ADEQUATE SUBCATEGORIES

BY

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Introduction

This paper introduces and studies the notion of a *left adequate subcategory* of an arbitrary category (and the dual notion). Definition follows.

Let $\mathcal{A}$ be a full subcategory of $\mathcal{C}$. For any object X of $\mathcal{C}$, let Map $(\mathcal{A}, X)$ denote the contravariant functor on $\mathcal{A}$ into the category $\mathcal{U}$ of all sets and all functions which takes each object A of $\mathcal{A}$ to the set Map (A, X), and each mapping $f: A' \to A$ in $\mathcal{A}$ to the function from Map (A, X) to Map (A', X) defined by $[\text{Map}(\mathcal{A}, X)(f)](g) = gf$. Observe that every mapping $h: X \to Y$ in $\mathcal{C}$ induces a natural transformation from Map $(\mathcal{A}, X)$ to Map $(\mathcal{A}, Y)$ by multiplication. We call $\mathcal{A}$ *left adequate* if every natural transformation between these functors is induced by a mapping in $\mathcal{C}$ and distinct mappings induce distinct natural transformations. *Right adequate* is defined dually.

A little thought will show that the phrase "$\mathcal{A}$ is right adequate in $\mathcal{C}$" is a natural formulation of the somewhat variable idea "every object of $\mathcal{C}$ has sufficiently many mappings into objects of $\mathcal{A}$". Then the main results of this paper are the examples. Using the usual mappings (for topological spaces, the continuous functions, and similarly for other objects), we have the following. In a category of algebras with n-ary operations, the free algebra on n generators is left adequate. In compact spaces, the 2-cell is right adequate. In sets, a single point is left adequate. The duals of these examples are less neat. For sets, a countably infinite set is right adequate if and only if no measurable cardinals exist. For compact spaces, no set of them is left adequate. The 1-cell is left adequate for Peano spaces, and for all products of Peano spaces up to the first weakly inaccessible cardinal. No nontrivial instance is found of a single algebra being right adequate for a large class, excepting the few which come from the examples mentioned by duality.

The first part of the paper is devoted mainly to inverting the notions of adequacy in order to obtain a reasonable closure operation on arbitrary categories. Left or right adequacy alone is unsuitable because a left adequate subcategory of a left adequate subcategory is not generally left adequate. Further, if $\mathcal{A}$ is a left adequate subcategory of $\mathcal{C}$, almost no useful restriction on $\mathcal{C}$ can be inferred from restrictions on $\mathcal{A}$. Neither of these objections applies to the notion of left *and* right adequacy. This does yield a closure operation. The main drawback, as the examples show, is that the operation is rather feeble.

Received July 10, 1959; received in revised form January 22, 1960.

[1] Supported in part by the National Science Foundation and in part by the Office of Naval Research.

To handle the case that the smaller category $\mathcal{A}$ is a proper class, we define $\mathcal{A}$ to be *properly left adequate* in $\mathcal{C}$ if $\mathcal{A}$ is left adequate and for each object X of $\mathcal{C}$ there is a set of objects A_α of $\mathcal{A}$ such that every mapping $B \to X$, B in $\mathcal{A}$, can be factored in the form $B \to A_\alpha \to X$. *Properly right adequate* is defined dually, *properly adequate* by conjunction. Then for every category $\mathcal{A}$ there is a largest (in the sense of equivalence) category which contains $\mathcal{A}$ as a properly left adequate subcategory. Consequently there is also a largest category $\mathcal{R}(\mathcal{A})$ in which $\mathcal{A}$ is properly adequate. $\mathcal{R}(\mathcal{R}(\mathcal{A}))$ is equivalent to $\mathcal{R}(\mathcal{A})$. If $\mathcal{A}$ can be isomorphically represented as a category of algebras with certain operations, satisfying specified identities and identical implications, then so can $\mathcal{R}(\mathcal{A})$. Similar statements are true for topological spaces and topological algebras.

I am indebted to J. P. Jans, N. Kimura, and R. J. Nunke for refuting some naive conjectures on right adequacy in algebraic categories, and to S. Mac Lane, J. H. Walter, and the referee for suggestions which improved the definitions.

1. Set functors

Terminology not explained here is from Grothendieck [3]. A subcategory $\mathcal{A}$ of a category $\mathcal{C}$ is called *full* if every mapping in $\mathcal{C}$ whose domain and range are in $\mathcal{A}$ is in $\mathcal{A}$. A *skeleton* is a full subcategory containing exactly one representative of each equivalence class of isomorphic objects. We shall use the easily verified remark that two categories are *equivalent* in the sense of [3] if and only if they have isomorphic skeletons (i.e., if and only if they are *coextensive* in the sense of [4]).

Since the categories will be mostly non-abelian we write Map (X, Y) instead of Hom (X, Y). If A is a full subcategory of $\mathcal{C}$ and X is an object of $\mathcal{C}$, Map $(\mathcal{A}, X)$ denotes the contravariant functor on $\mathcal{A}$ into the category of all sets and functions which assigns to each object A of $\mathcal{A}$ the set Map (A, X) and to each $f: A' \to A$ the function which takes each $g \in$ Map (A, X) to $gf \in$ Map (A', X). The functor Map $(X, \mathcal{A})$ is defined dually.

For short, we shall call a functor on $\mathcal{A}$ into the category of all sets and all functions a *set functor* on $\mathcal{A}$. We call a natural transformation from a contravariant set functor T on $\mathcal{A}$ to another such set functor U a *right transformation* from T to U. We extend this usage to call a natural transformation from Map $(\mathcal{A}, X)$ to Map $(\mathcal{A}, Y)$ a *right transformation from X to Y over* $\mathcal{A}$. For covariant set functors, a natural transformation from U to T will be called a *left transformation from T to U*; and similarly for Map $(X, \mathcal{A})$.

Note that a mapping $f: X \to Y$ induces natural transformations from Map $(\mathcal{A}, X)$ to Map $(\mathcal{A}, Y)$ and from Map $(Y, \mathcal{A})$ to Map $(X, \mathcal{A})$ by multiplication. Both of these induced transformations (right and left) are *from X to Y* as defined above.

We define $\mathcal{A}$ to be a *left adequate* subcategory of $\mathcal{C}$ provided $\mathcal{A}$ is a full subcategory of $\mathcal{C}$, every right transformation between objects of $\mathcal{C}$ over $\mathcal{A}$ is induced by a mapping, and different mappings induce different right transformations. We define $\mathcal{A}$ to be *properly left adequate* if $\mathcal{A}$ is left adequate and for each object X of $\mathcal{C}$ there is a set S of objects of $\mathcal{A}$ such that every mapping from an object A of $\mathcal{A}$ to X can be factored $A \to B \to X$ over some object B in S.

In the important special case that $\mathcal{A}$ has only a set of objects, left adequacy is the same as proper left adequacy; and in this situation we shall usually omit the "proper".

1.1. *Every skeleton of a category is a properly left adequate subcategory. Every full subcategory containing a (properly) left adequate subcategory is (properly) left adequate.*

The proof is omitted.

1.2. *A left adequate subcategory of a left adequate subcategory need not be left adequate.*

To prove this take three objects, W, X, Y, with the nine sets Map (A, B) as follows: Map (W, W) is a cyclic group of order 2; Map (X, Y) is a two-element set; Map (X, W), Map (Y, W), and Map (Y, X) are empty; the others are one-element sets. This determines the multiplication. The full subcategory whose objects are W and X is left adequate; in it the full subcategory whose only object is W is left adequate; but transitivity fails.

For any category $\mathcal{A}$, the *principal* contravariant set functors on $\mathcal{A}$ are the functors Map $(\mathcal{A}, A)$, A an object of $\mathcal{A}$. Any contravariant set functor F on $\mathcal{A}$ is said to be *dominated* by a set S of objects of $\mathcal{A}$ if every set $F(A)$, A in $\mathcal{A}$, is a union of sets $F(f)(F(B))$, B ranging over the elements of S and f ranging over mappings in Map (A, B). If F is dominated by some set of objects, it is called *proper*. All the proper contravariant set functors on $\mathcal{A}$, with all their natural transformations, form a category $\mathcal{P}^*(\mathcal{A})$. The full subcategory whose objects are the principal functors is called the *regular representation* of $\mathcal{A}$. We shall use the same term for the natural isomorphism of $\mathcal{A}$ into $\mathcal{P}^*(\mathcal{A})$.

In most of this, duality calls for no special comment; but it should be noted that the principal covariant set functors form an isomorphic representation of the dual category $\mathcal{A}^*$ (in fact, the regular representation of $\mathcal{A}^*$).

1.3. *For a category $\mathcal{C}$ containing $\mathcal{A}$ as a full subcategory, $\mathcal{A}$ is properly left adequate in $\mathcal{C}$ if and only if the regular representation of $\mathcal{A}$ can be extended to an isomorphic representation of $\mathcal{C}$ in $\mathcal{P}^*(\mathcal{A})$.*

This is obvious.

1.4. *If $\mathcal{A}$ is a full subcategory of $\mathcal{B}$, every proper contravariant set functor on $\mathcal{A}$ can be extended over $\mathcal{B}$ in such a way as to embed $\mathcal{P}^*(\mathcal{A})$ isomorphically in $\mathcal{P}^*(\mathcal{B})$ as a full subcategory.*

The proof is omitted. There is no unique natural extension; one must make several choices.

A subcategory $\mathfrak{A}$ of a category $\mathfrak{C}$ is called *adequate* (*properly adequate*) if $\mathfrak{A}$ is both left and right (properly) adequate. This implies that for each object X of $\mathfrak{C}$, for each object A of $\mathfrak{A}$, Map (X, A) is in a natural one-to-one correspondence with all the natural transformations from Map $(\mathfrak{A}, X)$ to Map $(\mathfrak{A}, A)$, and dually, Map (A, X) corresponds with the natural transformations from Map $(X, \mathfrak{A})$ to Map $(A, \mathfrak{A})$. That is, either of the functors Map $(\mathfrak{A}, X)$, Map $(X, \mathfrak{A})$ can be derived from the other by taking natural transformations into principal set functors.

For any proper contravariant set functor F on a category $\mathfrak{A}$, we define the *conjugate set functor* F^* as follows. For each object A of $\mathfrak{A}$, $F^*(A)$ is the set of all natural transformations from F to Map $(\mathfrak{A}, A)$; for each mapping $f:A \to A'$ in $\mathfrak{A}$, $F^*(f)$ takes $F^*(A)$ to $F^*(A')$ by composing each element of $F^*(A)$ with the natural transformation which f induces from Map $(\mathfrak{A}, A)$ to Map $(\mathfrak{A}, A')$.

Thus the conjugate of a proper contravariant set functor is covariant (but it need not be proper). The conjugate of a proper covariant set functor is defined dually. If both F and F^* are proper, then there is a natural transformation from F to F^{**} defined by evaluation, as follows: for p in $F(A)$, p^* in $F^*(B)$, the mapping $p^*(p) \in$ Map (A, B) may be denoted by $\hat{p}(p^*)$, and $\hat{p}$ so defined is a natural transformation in $F^{**}(A)$. We call F *reflexive* provided F is proper, F^* is proper, and the evaluation from F to F^{**} is a natural equivalence.

1.5. *A properly left adequate subcategory $\mathfrak{A}$ of a category $\mathfrak{C}$ is properly right adequate if and only if all the functors* Map $(\mathfrak{A}, X)$ *are reflexive.*

The rest of the proof is omitted.

In contrast with 1.2 and 1.4, we have

1.6. *An adequate subcategory of an adequate subcategory is adequate, and the same for properly adequate.*

1.7. *Not every reflexive set functor on a full subcategory has a reflexive extension.*

For 1.6 we shall prove a little more: *A left adequate subcategory of an adequate subcategory is left adequate.* The checking of domination for proper adequacy will be omitted.

Then let $\mathfrak{A}$ be a left adequate subcategory of $\mathfrak{B}$, which in turn is an adequate subcategory of $\mathfrak{C}$. We show first that any two different mappings $f:X \to Y$, $g:X \to Y$, in $\mathfrak{C}$ induce different right transformations over $\mathfrak{A}$. Since $\mathfrak{B}$ is adequate in $\mathfrak{C}$, there exist W in $\mathfrak{B}$ and $p:W \to X$ such that $fp \neq gp$. Similarly there exist Z in $\mathfrak{B}$ and $q:Y \to Z$ such that $q(fp) \neq q(gp)$. Then, since $\mathfrak{A}$ is left adequate in $\mathfrak{B}$, there is $e:U \to W$, U in $\mathfrak{A}$, with $qfpe \neq qgpe$.

Therefore $fpe \neq gpe$, and f and g induce different right transformations over $\mathfrak{A}$.

Now consider the special case of an object X in $\mathfrak{B}$, Y in $\mathfrak{C}$, and a right transformation T from X to Y over $\mathfrak{A}$. We shall determine the corresponding mapping $g:X \to Y$ by constructing the left transformation G which it induces over $\mathfrak{B}$. For any mapping $h:Y \to Z$, Z in $\mathfrak{B}$, $G(h)$ must be a mapping $k:X \to Z$. Since X and Z are in $\mathfrak{B}$, k can be determined by defining the right transformation K which it induces over $\mathfrak{A}$. Let $e:U \to X$ be any mapping with domain U in $\mathfrak{A}$; we define $K(e) = hT(e)$. Since T is a right transformation, so is K; then K is induced by a unique mapping k, and the values of G are defined. G is a left transformation because $G(qh)$ and $qG(h)$, for any q, induce the same right transformations over $\mathfrak{A}$; thus G is induced by $g:X \to Y$. From the definitions we have $hT(e) = ke = hge$ for all $e:U \to X$, U in $\mathfrak{A}$, and all $h:Y \to Z$, Z in $\mathfrak{B}$; that is, $T(e)$ and ge induce the same left transformations over $\mathfrak{B}$, and T is indeed induced by g.

The general case of T from X to Y over $\mathfrak{A}$, X and Y in $\mathfrak{C}$, reduces to the previous case. For each $f: W \to X$, W in $\mathfrak{B}$, there is a mapping $T'(f): W \to Y$ corresponding to the right transformation over $\mathfrak{A}$ which takes each $e:U \to W$ to $T(fe)$; and T' is a right transformation over $\mathfrak{B}$ whose restriction to $\mathfrak{A}$ coincides with T. Hence T is induced by a mapping.

Proof of 1.7. Let the category $\mathfrak{B}$ have two objects X, Y. Map (X, X) is the set of all 2 by 2 integral matrices, Map (X, Y) the set of all 3 by 2 integral matrices; Map (Y, Y) has just one element, and Map (Y, X) is empty. Multiplication is matrix multiplication. Let $\mathfrak{A}$ be the full subcategory with object X. Let F be the covariant set functor on $\mathfrak{A}$ which assigns to X the set of all 2 by 3 integral matrices and to e in Map (X, X) the operation of left multiplication by e. We omit the verification that F is reflexive.

For any extension G of F over $\mathfrak{B}$, $G^*(Y)$ is empty, since $G(X)$ is nonempty and Map (Y, X) is empty. Now consider the set $G(Y)$. The subset S which is the union of all $G(f)(G(X))$, f in Map (X, Y), is nonempty; the remainder T might be empty. Turning to $G^*(X)$, an element j must assign to each matrix d in $G(X)$ a matrix $j(d)$ in Map (X, X), with $j(ed) = ej(d)$ for all 2 by 2 matrices e; this implies $j(d) \equiv dj_0$ for some 3 by 2 matrix j_0. Further, j must assign to each $G(f)(d)$ in S the matrix $fj(d)$ in Map (X, Y); finally, j maps T to Map (X, Y) quite arbitrarily. Assuming G is reflexive, we can exhibit an element k of T in $G^{**}(Y)$. Since $G^*(Y)$ is empty, we need only define $k(j)$ for j in $G^*(X)$; let $k(j)$ be the matrix j_0, considered as an element of Map (X, Y). It follows that there are two natural transformations $\hat{\jmath}, j'$ in $G^{***}(X)$ differing only on k; since $j_0' = \hat{\jmath}_0$, this is a contradiction.

From 1.5 and 1.6 we obtain the following theorem.

1.8. THEOREM. *The reflexive contravariant set functors on a category* $\mathfrak{A}$ *form a category* $\mathfrak{R}(\mathfrak{A})$ *containing the regular representation of* $\mathfrak{A}$ *as a properly adequate subcategory. Every category in which* $\mathfrak{A}$ *is properly adequate is equivalent to a subcategory of* $\mathfrak{R}(\mathfrak{A})$, *and* $\mathfrak{R}(\mathfrak{R}(\mathfrak{A}))$ *is equivalent to* $\mathfrak{R}(\mathfrak{A})$.

This concludes the presentation of the concepts. We turn now to illustrations, and this almost forces us to turn to examples. However, there is an exception.

1.9. *For any mapping $p:X \to X$ in a category $\mathfrak{A}$ such that $pp = p$, there is a reflexive contravariant set functor on $\mathfrak{A}$ which is dominated by X and has a unique reflexive extension over any category containing $\mathfrak{A}$ as a full subcategory.*

The functor F referred to in 1.9 associates to each object A the set of all elements of Map (A, X) of the form pf. We omit the verification of its properties. The result may be restated roughly: any object is right and left adequate for its own retracts.

2. Examples

A few results on adequacy can be established for abstract algebras in the most general sense. It is possible to give a definition so general as to include compact Hausdorff spaces, but the following seems to be nearly standard. A *family of operations* is a well-ordered family of finite ordinal numbers n_α; an *algebra* having that family of operations consists of a ground set A and a well-ordered family of functions Q_α, each Q_α mapping A^{n_α} into A. The n_α are allowed to take the value 0; a 0-ary operation is just a distinguished element.

A *full category of algebras* is a concrete category of sets and functions whose sets can be taken as the ground sets of certain algebras, all having the same family of operations, in such a way that the mappings of the category are exactly the homomorphisms. The concepts of subalgebra and direct product require no explanation. Similar terminology (full category, subobject, direct product) will be used for topological spaces, topological algebras, uniform spaces, and categories. The meanings are obvious except perhaps for the following. In the topological cases, by *subobject* we mean a closed subspace (closed subalgebra). The objects of a full category of categories are of course rather small categories, since they must be sets; the mappings are the covariant functors. Note that for two functors, f, g, on a category $\mathfrak{C}$ into a category $\mathfrak{D}$, the set of all mappings x in $\mathfrak{C}$ such that $f(x) = g(x)$, with their domains and ranges, forms a subcategory. The corresponding proposition for topological spaces is true in the case of Hausdorff spaces.

2.1. THEOREM. *If $\mathfrak{A}$ is isomorphic with a full category of algebras, categories, Hausdorff spaces, Hausdorff topological algebras, or uniform spaces, then so is $\mathfrak{R}(\mathfrak{A})$; in fact, $\mathfrak{R}(\mathfrak{A})$ is isomorphic with a full category of subobjects of direct products of objects of $\mathfrak{A}$.*

Proof. We give the proof only for the case of algebras; trivial modifications suffice for the other cases. Since any full subcategory of $\mathfrak{R}(\mathfrak{A})$ containing the regular representation of $\mathfrak{A}$ is properly adequate, we can proceed by transfinite induction; and we need only prove that if $\mathfrak{B}$ is a full category of algebras, any reflexive contravariant set functor F on $\mathcal{B}$ is in fact Map $(\mathcal{B}, X)$ for some suitable algebra X, such that $\mathcal{B}$ is properly adequate in the full category of algebras formed by adjoining X to $\mathcal{B}$.

Let S be a set of objects of $\mathcal{B}$ dominating both F and F^*. Let I denote the union of all the product sets $W \times F(W)$, W an element of S. For each $g \in F(W)$ let $\bar{g}$ denote the function on the algebra W into the index set I defined by $\bar{g}(w) = (w, g)$ for each $w \in W$. Let P be the direct product of algebras defined as follows. The index set H is the union of all $F^*(Z)$, Z in S; the factor algebra corresponding to each $h \in F^*(Z)$ is a copy of Z. We define a function $\lambda : I \to P$ by its coordinates λ_h : for each (w, g) in I, $\lambda_h(w, g) = hg(w)$. Let X be the smallest subalgebra of P which contains $\lambda(I)$.

We shall define a natural transformation α from F to Map $(\mathcal{B}, X)$ by $\alpha(g) = \lambda\bar{g}$; $\lambda\bar{g}$ is a homomorphism since all its coordinates are homomorphisms. It remains to verify several things, most of which are immediate. For W in $\mathcal{B}$, not in S, each e in $F(W)$ is $F(f)(g)$ for some g in $F(V)$, V in S, and some $f : W \to V$ in $\mathcal{B}$; $\alpha(e)$ is defined as $\alpha(g)f = \lambda\bar{g}f$ for any such representation. This is independent of the choice of g and f because each h^{th} coordinate of $\lambda\bar{g}f$ is just $h(g)f = h(e)$. For the same reason, α is a natural transformation. It maps each $F(W)$ one-to-one into Map (W, X) since F is reflexive and S dominates F^*. On the other hand, for any homomorphism $m : W \to X$, W in $\mathcal{B}$, define a natural transformation μ in $F^{**}(W)$ as follows. For h in $F^*(Z)$, Z in S, $\mu(h)$ is the h^{th} coordinate $\pi_h m$ of m; for other values (Z not in S) μ is defined by means of representations $h = F^*(k)(j)$, where j is in $F^*(Y)$ for some Y in S. Since $F^{**} = F$, we have μ in $F(W)$ and $\alpha(\mu)$ is m. Thus α is a natural equivalence.

Entirely similar arguments show that Map $(X, \mathcal{B})$ is naturally equivalent to F^*. Finally, a natural transformation ν from F^* to F^* induces a homomorphism on X into the product algebra P, by applying ν to the coordinate indices; this homomorphism may be verified to take the generating set $\lambda(I)$ into itself, and therefore it maps X into X. Conversely, a homomorphism $n : X \to X$ induces a natural transformation from Map $(\mathcal{B}, X)$ to itself, and one may verify the equivalence.

Concerning 2.1 it is obvious that axiomatic versions of the theorem exist. A good axiomatization should give a dual conclusion also. For algebras (to justify a statement in the introduction) note that identities and identical implications are preserved in forming products and subalgebras.

In a full category of algebras, an algebra F generated by a set G of its elements is called *free* on G if every function from G to any algebra A in the category can be extended to a homomorphism from F into A. (Existence of a free algebra with given generators can be proved e.g. if the category is closed under forming products and subalgebras. For an axiomatic study, see [6].)

2.2. *If $\mathcal{A}$ is a full category of algebras with operations at most n-ary, and N* *is a free algebra on n generators in* $\mathfrak{A}$, *then N with its endomorphisms forms a left adequate subcategory of* $\mathfrak{A}$.

(It is assumed that $n \geqq 1$.)

Proof. Evidently, since a generator of N can be mapped anywhere by a homomorphism, two different homomorphisms induce different right transformations over N. Then let α be a right transformation from X to Y over N. For each x in X, there is a unique homomorphism $f_x : N \rightarrow X$ which takes all the generators to x. Moreover, $\alpha(f_x)$ takes all the generators to the same element $a(x)$ of Y; for the homomorphisms from N to Y which are constant on the set of generators are characterized as those which are unchanged by composition with any endomorphism of N permuting the generators, and this condition is preserved under right transformations. Next consider any g in Map (N, X). For each generator s of N, $\alpha(g)(s)$ is $ag(s)$; if we let $e:N \rightarrow N$ take all generators to s, we have

$$\alpha(g)(s) = \alpha(g)e(s) = \alpha(ge)(s) = ag(s).$$

Next we check that a is a homomorphism. Let Q be an algebraic operation—m-ary, $m \leqq n$—and let $x_0 = Q(x_1, \cdots, x_m)$ in X. Let $g:N \rightarrow X$ take some m generators $s_1, \cdots, s_m$ to $x_1, \cdots, x_m$, and let $q:N \rightarrow N$ take each s_i to $Q(s_1, \cdots, s_m)$. Then gq takes all generators to x_0, and $a(x_0)$ is the value of $\alpha(gq)$ on all (any) s_i. Also $\alpha(gq) = \alpha(g)q$; $\alpha(g)$ takes the first m generators s_i to $a(x_i)$, and $\alpha(g)q$ takes every generator to

$$Q(a(x_1), \cdots, a(x_m)).$$

Thus a is a homomorphism. For every g in Map (N, X), then, ag is a homomorphism coinciding with $\alpha(g)$ on the generators; consequently α is induced by a.

Concerning right adequacy for algebras, we have little information. For abelian groups the construction Hom (G, H) is available; but it is closer to the mark to consider the ring E of all endomorphisms of H, which acts on Hom (G, H) by composition, converting the group into an E-module G_H^*. Let G_H^{**} be the group consisting of all E-homomorphisms of G_H^* into H. There is a natural evaluation homomorphism $e:G \rightarrow G_H^{**}$, and we write $G_H^{**} = G$ if e is an isomorphism.

2.3. *In a full category of abelian groups containing an infinite cyclic group Z and a group H such that* $G_H^{**} = G$ *for all G in the category, the group* $H \oplus H$ *(if in the category) with its endomorphisms forms a right adequate subcategory. In any additive category composed of finite direct sums of objects* A_α, *the full subcategory whose objects are the sums* $A_\alpha \oplus A_\alpha$ *is adequate.*

The proofs are omitted; they are on the same lines as the proof of 2.2.

We turn to topological spaces. We assume acquaintance with the notion of a *measurable* cardinal number [8]. Ulam's results on nonexistence of measures have been extended in one direction by Mazur as follows. An infinite cardinal $\mathfrak{m}$ is *strongly accessible* if every limit cardinal $\aleph_\alpha$ satisfying $\aleph_0 < \aleph_\alpha \leqq \mathfrak{m}$ can be represented as a sum of fewer than $\aleph_\alpha$ cardinals each smaller than $\aleph_\alpha$. Mazur's theorem: a sequentially continuous real-valued function on a product of a strongly accessible number of separable metric spaces is continuous [7].

2.4. *If $\mathfrak{A}$ is a full category of topological spaces and $\mathfrak{B}$ a full subcategory of $\mathfrak{A}$ including a nonempty space, then for any objects X and Y of $\mathfrak{A}$, each right transformation from X to Y over $\mathfrak{B}$ is $f \to gf$ for some function g, not necessarily continous, on X to Y.*

Proof. First, a right transformation J takes every constant function to a constant function, since the constants $k: W \to X$ are precisely the functions unchanged by composition with any $e: W \to W$. Observe next that if Y is the empty space, there are no right transformations from X to Y, unless X is also empty. In the remaining case, select a nonempty space W in $\mathfrak{B}$ and define $g(p)$ for each point p of X as the constant value of $J(k)$, where k is the constant function on W to X whose value is p. The verification that $J(f) = gf$ for each $f: V \to X$, V in $\mathfrak{B}$, proceeds pointwise. For each v in V, let $p = f(v)$, and let $k: W \to X$ be the constant with value p. Let $e: V \to W$ be some constant function. Then $J(ke) = J(k)e$ is the constant function on V to Y with value $g(p)$. Consider the constant $d: V \to V$ with value v; since $fd = ke$, $J(f)d$ has the constant value $g(p) = gf(v)$, which on the other hand is $J(f)(v)$. Since v is arbitrary, the proof is complete.

2.5. *In the category of discrete spaces, a space of one point is left adequate. A countable space is right adequate if and only if no measurable cardinals exist.*

Proof. The first statement follows from 2.4 and the observation that two different functions with the same domain and range differ on some point. Next suppose there is an Ulam measure μ on a discrete space S. Let P be the one-point space and N a countable space. We define a left transformation J from P to S over N as follows. For each $h: S \to N$, the inverse sets $h^{-1}(n)$, $n \in N$, form a countable partition of S; thus there is exactly one n such that $\mu(h^{-1}(n)) = 1$. Let $J(h)$ be the mapping of P to N which takes the value n. Obviously J is an inverse transformation, and since μ vanishes on points, J is not induced by any mapping of P to S.

Now suppose there are no measurable cardinals. It is obvious that two different mappings $X \to Y$ induce different left transformations over N. On the other hand, let J be a left transformation from X to Y over N, and p a point of X. Consider the set function μ defined on subsets S of Y, as follows: $\mu(S) = 1$ if for all h in $M(Y, N)$, $J(h)(p) \in h(S)$; otherwise $\mu(S) = 0$. Evidently μ is monotone and superadditive. To see that μ is countably additive (and not identically zero) consider any indexed countable partition $\{K_i\}$ of Y. Let us identify N with the positive integers, and let $h: Y \to N$

take the value i on each K_i. The point $J(h)(p)$ is one of the values i of h, for it is fixed under all $g:N \to N$ which satisfy $gh = h$. For that i, I claim $\mu(K_i) = 1$. Suppose on the contrary that f is a mapping from Y to N, $J(f)(p) \notin f(K_i)$. Consider the mappings d and e: $d(y) = 1$ if $f(y) \in f(K_i)$, $d(y) = 2$ otherwise; $e(y) = 2h(y) + d(y)$. Then we have $d = rf = se$ for suitable mappings r, s, of N to N. Moreover, h has the form te; and e never takes on the value $2i + 2$, so that (though $t(n)$ must be the greatest integer in $(n - 1)/2$ for most values of n) we may define $t(2i + 2)$ arbitrarily. Now $J(d)(p) = rJ(f)(p) = 2 = sJ(e)(p)$, so that $J(e)(p)$ is even; but $i = J(h)(p) = tJ(e)(p)$, which yields a contradiction if we specify $t(2i + 2) = i + 1$. This establishes $\mu(K_i) = 1$; μ is countably additive; hence there is a point y in Y such that $\mu(\{y\}) = 1$. We define $g(p) = y$. The fact that $\mu(g(p)) = 1$ means precisely $J(h)(p) = hg(p)$ for all h; defining g for all p, we have the mapping g which induces J.

2.6. THEOREM. *A square (2-cell) is right adequate for compact spaces; an arc (1-cell) is left adequate for products of* $\mathfrak{m}$ *Peano spaces if* $\mathfrak{m}$ *is strongly accessible. A plane is right and left adequate for products of* $\mathfrak{m}$ *real lines if* $\mathfrak{m}$ *is strongly accessible. A 1-cell is not right adequate for compact spaces.*

Proof. Let us take first the plane W; let X and Y be products of accessibly many real lines. By Mazur's theorem [7], a sequentially continuous real-valued function on X is continuous. Then the same is true for a function on X to Y, since it is continuous if and only if all its coordinate projections are continuous. (The fact that Y is a full product space is not needed here.) Now a right transformation from X to Y over W is induced by a function $g:X \to Y$, by 2.4. If g is not continuous, there is a convergent sequence $p_i \to p$ in X with $g(p_i)$ not converging to $g(p)$. Since X is a linear topological space, there are line segments joining p_i to p_{i+1}; since X is locally convex, these line segments and p make up a compact subspace which is a continuous image of a closed interval. Then we may retract W upon some closed interval contained in it to obtain a continuous $f: W \to X$ such that gf is discontinuous, a contradiction.

There remains the case of a left transformation J from X to Y over W. Here we need the structure of pairs of real numbers, which we bring in by passing to the rings of real-valued continuous functions, $C(X)$, $C(Y)$. If we can exhibit a homomorphism j^* from $C(Y)$ to $C(X)$ satisfying $j^*(1) = 1$, we can conclude that j^* is induced by a continuous function $g:X \to Y$, the values $j^*(f)$ being fg [5].

Accordingly choose definite coordinate axes in the plane W, and let us use the following notation for particular functions on W to W:

$$t(x, y) = (y, x), \qquad q_1(x, y) = (x, 0), \qquad q_2(x, y) = (0, y),$$
$$+(x, y) = (x + y, 0), \qquad m(x, y) = (xy, 0).$$

Let i be the function on the line to the plane defined by $i(x) = (x, 0)$, and π on the plane to the line, $\pi(x, y) = x$. For each f in $C(Y)$, let $j^*(f)$ be $\pi J(if)$.

Now for a pair of functions, h, k, in $C(Y)$, let $[h, k]$ be the function on Y to W defined by $\pi[h, k] = h$, $\pi t[h, k] = k$. We have $J([h, k]) = [j^*(h), j^*(k)]$. For $q_1 J([h, k]) = J(q_1[h, k]) = J(ih)$, and $q_2 J([h, k]) = J(tik)$ similarly. Then

$$j^*(h + k) = \pi J(i(h + k)) = \pi J(+[h, k]) = \pi(+(J([h, k])))$$
$$= \pi(+[j^*(h), j^*(k)]) = j^*(h) + j^*(k);$$

and the same computation goes for multiplication. One may verify $j^*(1) = 1$ e.g. by rotating W around the point $(1, 0)$. Then j^* is a homomorphism. This yields continuous $g:X \to Y$ satisfying $j^*(f) = fg$ for all f in $C(Y)$. But every h in $M(Y, W)$ is $[e, f]$ for some two elements e, f of $C(Y)$; and we have

$$J([e, f]) = [j^*(e), j^*(f)] = [eg, fg] = [e, f]g.$$

This completes the proof of the statements about the plane.

The left adequacy of the arc for products of accessibly many Peano spaces is proved in almost the same way. By Mazur's theorem and 2.4, it suffices to show that in such a product X of factors X_α, every convergent sequence is contained in a Peano subspace. The analogue of local convexity is supplied by Bing's theorem [2] that each Peano space X_α has a metric d_α in which it is metrically convex, i.e., any two points can be joined by an arc isometric to a real interval. Choose such a metric for each factor, and let $p^i \to p$ in the product space X. We select intervals I_i joining p^i to p^{i+1} so that the projections joining the coordinates p_α^i to p_α^{i+1} are isometric to real intervals (no more care is needed). We must verify that every neighborhood of the limit p contains almost all I_i. Considering the definition of the product topology, it suffices to verify that each coordinate p_α has a basis of neighborhoods U_n which have the property that any isometric interval in X_α whose ends are in U_n lies wholly in U_n. But if V_{2n} is the spherical $(1/2n)$-neighborhood, and U_n the union of all intervals whose ends are in V_{2n}, then U_n is a neighborhood contained in V_n; and the U_n form a basis.

For right adequacy, let X and Y be compact spaces and J a left transformation from X to Y over the square S. Let us regard S as the product of two closed intervals $[-1, 1]$. For those functions f in $C(Y)$ which satisfy $|f| \leqq 1$ we can repeat the previous construction. For each real constant θ between 0 and 1, let $c_\theta : S \to S$ be the contraction $(x, y) \to (\theta x, \theta y)$. Observe that $J(c_\theta h) = c_\theta J(h)$. Then since Y is compact, there is for every h in $C(Y)$ some $\theta > 0$ such that $| \theta h | \leqq 1$; if we define $j^*(h)$ as $\theta^{-1} j^*(eh)$, the definition is independent of θ and yields a homomorphism j^* as before. Then j^*, hence also J, is induced by a continuous function $g:X \to Y$; and the proof, as to S, is complete.

Finally we shall exhibit a left transformation from a point P to a 2-cell S over a 1-cell I, which is not induced by a continuous function on P to S. Consider S as a triangle abc. To each mapping $h:S \to I$ we associate the mapping $J(h):P \to I$ whose value is that unique x in I such that the set

$h^{-1}(x)$ has a component K whose intersection with each of the three sides, ab, ac, bc, is nonempty. Evidently, if x always exists, J will be the required counterexample. The existence of such an x is routine and may be known; a proof follows.

First, one easily sees that if K and L were two disjoint closed connected subsets of S each meeting all three sides, then in the 1-sphere which is the boundary of S some two points of K would separate two points of L. This conflicts (e.g.) with the Jordan curve theorem; if we replace K with an arc in the locally arcwise connected space $S - L$, joining two suitable points, p, q, we can then embed S in the plane and continue the arc to a Jordan curve separating the given points, r, s, of L. Thus there is at most one such x. On the other hand, consider the set of all components K of inverse sets $h^{-1}(x)$ which have nonempty intersections with both ab and ac. There is such a K, namely the component of a in $h^{-1}(h(a))$. Evidently for any two such K, one separates a from the other; and by continuity and compactness, there is a K_0 farthest from a. Suppose K_0 fails to meet bc, and x_0 is the value of h on K_0. There is a relatively open-closed subset A of the compact space $h^{-1}(x_0)$ which contains K_0 and is disjoint from bc. Let R be the component of bc in $S - A$, $T = R \cup A$. On a neighborhood U of A in T, h does not take the value x_0. Since R is connected, $h - x_0$ has constant sign on U. We may suppose $h \geqq x_0 + \varepsilon$ on the boundary of U relative to T. Then $h^{-1}(x_0 + \varepsilon)$ separates A, and in particular K_0, from bc in the space T; therefore $h^{-1}(x_0 + \varepsilon)$ separates K_0 from bc in the 2-cell S. This implies that some component K_1 of $h^{-1}(x_0 + \varepsilon)$ separates K_0 from bc [1], and therefore meets both ab and ac, lying farther from a than K_0. This contradiction completes the proof.

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