ONE-DIMENSIONAL TOPOLOGICAL SEMILATTICES

BY

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1. Introduction

In [5] A. D. Wallace proved that a compact, connected mob with zero and unit has trivial cohomology groups for $n > 0$. It is implicit in this result that if such a mob is one-dimensional[2] and locally connected, then it is a tree. For, if X is a continuum, $\dim X = 1$, and $H^1(X) = 0$, then X is hereditarily unicoherent; thus, if X is locally connected, it is a tree [8]. In the main theorem of this note we modify Wallace's result so as to eliminate the necessity of hypothesizing a unit. Specifically, we prove

THEOREM. *A compact, connected, locally connected, one-dimensional, idempotent, commutative mob is a tree.*

2. Preliminaries

A *topological semilattice* (= TSL) is an idempotent commutative mob. A TSL can be endowed with a natural partial ordering by letting $x \leqq y$ if $xy = x$. Thus $xy = \text{g.l.b.}\ (x, y)$, denoted hereafter by $x \wedge y$, and this partial ordering is continuous in the sense that its graph $(= \{(x, y) : x \leqq y\})$ is closed. It is easy to see that a compact TSL is $\wedge$-complete and therefore has a zero. Also a $\wedge$-complete TSL with unit is an algebraic lattice (but not necessarily topological).

A *tree* is a continuum (= compact connected Hausdorff space) in which every two points are separated by a third point. A tree admits a partial ordering as follows: Select a point x_0, and define $x \leqq y$ if and only if $x = x_0$, or $x = y$, or x separates x_0 and y. This partial ordering is called the *cutpoint ordering* of a tree [6]. We recall [7] that a compact Hausdorff space X is a tree if, and only if, X admits a partial ordering, $\leqq$, such that for each $a, b \in X$

(i) $L(a)$ and $M(a)$ are closed,[3]

(ii) if $a < b$, then there exists $c \in X$ with $a < c < b$,

(*) (iii) $L(a) \cap L(b)$ is a nonvoid chain,

(iv) $M(a) - \{a\}$ is open.

3. Proof of the theorem

Throughout this section, S will denote a compact, connected, locally con-

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[2] The dimension function employed throughout this note is *codimension* as expounded by Haskell Cohen [3]. For a compact Hausdorff space, the codimension (with the integers as coefficient group) and the covering dimension agree.

[3] In a partially ordered set we write $L(a) = \{x : x \leqq a\}$ and $M(a) = \{x : a \leqq x\}$.

nected, one-dimensional TSL. The proof of the theorem will be accomplished by a series of lemmas.

LEMMA 1. *If* $x \in S$, *then there is a unique closed connected chain* $C(0, x)$ *such that* 0 *and* x *are elements of* $C(0, x)$ *and* $C(0, x) \subset L(x)$.

Proof. Since $L(x)$ has a unit, it is a tree. Let $\Gamma_c(x)$ denote the graph of the cutpoint ordering of the tree $L(x)$. If $(a, b) \in \Gamma_c(x)$, then $a = 0$, or $a = b$, or a separates 0 and b in $L(x)$. In the first two cases $a \leqq b$ is obvious. If a separates 0 and b in $L(x)$, then, since $L(b)$ is a connected subset of $L(x)$ containing 0 and a, again we have $a \leqq b$. Thus $\Gamma_c(x)$ is a subset of $\Gamma(x)$, the graph of the semilattice ordering of $L(x)$. Since there is a closed connected chain from 0 to x in the cutpoint ordering of the tree $L(x)$, so there is also one in the semilattice ordering. The uniqueness of this chain follows from the fact that $L(x)$ is hereditarily unicoherent.

We define a new relation $\triangle$ on the elements of S by $x \triangle y$ if and only if $x \leqq y$ and there exists a closed connected chain $C(x, y)$ such that x and y are elements of $C(x, y)$ and $C(x, y) \subset M(x) \cap L(y)$. It is clear that $\triangle$ is an order-dense partial order, and by Lemma 1, $0 \triangle x$ for each $x \in S$. Moreover $C(x, y)$ is unique for each x and y in S such that $x \triangle y$.

In order to distinguish between these relations let

$$L_\triangle(x) = \{y \in S : y \triangle x\},$$

$$M_\triangle(x) = \{y \in S : x \triangle y\},$$

$$L_c(x; y) = \{z \in L(y) : (z, x) \in \Gamma_c(y)\},$$

$$M_c(x; y) = \{z \in L(y) : (x, z) \in \Gamma_c(y)\}.$$

LEMMA 2. *If* $y \in S$, *then the cutpoint ordering of* $L(y)$ *is identical with* $\triangle$ *on* $L(y)$.

Proof. It is sufficient to prove that $M_c(x; y) = M_\triangle(x) \cap L(y)$ for $x \in L(y)$. If $p \in M_c(x; y)$, then $x = 0$, or $x = p$, or x separates 0 and p in $L(y)$. If $x = 0$, then $x \triangle p$ by Lemma 1. If $x = p$, then $x \triangle p$ is trivial. If x separates 0 and p in $L(y)$, then $x \in C(0, p)$, and hence $C(x, p) = M(x) \cap C(0, p)$ is the desired chain. In any event, $M_c(x; y) \subset M_\triangle(x) \cap L(y)$. To prove the reverse inclusion suppose $p \in M_\triangle(x) \cap L(y)$, i.e., there exists a closed connected chain $C(x, p)$. By the uniqueness of $C(0, p)$ and the existence of $C(0, x)$ it follows that $C(x, p) \subset C(0, p)$, and hence $x \in C(0, p)$. Since $L(y)$ is a tree, this implies that x separates 0 and p in $L(y)$.

LEMMA 3. *If* $y \in S$, *then* $L_\triangle(y)$ *is a closed chain containing* 0.

Proof. By Lemma 2, $L_\triangle(y) = L_c(y; y) = C(0, y)$.

LEMMA 4. *If* $y \in S$, *then* $M_\triangle(y)$ *is a closed set.*

Proof. Let $x \in (M_\triangle(y))^*$ and choose open sets U and V such that

$x \in V \subset U$, $V \wedge V \subset U$. If $z \in V \cap M_\triangle(y)$, then $x \wedge z \in U \cap L(x)$, and there exists a connected chain $C(y, z) \subset M_\triangle(y)$. It follows that $x \wedge C(y, z) = C(x \wedge y, x \wedge z) = C(y, x \wedge z)$ since $x \in (M_\triangle(y))^* \subset M(y)$. Therefore $x \wedge z \in U \cap L(x) \cap M_\triangle(y)$, and hence $x \in (L(x) \cap M_\triangle(y))^*$. Since $L(x) \cap M_\triangle(y) = M_c(y; x)$, a closed subset of $L(x)$, we have $x \in M_\triangle(y)$.

LEMMA 5. *If $x \in S$, then $M_\triangle(x) - \{x\}$ is open.*

Proof. If $y \in M_\triangle(x) - \{x\}$, then by Lemma 2, $L(y) \cap (M_\triangle(x) - \{x\})$ is open in the tree $L(y)$. Define $f\colon S \to L(y)$ by $f(z) = y \wedge z$. Since S is locally connected and f is continuous, there is a connected open set U such that

$$y \in U \subset U^* \subset f^{-1}(L(y) \cap (M_\triangle(x) - \{x\})).$$

Suppose there exists $z \in U^* - (M_\triangle(x) - \{x\})$. Then $x \in S - C(0, z)$, and since $C(0, z)$ and $C(0, y \wedge z)$ have a nonempty intersection, there exists

$$w = \sup(C(0, z) \cap C(0, y \wedge z)).$$

Since $z \in U^*$, we have $z \wedge y \in M_\triangle(x) - \{x\}$; therefore there is a connected chain $C(x, z \wedge y)$, and thus $x \in C(0, z \wedge y)$. Because $x \in S - C(0, z)$, it follows that $w < x$ and $x \in C(w, y \wedge z)$. Moreover, $C(w, z) \cup C(w, y \wedge z)$ is an irreducible continuum between z and $y \wedge z$. Since $L(z)$ is hereditarily unicoherent, it follows that $(C(w, z) \cup C(w, y \wedge z)) \cap (z \wedge U^*)$ is connected. Since $z \in z \wedge U^*$ and $y \wedge z \in z \wedge U^*$, we infer that

$$C(w, z) \cup C(w, y \wedge z) \subset z \wedge U^*,$$

and hence $x \in z \wedge U^*$.

We have proved that if $x \in U^* - (M_\triangle(x) - \{x\})$ then $x \in z \wedge U^*$. Now $y \wedge U^* \subset S - \{x\}$, an open set, and hence there is an open set V with $y \in V$ such that $V \wedge U^* \subset S - \{x\}$. In particular, $V \cap U$ is an open set containing y and $V \cap U \subset M_\triangle(x) - \{x\}$. (Otherwise $z \in V \cap U - (M_\triangle(x) - \{x\})$ implies $x \in z \wedge U^* \subset V \wedge U^* \subset S - \{x\}$.) Therefore $M_\triangle(x) - \{x\}$ is an open set.

Lemmas 1–5 show that the relation $\triangle$ satisfies all of the conditions (*), and hence S is a tree.

4. Order-dense and locally order-dense TSL's

A partially ordered set P is order-dense if for each $x, y \in P$ such that $x < y$ there exists $z \in P$ such that $x < z < y$. A subset C of P is *convex* if $x, y \in C$ implies $M(x) \cap L(y) \subset C$. A POTS[4] is *locally order-dense* (*locally convex*) if the topology has a base consisting of order-dense (convex) sets.

Nachbin [4] has observed that every compact POTS is locally convex, and thus a compact TSL is locally convex. In [2] it is shown that a locally com-

[4] A POTS is a partially ordered topological space, i.e., a topological space S with a partial order such that $L(x)$ and $M(x)$ are closed sets, for each $x \in S$.

pact connected topological lattice is locally convex. It is not known if this result is valid if "topological lattice" is replaced by "TSL".

It is known [1] that a locally convex connected topological lattice is locally connected. This is not true in a TSL (e.g., see Example 1).

However, we have the

LEMMA. *A locally compact, locally convex, locally order-dense TSL is locally connected.*

Proof. Let S satisfy the conditions of the lemma, and let $x \epsilon U \subset S$ with U an open set. Let U_1, U_2, U_3, and U_4 be open sets containing x such that U_1 is order-dense, U_2^* is compact, U_3 is convex, and

$$U_4 \wedge U_4 \subset U_3 \subset U_2 \subset U_2^* \subset U_1 \subset U.$$

Now if $y, z \epsilon U_4$, then $L(y) \cap M(y \wedge z) \subset U_3$ and is a compact order-dense POTS with zero and hence is connected [6]. Thus S is locally connected.

The following corollary follows directly from the theorem.

COROLLARY 1. *A compact, connected, locally order-dense, one-dimensional TSL is a tree.*

It is easy to see that a locally convex, order-dense TSL is locally order-dense. Thus we have

COROLLARY 2. *A compact, order-dense, one-dimensional TSL is a tree. Moreover the cutpoint ordering agrees with the semilattice ordering.*

5. Examples

Each of the following examples is a subset of the Euclidean plane with the usual topology. The semilattice operation in all cases is given by $(x, y) \wedge (x', y') = (\min(x, x'), \min(y, y'))$.

Example 1. For each positive integer n let

$$A_n = \{(x, y) : x = 1/n \text{ and } 0 \leqq y \leqq 1\};$$

$$B = \{(x, y) : 0 \leqq x, y \leqq 1 \text{ and } xy = 0\}.$$

Setting $S = B \cup \bigcup_{n=1}^{\infty} \{A_n\}$ we have a TSL which is compact and connected but not locally connected.

Example 2. For each positive integer n let

$$A_n = \{(x, y) : (n - 1)/n \leqq x \leqq 1 \text{ and } y = (n - 1)/n\};$$

$$B = \{(x, y) : 0 \leqq x \leqq 1 \text{ and } y = x\}.$$

Then $S = B \cup \bigcup_{n=1}^{\infty} \{A_n\}$ is a compact connected locally connected TSL which is not locally order-dense. We observe that S is a distributive lattice but is not a topological lattice.

Example 3. If $S = \{(x, y): 0 \leqq x, y \leqq 1 \text{ and } xy = 0 \text{ or } y = 1\}$, then S is a locally connected, locally order-dense TSL which is not order-dense.

Example 4. For each positive integer n let

$$A_n = \{(x, y): y = 0 \text{ and } 1/(n+1) < x < 1/n\},$$

and set $S = \{(0, 0)\} \cup \bigcup_{n=1}^{\infty} \{A_n\}$. Then S is a locally convex, locally order-dense TSL which is not locally connected and not locally compact.

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