BEHAVIOR OF INTEGRAL GROUP REPRESENTATIONS UNDER GROUND RING EXTENSION

BY

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1. Let K be an algebraic number field, and let R be a subring of K containing 1 and having quotient field K. Of primary interest will be the cases

(i) $R = K$,

(ii) $R =$ alg. int. $\{K\}$, the ring of all algebraic integers in K.

(iii) $R =$ valuation ring of a discrete valuation of K.

Given a finite group G, we denote by RG its group ring over R. By an RG-*module* we shall mean a left RG-module which as R-module is finitely generated and torsion-free, and upon which the identity element of G acts as identity operator. Each RG-module M is contained in a uniquely determined smallest KG-module

$$K \otimes_R M,$$

hereafter denoted by KM. For a pair M, N of RG-modules, we write

$$M \sim_R N$$

to denote the fact that $M \cong N$ as RG-modules. The notation

$$M \sim_K N$$

shall mean that $KM \cong KN$ as KG-modules.

Now let K' be an algebraic number field containing K, and let R' be a subring of K' which contains 1 and has quotient field K'. Suppose further that R' is a finitely generated R-module such that

$$R' \cap K = R.$$

Each RG-module M then determines an $R'G$-module denoted by $R'M$, given by

$$R'M = R' \otimes_R M.$$

If M, N are a pair of RG-modules, we write $M \sim_{R'} N$ if $R'M \cong R'N$ as $R'G$-modules. Surely

$$M \sim_R N \quad \Rightarrow \quad M \sim_{R'} N.$$

The reverse implication is false, as we shall see. We propose to investigate more closely the connection between R- and R'-equivalence.

As a first step we may quote without proof a well-known result [9, page 70] which is a consequence of the Krull-Schmidt theorem for KG-modules.

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THEOREM 1. *Let M, N be KG-modules, and let K' be an extension field of K. Then*

$$M \sim_{K'} N \quad \Rightarrow \quad M \sim_K N.$$

Remark. This result is valid for any pair of fields $K \subset K'$, even for those of nonzero characteristic.

COROLLARY. *If M, N are RG-modules, then*

$$M \sim_{R'} N \quad \Rightarrow \quad M \sim_K N.$$

2. An RG-module M is called *irreducible* if it contains no nonzero submodule of smaller R-rank. As is known [10], M is irreducible if and only if KM is irreducible as KG-module. Call M *absolutely irreducible* if for every field $K' \supset K$, the module $K'M$ is irreducible as $K'G$-module. Repeated use will be made of the following result [9, page 52]:

M is absolutely irreducible if and only if every KG-endomorphism of KM is given by a scalar multiplication

$$x \rightarrow ax, \qquad x \in KM,$$

for some $a \in K$.

As a first result, we prove

THEOREM 2. *Let R be a principal ideal ring, and let M, N be a pair of absolutely irreducible RG-modules. Then*

$$M \sim_{R'} N \quad \Rightarrow \quad M \sim_R N.$$

Proof. The preceding corollary shows that $M \sim_K N$. After replacing N by some new RG-module which is RG-isomorphic to N, we may in fact assume that $M \supset N$.

The isomorphism $R'M \cong R'N$ can be extended to an isomorphism $K'M \cong K'N$. As a consequence of the absolute irreducibility of M, and the fact that $K'M = K'N$, this latter isomorphism must be given by a scalar multiplication. Consequently there exists a scalar $\alpha \in K'$ such that

$$R'N = \alpha \cdot R'M. \tag{1}$$

Since R is a principal ideal ring, we may find an R-basis $\{m_1, \cdots, m_k\}$ of M, and nonzero elements $a_1, \cdots, a_k \in R$, such that

$$M = Rm_1 \oplus \cdots \oplus Rm_k, \tag{2}$$

$$N = Ra_1 m_1 \oplus \cdots \oplus Ra_k m_k. \tag{3}$$

Then

$$R'M = \sum R'm_i, \qquad R'N = \sum R'a_i m_i = \sum R'\alpha m_i. \tag{4}$$

Let $u(R')$ be the group of units of R', and $u(R)$ that of R. Then (4) implies the existence of $\beta_1, \cdots, \beta_k \in u(R')$ such that

$$a_i = \beta_i \alpha, \qquad 1 \leqq i \leqq k.$$

Therefore

$$a_i/a_1 = \beta_i/\beta_1 \in u(R'),$$

and so

$$b_i = a_i/a_1 \in u(R') \cap K = u(R).$$

Therefore

$$N = \sum Ra_i m_i = a_1 \sum Rb_i m_i = a_1 M,$$

which shows that N, M are R-equivalent, Q.E.D.

We next give an example to show that the result stated in Theorem 2 need not hold when R is not a principal ideal ring. Set

$$\mathfrak{o} = \text{alg. int. } \{K\}, \qquad \mathfrak{o}' = \text{alg. int. } \{K'\},$$

where $\mathfrak{o}$ is not a principal ideal ring. It is possible to choose K' so that for each ideal $\mathfrak{a}$ in $\mathfrak{o}$, the induced ideal $\mathfrak{o}'\mathfrak{a}$ in $\mathfrak{o}'$ is principal (see [4]). Now let M be any absolutely irreducible $\mathfrak{o}G$-module, $\mathfrak{a}$ any nonprincipal ideal in $\mathfrak{o}$, and set $N = \mathfrak{a}M$. Then M, N cannot be $\mathfrak{o}$-equivalent, since by the above remarks the isomorphism $M \cong N$ would imply that $N = aM$ for some $a \in K$. On the other hand,

$$\mathfrak{o}'N = \mathfrak{o}'\mathfrak{a}M = \alpha'\mathfrak{o}'M$$

for some $\alpha' \in K'$, and so M, N are $\mathfrak{o}'$-equivalent.

If M, N are $\mathfrak{o}G$-modules, we say that M, N are in the same *genus* (notation: $M \vee N$) if $RM \cong RN$ for each valuation ring R of a discrete valuation of K (see [5, 6]).

COROLLARY. *Let M, N be absolutely irreducible $\mathfrak{o}G$-modules. Then*

$$M \sim_{\mathfrak{o}'} N \Rightarrow M \vee N.$$

Proof. Let R be a valuation ring of a discrete valuation ϕ of K, and let ϕ' be an extension of ϕ to K', with valuation ring R'. Then R is a principal ideal ring, and so

$$M \sim_{\mathfrak{o}'} \mathrm{N} \Rightarrow \mathrm{M} \sim_{R'} \mathrm{N} \Rightarrow \mathrm{M} \sim_{R} \mathrm{N}$$

by Theorem 2, Q.E.D.

Maranda [5] showed that a pair of absolutely irreducible $\mathfrak{o}G$-modules M, N are in the same genus if and only if $M \cong \mathfrak{a}N$ for some $\mathfrak{o}$-ideal $\mathfrak{a}$ in K. But then $\mathfrak{o}'M \cong \mathfrak{o}'\mathfrak{a}N$, so M, N are $\mathfrak{o}'$-equivalent if and only if $\mathfrak{o}'\mathfrak{a}$ is a principal ideal in K'. Thus, the converse of the above corollary holds if and only if every ideal in $\mathfrak{o}$ induces a principal ideal in $\mathfrak{o}'$.

3. Throughout this section let R be the valuation ring of a discrete valuation ϕ of K, with unique maximal ideal P, and residue class field $\bar{K} = R/P$. Let ϕ' be an extension of ϕ to K', with valuation ring R', maximal ideal P', residue class field $\bar{K}' = R'/P'$. We shall give some *sufficient* conditions for the validity of the implication:

$$(5) \qquad M \sim_{R'} N \Rightarrow M \sim_R N,$$

where M, N denote RG-modules.

THEOREM 3. *If the group order* $(G:1)$ *is a unit in* R, *then* (5) *is valid.*

Proof. Use Theorem 1, together with the result [5] that if $(G:1)$ is a unit in R, then

$$M \sim_R N \quad \text{if and only if} \quad M \sim_K N.$$

THEOREM 4. *If* $\bar{K}' = \bar{K}$, *then* (5) *holds.*

Proof. Since R, R' are principal ideal rings, we may use matrix terminology. Let M, N be R-representations of G such that $M \sim_{R'} N$. Set

$$C = \{X \text{ over } R: M(g)X = XN(g),\ g \in G\},$$

$$C' = \{X \text{ over } R': M(g)X = XN(g),\ g \in G\}.$$

Since C is a finitely generated torsion-free R-module, we may choose an R-basis $\{X_1, \cdots, X_n\}$ of C. It is easily verified that this is also an R'-basis of C'.

The hypothesis $M \sim_{R'} N$ is equivalent to the statement that there exist elements $\alpha_1, \cdots, \alpha_n \in R'$ such that

$$\alpha_1 X_1 + \cdots + \alpha_n X_n$$

is unimodular over R', that is, has entries in R' and satisfies

$$| \alpha_1 X_1 + \cdots + \alpha_n X_n | \in u(R') \quad \text{(the group of units of } R').$$

Since $\bar{K}' = \bar{K}$, we may choose $a_1, \cdots, a_n \in R$ such that

$$a_i \equiv \alpha_i \pmod{P'}, \qquad 1 \leqq i \leqq n.$$

In that case,

$$a_1 X_1 + \cdots + a_n X_n \in C,$$

and is unimodular over R. Therefore $M \sim_R N$, Q.E.D.

In particular, suppose that K' is an *Eisenstein extension* of K relative to the valuation ϕ, that is, suppose that $K' = K(\alpha)$ where

$$\text{Irr}\,(\alpha, K) = x^m + b_1 x^{m-1} + \cdots + b_m$$

with $b_1, \cdots, b_m \in P$, $b_m \notin P^2$ (see [3]). In this case ϕ is uniquely extendable to K', and $\bar{K}' = \bar{K}$, so that (5) is true. We shall apply this later on.

Let us call a matrix of the form

$$\begin{bmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{bmatrix}$$

a *translation*; by such a notation, we mean to imply that the elements below the main diagonal are all zero. If M, N are R-representations of G, we write $M \approx N$ to indicate that M, N can be intertwined by a translation matrix.

On the other hand, suppose that

$$(6) \qquad M = \begin{bmatrix} M_1 & & * \\ & \ddots & \\ & & M_k \end{bmatrix}, \qquad N = \begin{bmatrix} M_1 & & * \\ & \ddots & \\ & & M_k \end{bmatrix}$$

are a pair of R-representations of G in which the $\{M_i\}$ are distinct (that is, not K-equivalent) and absolutely irreducible. If M, N can be intertwined by a matrix X over R of the form

$$(7) \qquad X = \begin{bmatrix} a_1 I & & * \\ & \ddots & \\ & & a_k I \end{bmatrix},$$

in which $a_i \in u(R)$, the group of units of R, then we shall say that M, N are *i-intertwinable*. Call M, N *everywhere intertwinable* if for each i, $1 \leqq i \leqq k$, M, N are i-intertwinable. Clearly if M, N are i-intertwinable, and if[2]

$$M \approx M', \qquad N \approx N',$$

then also M', N' are i-intertwinable.

LEMMA. *Let M, N be given by* (6), *and suppose the $\{M_i\}$ distinct and absolutely irreducible. Suppose that M, N are everywhere intertwinable, and further that they are intertwined by a matrix X given by* (7) *for which*

$$(8) \qquad a_1, \cdots, a_r \notin u(R), \qquad a_{r+1}, \cdots, a_k \in u(R).$$

Then

$$(9) \quad M \approx \left[\begin{array}{ccc|ccc} M_1 & & * & & & \\ & \ddots & & & 0 & \\ & & M_r & & & \\ \hline & & & M_{r+1} & & * \\ & & & & \ddots & \\ & & & & & M_k \end{array}\right], \qquad N \approx \left[\begin{array}{ccc|ccc} M_1 & & * & & & \\ & \ddots & & & 0 & \\ & & M_r & & & \\ \hline & & & M_{r+1} & & * \\ & & & & \ddots & \\ & & & & & M_k \end{array}\right].$$

Proof. Use induction on r. The result is trivial when $r = 0$, so assume $r \geqq 1$, and write

$$M = \begin{bmatrix} M_1 & * & * \\ & M' & \Lambda \\ & & M'' \end{bmatrix}, \qquad N = \begin{bmatrix} M_1 & * & * \\ & N' & \Delta \\ & & N'' \end{bmatrix},$$

[2] We use ${}^tM$ to denote the transpose of M; thus, M' is just another representation in this context.

where

$$M' = \begin{bmatrix} M_2 & & * \\ & \ddots & \\ & & M_r \end{bmatrix}, \quad M'' = \begin{bmatrix} M_{r+1} & & * \\ & \ddots & \\ & & M_k \end{bmatrix}, \quad \text{(submatrices of } M\text{)},$$

$$N' = \begin{bmatrix} M_2 & & * \\ & \ddots & \\ & & M_r \end{bmatrix}, \quad N'' = \begin{bmatrix} M_{r+1} & & * \\ & \ddots & \\ & & M_k \end{bmatrix}, \quad \text{(submatrices of } N\text{)}.$$

Then also

$$\begin{bmatrix} M' & \Lambda \\ & M'' \end{bmatrix}, \quad \begin{bmatrix} N' & \Delta \\ & N'' \end{bmatrix}$$

are everywhere intertwinable, and furthermore are intertwined by

$$\begin{bmatrix} a_2 I & & * \\ & \ddots & \\ & & a_k I \end{bmatrix},$$

a submatrix of X. It follows from the induction hypothesis that by transforming M, N by suitable translation matrices, we can make $\Lambda = \Delta = 0$. The new M, N will still be everywhere intertwinable, and also intertwinable by a new X for which (8) still holds.

Let us write

$$M = \left[\begin{array}{c|c|c} M_1 & * & \Lambda_{r+1} \cdots \Lambda_k \\ \hline & M' & 0 \\ \hline & & M'' \end{array}\right], \quad N = \left[\begin{array}{c|c|c} M_1 & * & \Delta_{r+1} \cdots \Delta_k \\ \hline & N' & 0 \\ \hline & & N'' \end{array}\right],$$

$$X = \left[\begin{array}{c|c|c} a_1 I & * & T_{r+1} \cdots T_k \\ \hline & X' & T \\ \hline & & X'' \end{array}\right], \quad X'' = \begin{bmatrix} a_{r+1} I & & * \\ & \ddots & \\ & & a_k I \end{bmatrix}.$$

Then

$$\begin{bmatrix} M' & 0 \\ & M'' \end{bmatrix}\begin{bmatrix} X' & T \\ & X'' \end{bmatrix} = \begin{bmatrix} X' & T \\ & X'' \end{bmatrix}\begin{bmatrix} N' & 0 \\ & N'' \end{bmatrix},$$

whence $M'T = TN''$. Since M', N'' have no common irreducible constituent, we conclude that $T = 0$.

It now follows that

$$(10) \qquad \begin{bmatrix} M_1 & \Lambda_{r+1} \\ & M_{r+1} \end{bmatrix}, \quad \begin{bmatrix} M_1 & \Delta_{r+1} \\ & M_{r+1} \end{bmatrix}$$

are R-representations intertwined by

$$\begin{bmatrix} a_1 I & T_{r+1} \\ & a_{r+1} I \end{bmatrix}. \tag{11}$$

This implies that

$$M_1 T_{r+1} + a_{r+1} \Lambda_{r+1} = a_1 \Delta_{r+1} + T_{r+1} M_{r+1},$$

and hence (since $a_{r+1} \in u(R)$),

$$\Lambda_{r+1} = b\Delta_{r+1} + M_1 U - UM_{r+1}, \qquad b = a_{r+1}^{-1} a_1 \notin u(R), \tag{12}$$

for some U over R. On the other hand, the hypothesis that M, N are 1-intertwinable guarantees the existence of a matrix of the form (11) which intertwines the representations given in (10), but for which the element playing the role of a_1 is a unit in R. Therefore we also have

$$\Delta_{r+1} = c\Lambda_{r+1} + M_1 V - VM_{r+1} \tag{13}$$

for some $c \in R$ and some V over R. Combining (12) and (13), we obtain

$$(1 - bc)\, \Lambda_{r+1} = M_1 W - WM_{r+1}$$

for some W over R. Since $(1 - bc) \in u(R)$, we conclude that

$$\Lambda_{r+1} = M_1 Y - YM_{r+1}$$

for some Y over R. Hence by a translation transformation of M, we can make $\Lambda_{r+1} = 0$. From (13) it follows that we can also make $\Delta_{r+1} = 0$ by a translation transformation of N. For this new M, N we must have $T_{r+1} = 0$.

But now we observe that

$$\begin{bmatrix} M_1 & \Lambda_{r+2} \\ & M_{r+2} \end{bmatrix}, \qquad \begin{bmatrix} M_1 & \Delta_{r+2} \\ & M_{r+2} \end{bmatrix}$$

are representations intertwined by

$$\begin{bmatrix} a_1 I & T_{r+2} \\ & a_{r+2} I \end{bmatrix}.$$

The above type of argument shows that we can make $\Lambda_{r+2} = \Delta_{r+2} = 0$, and therefore also T_{r+2} must be 0. By continuing this process, we establish the validity of (9), Q.E.D.

We may now prove one of the main results of this paper.

THEOREM 5. *Let M, N be RG-modules which are R'-equivalent, and suppose that the irreducible constituents of KM (which coincide with those of KN) are distinct from one another and are absolutely irreducible. Then also M, N are R-equivalent.*

Proof. Again use matrix terminology, and proceed by induction on the number k of irreducible constituents of KM. The result for $k = 1$ follows from Theorem 2; suppose it known up to $k - 1$, and let KM have k distinct absolutely irreducible constituents. There will be no confusion from our using M to denote both the module and the R-representation it affords. The R-representations of G afforded by the RG-modules M, N may be taken to be of the form[3]

$$(14)\qquad M = \begin{bmatrix} M_1 & & * \\ & \ddots & \\ & & M_k \end{bmatrix}, \qquad N = \begin{bmatrix} N_1 & & * \\ & \ddots & \\ & & N_k \end{bmatrix},$$

where the $\{M_i\}$ and $\{N_i\}$ are absolutely irreducible, and where

$$(15)\qquad M_i \sim_K N_i\,, \qquad M_i \nsim_K M_j\,, \qquad j \neq i, \quad 1 \leqq i \leqq k.$$

Since M, N are R'-equivalent, they are intertwined by a matrix X' unimodular over R'. From (15) we find readily (see [6]) that X' has the form

$$(16)\qquad X' = \begin{bmatrix} X'_1 & & * \\ & \ddots & \\ & & X'_k \end{bmatrix},$$

and necessarily each X'_i is also unimodular over R'. But we have then

$$(17)\qquad M_i X'_i = X'_i N_i\,, \qquad 1 \leqq i \leqq k,$$

so that M_i, N_i are R'-equivalent for each i. By the induction hypothesis it follows that for each i, $1 \leqq i \leqq k$, M_i and N_i are R-equivalent. Consequently for each i there exists a matrix Y_i unimodular over R which intertwines M_i and N_i. Setting $Y = \operatorname{diag}(Y_1, \cdots, Y_k)$, we deduce that

$$N \sim_R YNY^{-1} = \begin{bmatrix} M_1 & & * \\ & \ddots & \\ & & M_k \end{bmatrix} \qquad \text{(say)}.$$

Replacing N by YNY^{-1}, we may henceforth assume that $N_1 = M_1, \cdots, N_k = M_k$, that is, that M, N are given by (6).

From the R'-equivalence of M, N it follows that they are intertwined by a unimodular matrix X' over R', given by (16). Since now $M_i = N_i$, and M_i is absolutely irreducible, (17) implies that each X'_i is a scalar matrix, so that we may write

$$(18)\qquad X' = \begin{bmatrix} \alpha_1 I & & * \\ & \ddots & \\ & & \alpha_k I \end{bmatrix}, \qquad \alpha_1, \cdots, \alpha_k \in u(R').$$

Let us now set

$$R' = R\beta_1 \oplus \cdots \oplus R\beta_n\,, \qquad \beta_1 = 1, \quad n = (K':K).$$

[3] This really follows from [10].

Then we may write

$$X' = \sum_{\nu=1}^{n} X^{(\nu)}\beta_\nu , \qquad X^{(\nu)} \text{ over } R;$$

we note that

$$X^{(\nu)} = \begin{bmatrix} a_1^{(\nu)}I & & * \\ & \ddots & \\ & & a_k^{(\nu)}I \end{bmatrix}, \qquad 1 \leqq \nu \leqq n,$$

where

$$\alpha_i = \sum_\nu a_i^{(\nu)}\beta_\nu , \qquad a_i^{(\nu)} \in R. \tag{19}$$

Let us fix i, $1 \leqq i \leqq k$. Then $\alpha_i \in u(R')$, and so by (19) at least one of $a_i^{(1)}, \cdots , a_i^{(n)}$ is a unit in R. Since each $X^{(\nu)}$ intertwines M and N, and since $a_i^{(\nu)}$ occurs in the i^{th} diagonal block of $X^{(\nu)}$, we may conclude that M, N are i-intertwinable. This shows then that if M, N given by (6) are R'-equivalent, they must be everywhere intertwinable.

Since M, N are 1-intertwinable, there exists an X (over R) given by (7) which intertwines M and N, and for which $a_1 \in u(R)$. If also $a_2 , \cdots , a_k \in u(R)$, then X is unimodular over R, and so M, N are R-equivalent. For the remainder of the proof we may therefore suppose that not all of $a_2 , \cdots , a_k$ are units in R. Let us write

$$a_1 , \cdots , a_q \in u(R), \qquad a_{q+1} , \cdots , a_r \notin u(R), \qquad a_{r+1} , \cdots , a_s \in u(R) , \cdots .$$

Partition X accordingly, say

$$X = \begin{bmatrix} Y_1 & & * \\ & \ddots & \\ & & Y_t \end{bmatrix}, \quad Y_1 = \begin{bmatrix} X_1 & & * \\ & \ddots & \\ & & X_q \end{bmatrix}, \quad Y_2 = \begin{bmatrix} X_{q+1} & & * \\ & \ddots & \\ & & X_r \end{bmatrix}, \cdots .$$

Correspondingly partition M, N, say

$$M = \begin{bmatrix} \bar{M}_1 & \Lambda_{12} & \Lambda_{13} & & \\ & \bar{M}_2 & \Lambda_{23} & & \\ & & \bar{M}_3 & & * \\ & & & \ddots & \\ & & & & \bar{M}_t \end{bmatrix}, \quad N = \begin{bmatrix} \bar{N}_1 & \Delta_{12} & \Delta_{13} & & \\ & \bar{N}_2 & \Delta_{23} & & \\ & & \bar{N}_3 & & * \\ & & & \ddots & \\ & & & & \bar{N}_t \end{bmatrix}, \tag{20}$$

where

$$\bar{M}_1 = \begin{bmatrix} M_1 & & * \\ & \ddots & \\ & & M_q \end{bmatrix}, \quad \bar{N}_1 = \begin{bmatrix} M_1 & & * \\ & \ddots & \\ & & M_q \end{bmatrix}, \cdots .$$

By repeated use of the lemma, we may transform M, N by translations so as to make successively

$$\Lambda_{12} = \Delta_{12} = 0, \quad \Lambda_{23} = \Delta_{23} = 0, \quad \cdots , \quad \Lambda_{t-1,t} = \Delta_{t-1,t} = 0. \tag{21}$$

Such transformations do not affect the diagonal blocks of X, nor the R'-equivalence of M, N. We may therefore assume for the remainder of the proof that (21) holds. But in that case we see from (20) that

$$\begin{bmatrix} \bar{M}_1 & \Lambda_{14} \\ & \bar{M}_4 \end{bmatrix}, \quad \begin{bmatrix} \bar{N}_1 & \Delta_{14} \\ & \bar{N}_4 \end{bmatrix}$$

are R-representations of G, and again we may apply the lemma to conclude that M, N may be further transformed by translation matrices so as to make $\Lambda_{14} = \Delta_{14} = 0$, and so on. Continuing in this way, we find that

$$M \approx M' = \begin{bmatrix} \bar{M}_1 & & \Omega \\ & \ddots & \\ & & \bar{M}t \end{bmatrix}, \quad N \approx N' = \begin{bmatrix} \bar{N}_1 & & \Sigma \\ & \ddots & \\ & & \bar{N}t \end{bmatrix},$$

where $\Omega_{ij} = \Sigma_{ij} = 0$ whenever the diagonal entries of X associated with $\bar{M}_i$ are units, those with $\bar{M}_j$ nonunits, or vice versa. But we may then find a permutation matrix F such that

$$FM'F^{-1} = \begin{bmatrix} M^* & 0 \\ & M^{**} \end{bmatrix}, \quad FN'F^{-1} = \begin{bmatrix} N^* & 0 \\ & N^{**} \end{bmatrix},$$

where

$$M^* = \begin{bmatrix} \bar{M}_1 & & * \\ & \bar{M}_3 & \\ & & \ddots \end{bmatrix}, \quad M^{**} = \begin{bmatrix} \bar{M}_2 & & * \\ & \bar{M}_4 & \\ & & \ddots \end{bmatrix},$$

$$N^* = \begin{bmatrix} \bar{N}_1 & & * \\ & \bar{N}_3 & \\ & & \ddots \end{bmatrix}, \quad N^{**} = \begin{bmatrix} \bar{N}_2 & & * \\ & \bar{N}_4 & \\ & & \ddots \end{bmatrix}.$$

We now have

$$(22) \qquad M \sim_R \begin{bmatrix} M^* & 0 \\ & M^{**} \end{bmatrix}, \quad N \sim_R \begin{bmatrix} N^* & 0 \\ & N^{**} \end{bmatrix},$$

and so (since $M \sim_{R'} N$),

$$\begin{bmatrix} M^* & 0 \\ & M^{**} \end{bmatrix} \sim_{R'} \begin{bmatrix} N^* & 0 \\ & N^{**} \end{bmatrix}.$$

Since M^*, M^{**} have no common irreducible constituents, this latter equivalence implies that

$$M^* \sim_{R'} N^*, \qquad M^{**} \sim_{R'} N^{**}.$$

We may (at last) use the induction hypothesis to conclude from this that

$$M^* \sim_R N^*, \qquad M^{**} \sim_R N^{**}.$$

This, together with (22), implies that M, N are R-equivalent. Thus the theorem is proved.

4. We shall apply the preceding result to the case of p-groups.

THEOREM 6. *Let G be a p-group, where p is an odd prime. Let R be the ring of p-integral elements of the rational field Q. Suppose that K' is an algebraic number field, and R' any valuation ring of K' such that $R' \supset R$. Then for any pair of irreducible RG-modules M, N we have*

$$M \sim_{R'} N \;\Rightarrow\; M \sim_R N. \tag{23}$$

Proof. Set $(G:1) = p^m$, $m > 1$, and let ζ be a primitive $(p^m)^{\text{th}}$ root of 1 over Q. Let M, N be R'-equivalent irreducible RG-modules. As a first step, let us set $K_1 = K'(\zeta)$, and let R_1 be a valuation ring of K_1 such that $R_1 \supset R'$. Then since

$$M \sim_{R'} N \;\Rightarrow\; M \sim_{R_1} N,$$

we may now restrict our attention to K_1, R_1 instead of K', R'.

Next we note that

$$f(x) = \text{Irr}\,(\zeta, Q) = x^{p^{m-1}(p-1)} + x^{p^{m-1}(p-2)} + \cdots + x^{p^{m-1}} + 1,$$

and that $f(x+1)$ is an Eisenstein polynomial at the prime p. If we set $K_0 = Q(\zeta)$, it follows that K_0 contains a uniquely determined valuation ring R_0 such that $R_0 \supset R$, and further that the residue class fields corresponding to R_0, R coincide. We may therefore conclude from Theorem 4 that

$$M \sim_{R_0} N \;\Rightarrow\; M \sim_R N. \tag{24}$$

The proof will be complete as soon as we establish

$$M \sim_{R_1} N \;\Rightarrow\; M \sim_{R_0} N. \tag{25}$$

This is a consequence of Theorem 5, however, as we now proceed to demonstrate. The modules $R_0 M$, $R_0 N$ are (in general) no longer irreducible. Since K_0 is an absolute splitting field for G (see [1]), the irreducible constituents of $K_0 M$ and $K_0 N$ are all absolutely irreducible. The multiplicity with which any absolutely irreducible constituent of $K_0 M$ occurs is precisely the Schur index of that constituent relative to the rational field (see [7]). On the other hand, for p-groups (p odd) it is known [2, 8] that this Schur index is 1. Hence the irreducible constituents of $R_0 M$ and $R_0 N$ are distinct and absolutely irreducible. We may therefore apply Theorem 5, and obtain

$$R_1 M \cong R_1 N \;\Rightarrow\; R_0 M \cong R_0 N,$$

so that (25) is proved, Q.E.D.

The referee has kindly pointed out that the preceding theorem is also valid for the more general case in which R is a valuation ring of an algebraic number field K such that R lies over the ring of p-integral elements of the rational field. Indeed, the above proof requires only a minor modification for the more general case.

5. We conclude by listing a number of open questions.

A. If $R \subset R'$ are valuation rings, does (5) hold without any restrictive hypotheses?

B. Using the notation of Section 2, under what conditions does $\mathfrak{o}'M \vee \mathfrak{o}'N$ imply $M \vee N$, where M and N are $\mathfrak{o}G$-modules?

C. If $\mathfrak{o}$ is a principal ideal ring, does $\mathfrak{o}'$-equivalence imply $\mathfrak{o}$-equivalence?

It may be of interest to mention yet one more special case in which additional information may be obtained. Suppose that M and N are projective RG-modules, where R is the valuation ring of a discrete valuation of K. (For example, M and N might be direct summands of RG.) Then it is known[4] that $M \sim_R N$ if and only if $M \sim_K N$. Using Theorem 1 and its corollary, we conclude that (5) holds in this case.

In particular, if M and N are projective $\mathfrak{o}G$-modules, then $\mathfrak{o}'M \vee \mathfrak{o}'N$ surely implies that M and N are K'-equivalent, and hence by the above discussion that $M \vee N$.

Added in proof. In a recently completed paper [11], Zassenhaus and the author have shown that (5) holds without any restrictive hypotheses, assuming still that R and R' are valuation rings as in Section 3. This settles questions A and B, but C is still open.

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