GROUPS WITH REPRESENTATIONS OF BOUNDED DEGREE II

BY

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Kaplansky has initiated in [3] a study of infinite groups G all of whose irreducible representations are of bounded degree. It was shown in [3] that if G contains a normal abelian subgroup of finite index, then all the irreducible representations of G are of bounded degree, and a bound was obtained utilizing identities of finite matrix rings and the theory of Banach algebras.

With the additional information we have about identities of matrix rings and of discrete group algebras, we are able to obtain more concrete results in this direction. In particular we obtain among others the result that if G contains any abelian (not necessarily normal) subgroup of index n in G, then all representations of G are of degree $\leqq n$.

The converse is not true even for $n = 2$. In this case we determine all groups whose irreducible representation are finite-dimensional of degree $\leqq 2$, and show that they belong to two types: (1) groups G having a normal abelian subgroup of index 2; (2) groups G having a center N such that G/N is an abelian 2-group of order 8.

1. Finite-dimensional representations

In what follows all representations are considered over fields of characteristic zero.

The following notations and results will be used:

$$[x_1, x_2, \cdots, x_k] = \sum \pm x_{i_1} x_{i_2} \cdots x_{i_k},$$

where x_i are noncommutative indeterminates and the sum ranges over all permutations $(i_1, \cdots, i_k)$ of the first k letters and the sign is positive for even permutations and negative for odd permutations. It is well known [4] that matrix rings F_n satisfy the identity $[x_1, x_2, \cdots, x_{2n}] = 0$ and no identities of lower degree.

Following [3], a group G is said to have the property P_k if for any k elements $g_1, \cdots, g_k$ of G, the $k!/2$ products $g_{i_1} g_{i_2} \cdots g_{i_k}$ obtained from all even permutations is identical with the $k!/2$ products obtained from the odd permutations. The fact that G has property P_k is equivalent to the group ring $F[G]$ (for arbitrary field of characteristic zero) satisfying the identity

$$[x_1, x_2, \cdots, x_k] = 0.$$

Let V be a finite-dimensional vector space over some field C, and let it be also a representation space of G. V determines an *absolutely irreducible representation* of G, if $V \otimes_C F$ is G-irreducible for all field extensions F of C.

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A set $\{V_\alpha\}$ of representation spaces of G is said to be a *complete* set of representations of G if, for every element $\sum n_g g$ in the group ring $Z[G]$ over the integers, there exists at least one space V_α for which $(\sum n_g g) V_\alpha \neq 0$.

Our first result is

THEOREM 1. *The following are equivalent:*

(A) *G has a complete set of representations (not necessarily irreducible) of degree $\leqq n$.*

(B) *For any field F (of characteristic zero), $F[G]$ satisfies the polynomial identity $[x_1, \cdots, x_{2n}] = 0$.*

(C) *Every primitive image of $F[G]$, for every F, is a central simple algebra of dimension $\leqq n^2$ over its center.*

(D) *Every absolutely irreducible representation of G is of degree $\leqq n$, and the set of all its absolutely irreducible representations is complete.*

Proof. For any space V over a field C, $\mathcal{E}(V)$ will denote the ring of all linear transformations of V. The field of all rationals will be denoted by Q, and as we consider only fields of characteristic zero, all fields will be assumed to contain Q.

Let $\{V_\alpha\}$ be a complete set of representation spaces (possible over different fields) of G and of degree $\leqq n$. Each representation determines a homomorphism $\phi_\alpha : Q[G] \to \mathcal{E}(V_\alpha)$, and if $\mathfrak{P}_\alpha =$ kernel of ϕ_α, then $Q[G]/\mathfrak{P}_\alpha$ is isomorphic with a subring of $\mathcal{E}(V_\alpha)$. The latter by the preceding remarks satisfies the identity $[x_1, \cdots, x_{2n}] = 0$ ([4]), and thus also $Q[G]/\mathfrak{P}_\alpha$ satisfies the same relation. The fact that $\{V_\alpha\}$ is complete is equivalent to $\bigcap \mathfrak{P}_\alpha = 0$; consequently (A) implies that $Q[G]$ satisfies the relation $[x_1, \cdots, x_{2n}] = 0$. Now for arbitrary F, $F[G] = Q[G] \otimes_Q F$, and thus it follows that (A) $\Rightarrow$ (B).

The result (B) $\Rightarrow$ (C) is a simple consequence of the fact that every primitive image of $F[G]$ satisfies the same identity, and therefore, by a result of Kaplansky, it is of the form described in (C) (e.g., [2], Theorem 1, p. 226).

Clearly (D) $\Rightarrow$ (A), so it remains to show that (C) $\Rightarrow$ (D). To this end we show how in our case all absolutely irreducible representations of G are obtained.

Let V be a primitive representation of $Q[G]$ (i.e., $Q[G]$ is homomorphic with an irreducible ring of endomorphisms of the abelian group V). If Δ is the centralizer of G in the ring of endomorphisms of V, then it follows in view of (C) that $Q[G]/\mathfrak{P} \cong \Delta_r$, where $\mathfrak{P}$ is the kernel of this representation and Δ is a central simple algebra of order s^2 over its center $C = C_\Delta$ with $rs = m \leqq n$. Actually, it is known that V, when considered as a vector space over Δ, is a space of dimension r and $Q[G]/\mathfrak{P} \cong \mathcal{E}_\Delta(V)$. Since C is the center of Δ, V can be considered as a vector space over C, and Δ will remain the centralizer of $C[G]$ in the ring of endomorphisms of V. Consequently, the above representation can be extended to a primitive representation of $C[G]$, and we still have

$$C[G]/\mathfrak{P}_1 \cong \mathcal{E}_\Delta(V) \cong \Delta_r,$$

with the new kernel $\mathfrak{P}_1$. Now let $\bar{C}$ be the algebraic closure of C; then $\Delta_r \otimes_C \bar{C} \cong \bar{C}_m$, and $V \otimes_C \bar{C} = V_1 \oplus \cdots \oplus V_s$, where each V_i is an $rs=m$-dimensional vector space over $\bar{C}$. The preceding decomposition is actually a decomposition of V to s isomorphic absolutely-G-irreducible representations. We prefer to attack this question in a different way: Since $\Delta_r \otimes \bar{C} \cong \bar{C}_m \cong \mathcal{E}(W)$, where W is any $\bar{C}$-space of dimension m, the homomorphism $\pi: G \to \Delta_r$ can be extended to a representation of G into the ring of linear transformations $\mathcal{E}(W)$. Thus W becomes a G-representation space, and, in fact, it is an absolutely irreducible G-space. Indeed, let $F \supseteq \bar{C}$; then

$$\mathcal{E}(W \otimes_{\bar{C}} F) \cong \mathcal{E}(W) \otimes_{\bar{C}} F \cong \Delta_r \otimes_C F.$$

Since the set of elements $\pi(G)$ contains a C-base of Δ_r, it follows that it contains also a base of $\mathcal{E}(W \otimes F)$; hence $W \otimes F$ is G-irreducible as required, and $(W:\bar{C}) = m \leqq n$. Note also that the kernel of the homomorphism of $Q[G]$ into $\mathcal{E}(W)$ is the same ideal $\mathfrak{P}$ we have started from, since the restriction of the representation to $Q[G]$ is actually the homomorphism $Q[G] \to \Delta_r$ whose kernel is $\mathfrak{P}$.

It follows from [1], Theorem 1 that in our case $Q[G]$ is semisimple; hence the intersection of all primitive ideals in $Q[G]$ is zero, and, consequently, the set of all absolutely irreducible representations obtained in the previous manner constitute a complete set. Obviously, all absolutely irreducible representations of G will constitute a complete set, and the second part of (D) is proved.

To prove the first part of (D), we consider an arbitrary vector space V of a field C which is an absolutely irreducible representation of G. Clearly V can be considered as a primitive representation of the group ring $C[G]$, and if $\mathfrak{P}$ is the primitive ideal which is the kernel of the homomorphism π of $C[G]$ into the ring of endomorphisms of V, then as before it follows by (C) that $C[G]/\mathfrak{P} \cong \Delta_r$, where Δ is the centralizer of G (and of $C[G]$) in the ring of endomorphisms of V. Furthermore, $(V:\Delta) = r$ and $(\Delta:F) = s^2$, F the center of Δ, and $rs = m \leqq n$. We wish to show that $\Delta = C$. Indeed, let K be any subfield of Δ containing F $(\supseteq C)$. Since $K \subseteq \Delta$, the homomorphism $\rho: V \otimes_C K \to V$ defined by $\rho(v \otimes k) = vk$ (note that $k \in \mathcal{E}(V)$ so that $vk \in V$) is actually a G-homomorphism, where $g(v \otimes k) = gv \otimes k$ for $g \in G$. Since V is absolutely irreducible, ρ must be an isomorphism onto, and since K is C-free, this holds only if $K = C$. This implies that actually we have $\Delta = F = C$, and consequently V is an r-dimensional C-space, and $r \leqq n$ as required in (D). This completes the proof of our theorem.

We are now in position to prove

THEOREM 2. *Let H be a subgroup of G of finite index. If H has a complete set of representations of degree $\leqq m$, then G has a complete set of representations of degree $\leqq m(G:H)$.*

Proof. Let $\{W_\alpha\}$ be a complete set of representations of H. Consider the spaces $V_\alpha = Q[G] \otimes_H W_\alpha$, where the tensor product is taken with respect to

$Q[H]$. Each V_α is a vector space over the field C_α over which W_α is a space of dimension $\leqq m$, by defining $c(a \otimes w) = a \otimes cw$ for all $c \in C_\alpha$, $a \in Q[G]$ and $w \in W_\alpha$; and if $G = \bigcup_{i=1}^{r} g_i H$ where $r = (G:H)$, then

$$V_\alpha = g_1 \otimes W_\alpha \oplus \cdots \oplus g_r \otimes W_\alpha,$$

and thus $(V_\alpha : C_\alpha) \leqq rm$. V_α is also a G-representation space by setting $g(a \otimes w) = ga \otimes w$.

This set $\{V_\alpha\}$ constitute a complete set of representations of G. Indeed, every element $\sum n_g g \in Z[G]$ can be written in the form $\sum g_i x_i$ with $x_i \in Z[H]$; let $x_1 \neq 0$; then by assumption $x_1 W_\alpha \neq 0$ for some W_α, and thus

$$(\sum g_i x_i) V_\alpha \supseteq (\sum g_i x_i)(1 \otimes W_\alpha) = \sum g_i \otimes x_i W_\alpha \neq 0$$

since the sum is direct.

An immediate consequence of Theorems 1 and 2 is the following generalization of a result of Kaplansky ([3], Theorem 1):

COROLLARY 1. *Let H be a maximal abelian subgroup of G; if $(G:H) < \infty$, then all absolutely irreducible representations of G are $\leqq (G:H)$.*

Indeed, $Q[H]$ is semisimple (by [1]) and commutative; hence all its primitive images constitute a complete set of 1-dimensional representations. The corollary follows now from Theorem 2 and (D) of Theorem 1.

The converse of this corollary is not true as will be shown in the next section.

The following simple observation is of interest:

COROLLARY 2. *If G has a complete set of representations of degree $\leqq n$, then G has a faithful representation as a group of linear transformations of an n-dimensional space over a commutative ring.*

For let $\{V_\alpha\}$ be a complete set of representations of G, and V_α a vector space over a field C_α; then set $R = \sum C_\alpha$ the direct sum, and let V be an n-dimensional R-module. If e_α is the projection of R onto C_α, one can replace V_α by $e_\alpha V$ in an obvious way (with a possibility of increasing the dimension) and clearly turn V into a G-module on which G acts faithfully.

2. Groups with representations of degree $\leqq 2$

The object of this section is to prove the following.

THEOREM 3. *A group G has all its absolutely irreducible representations of degree $\leqq 2$ if and only if it is one of the following types:*

(A) *G is abelian,*

(B) *G has an abelian subgroup H, and $(G:H) = 2$,*

(C) *G has a center N, and G/N is an abelian 2-group of order* 8.

Proof. If G is abelian, all of its absolutely irreducible representations are of degree 1; for groups of type (B) this result is a special case of Theorem 2.

To prove the third case we shall show that these groups satisfy (B) of Theorem 1, namely, that $[x_1, x_2, x_3, x_4] = 0$ holds in $Q[G]$.

Since N is the center of G, it suffices to show that the preceding relation holds for x_i constituting different representatives of classes modulo N and not in N, since if two of the x_i represent the same class, the above relation evidently holds. Now let all $x_i N$ be different. As G/N is a 2-group, it follows that the group of classes generated by x_1 and x_2 consists of N and the three classes $x_1 N$, $x_2 N$, $x_1 x_2 N$; hence x_3 and x_4 cannot both belong to these classes, and so G/N is generated by x_1, x_2, and one of the other x_i. We have, therefore, to consider only the following two cases:

$$(1) \quad x_1 = a, \quad x_2 = b, \quad x_3 = c, \quad \text{and} \quad x_4 = ab, \qquad a, b, c \in G,$$

$$(2) \quad x_1 = a, \quad x_2 = b, \quad x_3 = c, \quad \text{and} \quad x_4 = abc.$$

First we observe that since G/N is commutative and a 2-group, we have the following relations:

$$(*) \qquad a^2 = \alpha, \quad b^2 = \beta, \quad c^2 = \gamma, \quad ba = abu, \quad ca = acv, \quad cb = bcw,$$

with α, β, γ, u, v, and w belonging to N. Note also that since $a^2, b^2, c^2 \in N$ and N is in the center, we must have $u^2 = v^2 = w^2 = e$.

Consider first case (2): $[a, b, c, abc]$ will consist of 24 terms which we divide into six groups each of 4 terms obtained from a product $abc \cdot y_1 \cdot y_2 \cdot y_3$ by cyclic permutation, and the six groups are obtained when y_1, y_2, y_3 range over all six permutations of a, b, c. Next note that in each class the cyclic permutation $y_3 \cdot abc \cdot y_1 \cdot y_2$ etc. changes the parity of the corresponding permutation; hence each class will contain two even and two odd permutations, but not considering the sign they are the same element, e.g.,

$$abc \cdot c \cdot b \cdot a = a \cdot abc \cdot c \cdot b = \cdots = \alpha\beta\gamma.$$

Consequently one readily observes that $[a, b, c, abc] = 0$.

To compute the first case we observe that

$$\begin{aligned}
[a, b, c, ab] &= [a, b][c, ab] - [a, c][b, ab] + [a, ab][bc] \\
&\qquad + [b, c][a, ab] - [b, ab][a, c] + [c, ab][a, ab] \\
&= abcab(1 - u)(1 - vw) - acbab(1 - v)(1 - u) \\
&\qquad + a^2b^2c(1 - u)(1 - w) + bca^2b(1 - w)(1 - u) \\
&\qquad - babac(1 - u)(1 - v) + caba^2b(1 - vw)(1 - u) \\
&= \alpha\beta c(1 - u)\{uwv(1 - vw) - uv(1 - v) + (1 - w) \\
&\qquad + w(1 - w) - u(1 - v) + u(1 - vw)\} \\
&= 0.
\end{aligned}$$

The proof of the converse of our theorem involves using a different identity which should hold in $Q[G]$ if all representations of G are of degree $\leqq 2$. In this case, since $Q[G]$ is semisimple, it must satisfy also all identities of the ring of all 2×2 matrices over Q, and in particular, the Wagner-Hall identity $\delta = [[xy]^2, z] = 0$; or in other words, for arbitrary $x, y \in Q[G]$,

$$[x, y]^2 = (xy - yx)^2$$

must belong to the center of G.

Another fact which will be used later is that an element $\sum n_g g \in Z[G]$ belongs to the center if and only if $n_g = n_{hgh^{-1}}$ for all $h, g \in G$.

We linearize δ by considering $[x + ty, z]^2 - [x, z]^2 - t^2[y, z]$, and we obtain that in $Q[G]$ the element

$$\begin{aligned} \delta &= [x, z][y, z] + [y, z][x, z] \\ &= xzyz + zxzy + yzxz + zyzx - xz^2y - zxyz - yz^2x - zyxz \end{aligned} \tag{2.1}$$

belongs to the center of $Q[G]$.

Put in (2.1) $x = ab^{-1}c^{-1}$, $y = bc^{-1}$ and $z = c$ $(a, b, c \in G)$; then (2.1) takes the form

$$\begin{aligned} \delta = a + cac^{-1} + bab^{-1} + (cb)a(cb)^{-1} - a\{b^{-1}, c\} - cac^{-1}\{c, b^{-1}\} \\ - \{b, c\}(cb)a(cb)^{-1} - \{c, b\}bab^{-1}, \end{aligned} \tag{2.2}$$

where $\{u, v\} = uvu^{-1}v^{-1}$ for $u, v \in G$. Let N_u denote the centralizer of $u \in G$ in G. We choose first $b \in N_a$ (i.e., $ba = ab$); then (2.2) takes the form

$$\delta_1 = 2a + 2cac^{-1} - a\{b^{-1}, c\} - cac^{-1}\{c, b^{-1}\} - \{b, c\}cac^{-1} - \{c, b\}a, \tag{2.2a}$$

and if also $c \in N_a$, then we get

$$\delta_2 = 4a - a\{b^{-1}, c\} - a\{c, b\} - \{b, c\}a - \{c, b\}a. \tag{2.2b}$$

Now if $\delta_2 \neq 0$ for some b and c, and since δ_2 belongs to the center of $Q[G]$' it follows that a must belong to the center N of G. If $\delta_2 = 0$ for all $b, c \in N_a$, then a must be equal to the elements with negative sign, from which one readily deduces that $bc = ca$. Summarizing, we have shown

LEMMA 1. *Under the condition of Theorem 3, if $a \notin N$-center of G, then its centralizer N_a is abelian.*

Next we want to show under the same condition

LEMMA 2. *If $a \notin N$, then either $(G:N_a) = 2$, or $N_a = N \cup Na$.*

Here we utilize (2.2a). Let $c \notin N_a$. If $\delta_1 \neq 0$ for some c, then again from the form of the elements of the center of $Z[G]$ and from the fact that δ_1 belongs to this center, it follows that either $a \in N$, or a has exactly two conjugates, and hence $(G:N_a) = 2$. If this is not the case, then $\delta_1 = 0$ for all $c \notin N_a$. It follows now from (2.2a) that one of the following holds:

(1) $a = a\{b^{-1}, c\}$ or $a = \{c, b\}a$,

(2) $a = cac^{-1}\{c, b^{-1}\}$ or $a = \{b, c\}cac^{-1}$.

The first case yields $bc = cb$, but since $ba = ab$ (i.e., $a, c \in N_b$), and $ac \neq ca$, it follows by Lemma 1 that $b \in N$.

In the second case we get $a = cab^{-1}c^{-1}b$ or $a = bcb^{-1}ac^{-1}$, which yield $ab^{-1}c = cab^{-1}$ or $(b^{-1}a)c = c(b^{-1}a)$. There both $b^{-1}a$ and $a^{-1}b$ commute with a since $b \in N_a$, so $c, a \in N_{b^{-1}a}$ or $N_{a^{-1}b}$; but as $ac \neq ca$, it follows again by Lemma 1 that $a^{-1}b$ or $b^{-1}a \in N$, and in any case $b \in Na$. The above argument being true for all $b \in N_a$ shows that $N_a = N \cup Na$, which concludes the proof of the lemma.

An immediate consequence of Lemma 2 is

LEMMA 3. *If G satisfies the condition of Theorem 3 and it is not of the type (A) or (B), then G/N is an abelian 2-group.*

Indeed, in this case for $a \notin N$, $N_a = N \cup Na$, and as $a^2 \in N_a$ we must have $a^2 \in N$. Furthermore, since

$$aba^{-1}b^{-1} = (ab)^2 b^{-1}a^{-2}b^{-1} = (ab)^2 b^{-2}a^{-2},$$

it follows that G/N is abelian.

To complete the proof of Theorem 3 we have to show that G/N has at most three generators, and to this end we linearize (2.1) one step more and obtain that the following element belongs to the center:

$$\delta = [x, u][y, v] + [x, v][y, u] + [y, u][x, v] + [y, v][x, u]. \tag{2.3}$$

Set $a = ab^{-1}$, $u = b$, $y = c$, $v = c^{-1}$ where $a, b, c \in G$; then

$$\begin{aligned}\delta_3 = a + c^{-1}ac + bab^{-1} + (cb)a(cb)^{-1} - ab^{-1}c^{-1}bc \\ - c^{-1}ab^{-1}cb - cbc^{-1}ab^{-1} - bcab^{-1}c^{-1}.\end{aligned} \tag{2.3a}$$

Assume that G is not of the type (A) or (B); then clearly in view of Lemma 3, G/N cannot be generated by two elements a, b since then G/N is of order 4 and $(G:N_a) = 2$. So let us choose a, b two elements (not in N) representing different classes in G/N (so that $b \notin N_a$), and choose

$$c \notin N \cup aN \cup bN \cup abN.$$

In this case we want to show that δ_3 of (2.3a) is never zero.

Indeed, if $\delta_3 = 0$, then we must have one of the following cases:

(1) $a = ab^{-1}c^{-1}bc$, which yields $bc = cb$; but then in view of Lemma 2, $c \in N \cup bN$, a contradiction.

(2) $a = c^{-1}ab^{-1}cb$; then $ab^{-1}c = cab^{-1}$ which implies by Lemma 2 that $c \in N \cup ab^{-1}N = N \cup abN$ since $b^{-1} \in bN$. Contradiction!

(3) $a = cbc^{-1}ab^{-1}$ which implies $c^{-1}ab^{-1} = b^{-1}c^{-1}a$; thus for the same reason $c^{-1}a \in N \cup bN$, and therefore $c \in aN \cup abN$ as $c^{-1}a \in ca^{-1}N$. Impossible!

(4) $a = bcab^{-1}c^{-1}$. This gives $acb = bca$. Multiplying both sides on the right by a and noting that a^2 belongs to the center, we get

$$(ac)(ba) = bca^2 = (ba)(ac),$$

so that for the same reason we obtain $ac \in N \cup baN$, and consequently $c \in aN \cup bN$, which is again impossible.

Now if $\delta_3 \neq 0$ for all c not in the above class, it follows from the fact that $\delta_3 \in$ center of $Q[G]$ that all conjugates of a are bab^{-1}, $c^{-1}ac$, and $(cb)a(cb)^{-1}$ for the b, c chosen in the preceding argument. Take now $d \in G$; then dad^{-1} must be one of the preceding conjugates. Thus if $dad^{-1} = a$, it follows that $d \in N \cup aN$; if $dad^{-1} = bab^{-1}$, then $b^{-1}d \in N \cup aN$, so that $d \in bN \cup abN$; if $dad^{-1} = c^{-1}ac$, we get $cd \in N \cup aN$, and thus $d \in cN \cup acN$. In the last case where $dad^{-1} = cba(cb)^{-1}$, we get $(cb)^{-1}d \in N \cup aN$, so that $d \in bcN \cup abcN$. Consequently a, b, and c generate the classes of G/N, and the proof of Theorem 3 is completed.

We conclude with a remark that one can construct groups of the type (C) which are not of type (A) and (B), and thus the converse of Theorem 2 does not hold. These groups are constructed by taking abelian groups N possessing three elements u, v, w such that $u^2 = v^2 = w^2 = e$ but none of the products u, v, w, uv, uw, vw, uvw equals e, and constructing G by the relations given in $(*)$ for arbitrary $\alpha, \beta, \gamma \in N$.

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