIOGRAPHY

1. E. E. FLOYD, *Fixed point sets of compact abelian Lie groups of transformations*, Ann. of Math. (2), vol. 66 (1957), pp. 30–35.
2. D. MONTGOMERY AND C. T. YANG, *Groups on S^n with principal orbits of dimension $n-3$*, Illinois J. Math., vol. 4 (1960), pp. 507–517.
3. D. MONTGOMERY, H. SAMELSON, AND C. T. YANG, *Exceptional orbits of highest dimension*, Ann. of Math. (2), vol. 64 (1956), pp. 131–141.
4. ———, *Groups on E^n with $(n-2)$-dimensional orbits*, Proc. Amer. Math. Soc., vol. 7 (1956), pp. 719–728.

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