GROUPS ON S^n WITH PRINCIPAL ORBITS OF DIMENSION $n-3$

BY

D. MONTGOMERY AND C. T. YANG[1]

Introduction

Throughout this paper X is to be S^n with the usual differentiable structure, and G is to be a compact connected Lie group acting differentiably on X, with principal orbits of dimension $n-3$. We begin without assuming the existence of a stationary point and obtain some information about the orbit space X^*. However our main concern is with the case where a stationary point exists. In this case we obtain rather complete information about X^*. The main result is that if $n > 4$, then X^* is a 3-manifold with boundary S^2, and furthermore D is null, where D is the set of points on exceptional orbits of dimension $n-3$.

Let G_x be the isotropy group at x. The orbit $G(x)$ is a principal orbit if it is $(n-3)$-dimensional and if for y in a neighborhood of x, G_y and G_x have the same number of components. If $G(x)$ is $(n-3)$-dimensional and there is no such neighborhood of x, then $G(x)$ is called an exceptional three-dimensional orbit. The principal orbits form a dense open connected subset of S^n whose complement has dimension at most $n-2$. For all $(n-3)$-dimensional orbits, it is true that dim G_x is constant. For all principal orbits $G(x)$ the number of components of G_x is constant.

Let U be the union of all principal (sometimes regular) orbits, D the union of exceptional $(n-3)$-dimensional orbits, and B the union of all singular orbits. Then

$$X = U \cup D \cup B,$$

and the sets U, B, D are invariant and mutually exclusive. Let p be the natural map from X to the orbit space X^*.

The number of components of G_x is denoted by $m(x)$ and $m(x^*) = m(x)$, for any x such that $p(x) = x^*$. It will be convenient to use $\rho(y^*)$ where

$$\rho(y^*) = m(y^*)/m(x^*), \qquad y^* \in D^*, \quad x^* \in U^*.$$

It is known that U^* is orientable and that every orbit in $U \cup D$ is orientable [8].

In the case $n = 4$, G a circle, there is the following special example with a stationary point Let the circle act on one plane with period $1/p$, $p > 1$, and on another plane with period $1/q$, $q > 1$. Then G acts on the product of the planes by defining

$$g(p_1, p_2) = (g(p_1), g(p_2)),$$

Received February 29, 1960.

[1] The second author is supported in part by the Army Office of Ordnance Research.

and the action becomes defined for S^4 by requiring the point at infinity to be a stationary point. In this case X^* is S^3, B^* consists of the origin and the point at infinity, and D^* consists of two open arcs joining these two points. Of course if $p = 1, q > 1, D^*$ contains one arc, and if $p = 1, q = 1$, D^* is null. This type of base space does not occur for any other dimension as we shall see.

In fact the main purpose of this paper is to prove the following:

THEOREM A. *Let G be a compact connected Lie group acting differentiably on S^n, $n > 4$, with principal orbits of dimension $n-3$ and with a stationary point α. Then X^* is a simply connected 3-manifold with boundary $B^* = S^2$, and the set D^* is null.*

As an illustration of the theorem consider the following example. Let G_1, G_2, G_3 be the proper orthogonal groups in dimension a, b, c, respectively. Let $G = G_1 \times G_2 \times G_3$ act on the product $R^a \times R^b \times R^c$ by the definition

$$(g_1, g_2, g_3)(p_1, p_2, p_3) = (g_1(p_1), g_2(p_2), g_3(p_3)),$$

and let X be the sphere formed by adding $\alpha = p_\infty$ as a stationary point of G. Then $\dim X = a + b + c$, and the principal orbits have dimension $a + b + c - 3$. It can be seen that D^* is null and that B^* is as described in Theorem A. In case $n = 4$, and principal orbits of dimension $n-3$, G must be a circle. Here much is known or can easily be deduced, but we do not deal with this case except in so far as Sections 2 and 3 are applicable. The nature of D and D^* in this case is not fully understood by any means. For example if closure D^* is a simple closed curve, can it be knotted?

1. $(n-2)$-dimensional orbits

We shall make use of results on $(n-2)$-dimensional orbits, and we describe and develop the ones we need in this section. We assume then in this section that G is a compact connected group acting differentiably on S^n with principal orbits of dimension $n-2$. If all orbits are $(n-2)$-dimensional, it can be seen by the use of a slice that X^* is a 2-manifold which, since it is simply connected, must be S^2.

If some orbit has dimension less than $n-2$, then the base space is a disc whose boundary is B^*; the set D^* is null. This can be seen by methods analogous to those in [6]. Moreover on B^*, the isotropy group is relatively continuous except at a finite set $z_1^*, \cdots, z_m^*$; the orbits, in each interval of B^* complementary to this set, are of constant type. The following theorem will also be useful.

THEOREM 1.1. *Let G be a compact connected Lie group acting effectively on $X = S^n$. If every orbit has dimension $n-2$, then $n = 3$ and G is a circle, or else $n = 2$ and $G = e$.*

In this case X/G is homeomorphic to S^2. It now follows from a theorem of Conner [3, p. 29] that G is a circle group, or else G is $SO(3)$ or its covering group and the isotropy groups are finite. To complete the proof, it must be shown that the latter case is impossible under the added assumption that all orbits have dimension $n-2$.

Hence assume G is $SO(3)$, or its covering group, and that all orbits have dimension $n-2$, so that $n = 5$. By another of Conner's theorems [3, p. 27] there is a spectral sequence $\{E_r^{s,t}\}$ with (using his notation in which $G = L$, Q = rationals)

$$E_2^{s,t} = H^s(L/B_x \times X/L; Q) \otimes H^t(\hat{B}_x ; Q)$$

whose E_∞ term is associated with $H^*(X; Q) = H^*(S^5; Q)$.

Since all isotropy groups are finite, it follows (again using Conner's notation) that $B_x = \hat{B}_x = G$; since $G = L$ and since G has the same homology over Q as S^3, we have

$$E_2^{s,t} = H^s(S^2; Q) \otimes H^t(S^3; Q).$$

Clearly the E_∞ term cannot be associated with $H^*(S^5; Q)$ since $E_\infty^{0,3} = Q$. This contradiction proves the theorem.

2. Properties of D^*

For the present we do not assume a stationary point exists nor that $n > 4$. We do assume that G is a compact connected Lie group acting differentiably on $X = S^n$ with principal orbits of dimension $n-3$.

For $y^* \in D^*$ select $y \in p^{-1}(y^*)$. Let V be a closed 3-cell which is a slice at y and on which G_y acts orthogonally. Then $H = G_y/G_V$ is a finite group acting effectively and orthogonally on V.

We note that every element of H preserves the orientation of V; this can be shown to follow from the fact that U^* is orientable. Thus H is a finite group of rigid motions leaving y fixed. Hence H is one of the following: (a) cyclic group; (b) dihedral group; (c) tetrahedral group; (d) octahedral group; (e) icosahedral group.

LEMMA 2.1. *$U^* \cup D^*$ is a 3-manifold.*

This follows from the fact that H is one of the groups of rigid motions as listed above, and can be seen by examining the actions of such groups.

Since X is a sphere, it is known that dim $D^* \leqq 1$ [5]. It is also known that D^* contains no isolated points, and that $U^* \cup D^*$ is simply connected [6].

THEOREM 2.1. *If G is a compact connected group acting differentiably and effectively on $X = S^n$ with every orbit of dimension $n-3$, then $G = e$ and $n = 3$.*

Assume an action possible with $n > 3$. Then X^* is simply connected and is also a 3-manifold by the preceding lemma. Hence it has the same cohomology groups for Z, and hence for Q, as S^3.

Referring to Conner's theorem [3] as applied earlier, we see in this case that

G is not a circle because a circle cannot act on S^4 with all its orbits one-dimensional. Moreover G cannot be $SO(3)$ or its covering group by the same argument as in the earlier case. This completes the proof of the theorem.

LEMMA 2.2. *Every $y^* \in D^*$ has a compact neighborhood V^* which may be regarded as a solid sphere of center y^* such that $V^* \cap D^*$ is a finite union of radii. If $y^* y_1^*$ is a radius in $V^* \cap D^*$ and $\rho(y^*)/\rho(y_1^*)$ is not even, then we can find a second radius $y^* y_2^*$ such that $(G, p^{-1}(y_1^* \, y^* \, y_2^*))$ has a cross-section L with G_x being constant on $L - p^{-1}(y^*)$.*

Let $y \in p^{-1}(y^*)$, and let V be a closed 3-cell which is a slice at y and on which G_y acts orthogonally. Let $H = G_y/G_V$ as above. By Lemma 2.1

$$V^* = G(V)/G = V/G_y = V/H$$

is a compact neighborhood of y^* which is a closed cell.

Every $x \in V$ is on D if and only if the isotropy group H_x is not trivial. Hence $V \cap D$ is a finite union of diameters of V. It is possible that some element of H might reflect one of these diameters into itself, in which event it would become a radius in V^*. At any rate it is always true that

$$V^* \cap D^* = (V \cap D)/G_y$$

is a finite union of radii of V^*.

Let $y^* y_1^*$ be any radius of V^* in $V^* \cap D^*$, and let yy_1 be a radius of V with $p(yy_1) = y^* y_1^*$. Let $y_1 yy_2$ be the diameter of V containing yy_1. If an element of H takes yy_1 to yy_2, then the order of H is an even multiple of the order of H_{y_1}, and therefore $\rho(y^*)/\rho(y_1^*)$ is even. Thus if $\rho(y^*)/\rho(y_1^*)$ is not even, then no element of H maps yy_1 into yy_2. Hence $y^* y_2^*$ is a second radius of precisely the kind desired. This completes the proof.

Since $U^* \cup D^*$ is a 3-manifold, it can be triangulated (see Moise [5] and also Bing [1]). The preceding lemma shows that D^* is locally tame (see Bing [2]). It follows from Bing's result that D^* *is polyhedral* in some triangulation of $U^* \cup D^*$. This fact is used in 2.3 and in 3.4.

LEMMA 2.3. *There is no simple closed curve E^* in D^* such that for all $y^* \in E^*$, $\rho(y^*)$ is divisible by an integer $k > 1$ independent of y^*.*

It can be seen that a simple closed polyhedral path, in a simply connected manifold, is the boundary of an oriented surface of some genus. We are indebted to Papakyriakopoulos for pointing out this fact to us. See [4] for a proof when the 3-manifold is E^3.

Let Q^* be an oriented 2-manifold in $X^* - B^* = U^* \cup D^*$ with E^* as its boundary; we assume, as we may without loss of generality, that $(Q^* - E^*) \cap D^*$ is finite. Then $p^{-1}(Q^*) - p^{-1}(E^*)$, after a closed subset of dimension $\leqq n-3$ is removed, is an orientable $(n-1)$-manifold. Let z be the integral fundamental $(n-1)$-cycle of

$$[p^{-1}(Q^*), p^{-1}(E^*) \cup p^{-1}((Q^* - E^*) \cap D^*)].$$

Then ∂z is an $(n-2)$-cycle on $p^{-1}(E^*)$. Using the fact that $\rho(y^*)$ is divisible by k for all $y^* \in E^*$, we can see that $(1/k)\partial z$ is an integral cycle on $p^{-1}(E^*)$ which does not bound on $p^{-1}(Q^*)$. This shows that $p^{-1}(Q^*)$ has $(n-2)$-torsion. This is a contradiction which completes the proof of the lemma.

3. Properties of B^*

We continue without the blanket assumption of a stationary point and without assuming $n > 4$.

LEMMA 3.1. *Let G be a compact connected Lie group acting differentiably on S^n with a stationary point α and principal orbits of dimension $n-3$. If α is an isolated orbit in B, then B consists of two stationary points, G is a circle, and $n = 4$.*

There is a local coordinate system at α on which G acts orthogonally. Let Y be an $(n-1)$-sphere of center α in this local coordinate system. Since α is an isolated orbit of B, all orbits in Y must have dimension $n-3$. By Theorem 1.1, $n = 4$ and G/G_Y is the circle group. But G_Y can contain only the identity, and G must be a circle. Hence B is a stationary set of the circle G, and this set must have the cohomology of a sphere of some dimension. Since α is isolated, the sphere must be the zero-dimensional sphere consisting of α and one other point. This proves the lemma. It can be seen that X^* is a simply connected 3-manifold.

LEMMA 3.2. *Let G be a compact connected Lie group acting differentiably on S^n with principal orbits of dimension $n-3$. At an isolated orbit of B^*, X^* is locally a 3-cell.*

Let b^* be an isolated orbit of B^*, and choose $b \in p^{-1}(b^*)$ where

$$\dim G(b) = m.$$

Let K be a slice at b, and let Y be a sphere in K of center b; then

$$\dim Y = n - m - 1.$$

All G_b-orbits in Y have dimension $n-m-3$, so $n - m = 4$. Then for G^*, the identity component of G_b, we have

$$Y/G_b^* = S^2.$$

Now $F = G_b/G_b^*$ acts on S^2. Let F_1 be the elements of F which preserve orientation, so that $S^2/F_1 = S^2$. If $F_1 \neq F$, then S^2/F is a projective plane, and U^* would fail to be orientable. Hence $F_1 = F$ and

$$Y/G_b = S^2,$$

which completes the proof.

LEMMA 3.3. *Let G act differentiably on S^n with principal orbits of dimension $n-3$. Then B^* consists of isolated points and 2-manifolds which are 2-spheres.*

Moreover X^* is a 3-manifold with boundary, and the boundary consists of the 2-spheres in B^*.

If b^* is in B^* and not isolated in B^*, choose $b \in p^{-1}(b^*)$, and let K be a slice at b. Let $m = \dim G(b)$, and let Y be an invariant $(n-m-1)$-sphere in K with center b. The group G_b acts in Y with orbits of dimension $n-m-3$. Not all G_b-orbits in Y are of dimension $n-m-3$ because of the fact that b^* is not isolated. Therefore Y/G_b^* is a disc, and G_b/G_b^* acts on this disc, and the resulting orbit space is Y^*. Notice that no element of G_b/G_b^* can reverse the orientation of the disc, for if it did, we would have $\dim D^* = 2$, which is impossible. Hence Y^* is a disc, and $Y^* \cap B^*$ is a simple closed curve. This shows that B^* is locally a 2-cell at b^*, and that X^* is locally a 3-cell at b^*. The space X^* is known to be simply connected, and this implies that each component of the boundary is a 2-sphere [9]. This completes the proof.

Since X^* is a 3-manifold, it can be triangulated; see Moise [5] and also Bing [1]. Moreover this can be done so that $B^* \cup D^*$ is a subpolyhedron. This follows [2] because it can be seen by the use of slices that $B^* \cup D^*$ is locally tame.

When there is a stationary point, the case of main interest for this paper, then we can get much of the information about B^* which we need by using the results of Yang [10].

LEMMA 3.4. *There is no simple closed curve E^* in $B^* \cup D^*$ which does not lie entirely in B^* and such that for all $y^* \in E^* - B^*$, $\rho(y^*)$ is divisible by an integer $k > 1$ independent of y^*.*

Note first that if any such curve exists, then there is such a curve which is polyhedral, because E^* is polyhedral in D^* in any case and any arc in B^* can be replaced by a polyhedral arc.

The proof is very similar to the proof of Lemma 2.3 and will be omitted.

4. D^* is empty

We assume throughout the remainder of this paper the full hypothesis of Theorem A, that is, that G is a compact connected Lie group acting differentiably on S^n $(n > 4)$ with a stationary point α and with $(n-3)$-dimensional principal orbits. The assumption $n > 4$ implies that α is not an isolated orbit of B.

In order to show that D^* is empty, as we now wish to do, it will be convenient to define D_1^* to be the subset of D^* consisting of points y^* for which $\rho(y^*)$ is even.

By Lemma 3.4, $D_1^* \cup B^*$ cannot contain a simple closed curve unless the curve is completely contained in B^*.

LEMMA 4.1. *$X^* - (B^* \cup D_1^*)$ is simply connected.*

In order to prove the lemma, form a new space W^* by beginning with X^*

and adding to it, for each 2-sphere in B^*, a 3-cell with its boundary identified with the 2-sphere. Then W^* is a simply connected 3-manifold without boundary. Consider the set

$$A^* = B^* \cup D^* \cup \text{(all new 3-cells)}.$$

We wish to deform A^* over itself, and we begin by shrinking each of the new 3-cells to a point inside itself. Thus A^* becomes A_1^* which is a graph, and this graph carries no simple closed curve by Lemma 3.4. Any component of this graph can be shrunk to one of its end points. Thus in the original space, $X^* - (B^* \cup D_1^*)$ is simply connected as we wished to prove.

LEMMA 4.2. *Under the assumption of this section there can be no point of D in the vicinity of α.*

Choose coordinates, for a neighborhood of α, on which G acts linearly, and let Y be an invariant $(n-1)$-sphere in these coordinates with center α. The group G is connected, and G-orbits on Y have dimension $n-3$. The conclusion follows from the known results on connected groups on a sphere with principal orbits of dimension two less than the dimension of the sphere.

Let B_1^* be the union of the 2-spheres in B^*.

LEMMA 4.3. *$B_1^* \cap$ closure $D_1^* = \emptyset$.*

Assume this is false, and let b^* be a point of $B_1^* \cap$ closure D_1^*. By taking a slice at a point $b \in p^{-1}(b^*)$ we can see that there exists an arc $b^*y_1^*$ with $b^*y_1^* - b^* \subset D_1^*$ and such that $\rho(y^*)$ is constant on $b^*y_1^* - (b^* \cup y_1^*)$. Using Lemma 2.2 we can extend this arc, if $\rho(y^*)/\rho(y_1^*)$ is not even, to a larger arc with $\rho(y^*)$ constant on the larger arc except at a finite number of points. We can therefore find a maximal arc $l_1^* = b^*y_1^*z^*$ such that $\rho(y^*)$ is constant on this arc except at a finite number of points. By Lemma 2.3, l_1^* has no self-intersections, and by Lemma 3.4, l_1^* must end either at a point of $X^* - B^*$, or else on a different component of B^* from the component containing the initial point b^* of l_1^*.

If $z^* \in D^*$, $\rho(z^*)/\rho(y^*)$ is even for any neighboring point y^* of l_1^*, as otherwise the path would not be maximal. Hence, in any case, $p^{-1}(l_1^*)$ carries an $(n-2)$-cycle c mod 2 which is not bounding on $B \cup D$. Let c' be a 1-cycle mod 2 on U linked with c, and let c'' be the image of c' in U^*. We may assume that c'' is a simple closed curve. Moreover we may assume, by making a slight deformation if necessary, that c' is mapped homeomorphically onto c''.

Let Q^* be an oriented 2-manifold in $X^* - (B^* \cup D_1^*)$ with c'' as its boundary and such that $Q^* - c''$ intersects D^* at a finite number of points. Let Q^* be obtained from a polygon $P^* = \alpha_1^* \alpha_2^* \cdots \alpha_{4r+1}^*$ by identifying $\alpha_{4i+1}^* \alpha_{4i+2}^*$ with $\alpha_{4i+4}^* \alpha_{4i+3}^*$ and $\alpha_{4i+2}^* \alpha_{4i+3}^*$ with $\alpha_{4i+5}^* \alpha_{4i+4}^*$, $i = 0, \cdots, r-1$. Thus c'' is obtained from $\alpha_{4i+1}^* \alpha_1^*$ by identifying the end points.

Since the interior of P^* intersects D^* at only a finite number of points, these points of intersection are on an arc l^* which intersects the boundary of P^* at one end point which will be chosen as α^*_{4i+1}. In X let $\alpha_{4r+1}\,\alpha_1$ be the cross-section of $(G, p^{-1}(\alpha^*_{4r+1}\,\alpha^*_1))$ representing c'.

Let $\alpha_k\,\alpha_{k+1}$ be a cross-section of $(G, p^{-1}(\alpha^*_k\,\alpha^*_{k+1}))$ such that

(1) $\alpha_1, \alpha_2, \cdots, \alpha_{4r+1}$ are identical as points of $p^{-1}(Q^*)$,

(2) $\alpha_{4i+1}\,\alpha_{4i+2} = \alpha_{4i+4}\,\alpha_{4i+3}$ and $\alpha_{4i+2}\,\alpha_{4i+3} = \alpha_{4i+5}\,\alpha_{4i+4}$, $i = 0, \cdots, r-1$, when they are regarded as arcs in $p^{-1}(Q^*)$.

Since $P^* - l^*$ is contractible, there is a cross-section K of $(G, p^{-1}(P^* - l^*))$ such that

$$K \cap p^{-1}(\alpha^*_k\,\alpha^*_{k+1}) = \alpha_k\,\alpha_{k+1}, \quad k = 1, \cdots, 4r-1,$$

$$K \cap p^{-1}(\alpha^*_{4r}\,\alpha^*_{4r+1} - \alpha^*_{4r+1}) = \alpha_{4r}\,\alpha_{4r+1} - \alpha_{4r+1},$$

$$K \cap p^{-1}(\alpha^*_{4r+1}\,\alpha^*_1 - \alpha^*_{4r+1}) = \alpha_{4r+1}\,\alpha_1 - \alpha_{4r+1}.$$

As cycles in $p^{-1}(Q^*)$, $\alpha_{4r+1}\,\alpha_1$ is homologous to $\alpha_{4r+1}\,\alpha_1\,\alpha_2 \cdots \alpha_{4r}\,\alpha_{4r+1}$. By using the cross-section K it is easily seen that whenever N is a neighborhood on $p^{-1}(l^*)$, then $\alpha_{4r+1}\,\alpha_1\,\alpha_2 \cdots \alpha_{4r}\,\alpha_{4r+1}$ is homologous to a cycle in N. Hence on $p^{-1}(Q^*)$, c' is homologous to a cycle of $p^{-1}(l^*)$.

Let x^* be a point of l^*. We claim that the inclusion map i of $p^{-1}(x^*)$ into $p^{-1}(l^*)$ induces a homomorphism of $H_1(p^{-1}(x^*))$ onto $H_1(p^{-1}(l^*))$, where Z_2 is the coefficient group. The proof is by induction on the number of points in $l^* \cap D^*$. If $l^* \cap D^* = \emptyset$, we may regard $p^{-1}(l^*)$ as a cylinder on $p^{-1}(x^*)$. It follows that $p^{-1}(x^*)$ is a deformation retract of $p^{-1}(l^*)$. Hence our assertion follows. If $l^* \cap D^*$ contains only one point y^*, it is easily seen that $p^{-1}(y^*)$ is a deformation retract of $p^{-1}(l^*)$, and hence the inclusion map of $p^{-1}(y^*)$ into $p^{-1}(l^*)$ induces an isomorphism of $H_1(p^{-1}(y^*))$ onto $H_1(p^{-1}(l^*))$. Using the retracting deformation, we know that $p^{-1}(x^*)$ covers $p^{-1}(y^*)$ an *odd* number of times. Hence the inclusion map of $p^{-1}(x^*)$ into $p^{-1}(l^*)$ induces a homomorphism of $H_1(p^{-1}(x^*))$ onto $H_1(p^{-1}(l^*))$ as we use Z_2 for coefficients. In general if $l^* \cap D^*$ contains m points, $m > 1$, we divide l^* into two arcs l^*_1 and l^*_2 with a common end point such that $l^*_1 \cap D^*$ contains $m-1$ points and $l^*_2 \cap D^*$ contains one point. The Mayer-Vietoris sequence

$$0 \leftarrow H_1(p^{-1}(l^*)) \leftarrow H_1(p^{-1}(l^*_1)) + H_1(p^{-1}(l^*_2)) \leftarrow H_1(p^{-1}(l^*_1 \cap l^*_2))$$

is exact, and the homomorphisms

$$H_1(p^{-1}(l^*_1 \cap l^*_2)) \rightarrow H_1(p^{-1}(l^*_1)),$$

$$H_1(p^{-1}(l^*_1 \cap l^*_2)) \rightarrow H_1(p^{-1}(l^*_2))$$

are onto by the induction hypothesis. For any $e \in H_1(p^{-1}(l^*))$, there is an element (e_1, e_2) in $H_1(p^{-1}(l^*_1)) + H_1(p^{-1}(l^*_2))$ having e as its image. Let e' be an element of $H_1(p^{-1}(l^*_1 \cap l^*_2))$ with e_2 as its image in $H_1(p^{-1}(l^*_2))$. Let e'_1 be the image of e' in $H_1(p^{-1}(l^*_1))$. Then

$$(e_1 - e'_1, 0) = (e_1, e_2) - (e'_1, e_2)$$

has e as its image. Hence $H_1(p^{-1}(l_1^*)) \to H_1(p^{-1}(l^*))$ is onto, and consequently $H_1(p^{-1}(l_1^* \cap l_2^*)) \to H_1(p^{-1}(l^*))$ is onto. This completes the proof of the assertion.

From this result it follows that on $p^{-1}(Q^*)$, c' is homologous to a cycle on a principal orbit. By deforming the principal orbit toward the known stationary point, c' is seen to be homologous to zero in $X - D$. This contradicts the fact that c' is linked with c and completes the proof.

LEMMA 4.4. *$B_1^* \cap$ closure $D^* = \emptyset$.*

Suppose this is false, and let $b^* \in B_1^* \cap$ closure D^*. Let $b^*y_1^*$ be an arc with $b^*y_1^* - b^* \subset D^*$ and such that $\rho(y^*)$ is constant on $b^*y_1^* - b^*$. By the result above, $\rho(y^*)$ is odd on $b^*y_1^* - b^*$. Using Lemma 2.2, we can obtain an arc l^* beginning at b^* and ending at a point of $D_1^* \cup B^*$ such that on l^*, $\rho(y^*)$ is constant except at a finite number of points and such that no interior point on l^* is on D_1^*. By Lemma 3.4, l^* cannot end on the same component of B^* as the one from which it started.

As in the proof of the above, $p^{-1}(l^*)$ carries an $(n-2)$-cycle mod 2 which is not bounding on $B \cup p^{-1}(l^* \cup D_1^*)$. Since $B^* \cup l^* \cup D_1^*$ does not contain any nonbounding 1-cycle, an argument similar to the one above produces a contradiction. This completes the proof that

$$B^* \cap \text{closure } D^* = \emptyset.$$

LEMMA 4.5. *$D^* = \emptyset$.*

Let $J^* = b^*d^*$ be an arc such that $b^* \in B_1^*$, $d^* \in D^*$, $b^* \neq \alpha^*$, and

$$J^* - (b^* \cup d^*) \subset U^*.$$

Let x^* be an interior point of J^*, and choose $x \in p^{-1}(x^*)$.

We shall see first that any closed path γ in $U - p^{-1}(J^*)$ near x can be shrunk to a point in $X - p^{-1}(J^*)$. Let $p(\gamma) = \gamma^*$. Since $b^* \in B_1^*$, $b^* \in S^*$, where S^* is a 2-sphere in B_1^*. There is an S_1^* in X^* which is near S^* and such that V^*, the region between S^* and S_1^*, is a spherical shell. We can assume by the lemmas above that

$$V^* \cap D^* = \emptyset.$$

Now γ^* can be shrunk into V^* while staying in $X^* - (J^* \cup D^*)$ and then can be shrunk to a point in V^*, again while remaining in $X^* - (J^* \cup D^*)$. This shrinking of γ^* can be covered, and it follows that γ can be shrunk to a regular orbit, with the shrinking taking place in $U - p^{-1}(J^*)$. This regular orbit can be shrunk to α without touching $p^{-1}(J^*)$. Hence γ can be shrunk to a point in $X - p^{-1}(J^*)$.

Next let γ be any closed path in $X - p^{-1}(J^*)$. Then γ bounds a singular 2-cell σ in X, and since $\dim (p^{-1}(b^*) \cup p^{-1}(d^*)) \leqq n - 3$ and since $p^{-1}(b^*)$ and $p^{-1}(d^*)$ are differentiably imbedded, we may assume that

$$\sigma \cap (p^{-1}(b^*) \cup p^{-1}(d^*)) = \emptyset.$$

Hence we may assume that σ meets $p^{-1}(J^*)$ at only a finite number of points each of which is above an interior point of J^*. The preceding argument then shows that σ can be deformed so as to have no intersection with $p^{-1}(J^*)$, because near any such intersection we may replace a small sub-2-cell which intersects by another 2-cell which does not.

It follows from the above that $X - p^{-1}(J^*)$ is simply connected, and hence that it has trivial one-dimensional homology. On the other hand, $p^{-1}(J^*)$ carries an $(n-2)$-cycle mod k, and by duality this is a contradiction. This proves the lemma.

5. A lemma on B^*

LEMMA 5.1. *Let G be a compact connected Lie group acting differentiably on S^n, $n > 4$, with a stationary point α and with principal orbits of dimension $n-3$. Then $B^* = S^2$, and X^* is a simply connected 3-manifold with boundary S^2.*

In B^* there is at least one 2-sphere S^*, and this can be displaced to S_1^*, near S^*, $S_1^* \subset U^*$. We now deform S_1^* to S_2^*, where S_2^* lies in U^* except that at one point it coincides with α^*. Now S_2^* can be raised, because of α^*, to S^2 in X. Of course S^2 can be shrunk to a point in X, and moreover this can be done while avoiding isolated orbits in B, since each such orbit has dimension at most $n-4$ and there are at most a finite number of them. In this way we see that the fundamental cycle of S^* bounds in (X^* − isolated orbits of B^*). However the fundamental cycle of S^* cannot bound in a proper closed subset of X^*. This proves that there are no isolated orbits in B^* and also that $B^* = S^*$. This completes the proof of the lemma and also of Theorem A.

6. A lemma on U^*

LEMMA 6.1. *U^* has trivial homotopy groups in all dimensions.*

For the proof let

$$f^*: S^i \to U^*.$$

By deforming we obtain

$$f_1^*: S^i \to U^* \cup \alpha^*$$

with precisely one point going to α^*. This may be raised to

$$f_1: S^i \to U \cup \alpha.$$

The map f_1 is homotopic to a constant, that is, $f_1(S^i)$ is the boundary of a singular $(i+1)$-cell. Therefore the same is true for f_1^*. The singular $(i+1)$-cell in X^* may be deformed so it does not touch B^*. This completes the proof.

The above may be extended in part to the case of a compact connected group acting on S^n with orbits of any dimension with a fixed point α. *Then the continuous image of any sphere in U^* can be shrunk to a point in X^*.*

BIBLIOGRAPHY

1. R. H. BING, *An alternative proof that 3-manifolds can be triangulated*, Ann. of Math. (2), vol. 69 (1959), pp. 37-65.
2. ———, *Locally tame sets are tame*, Ann. of Math. (2), vol. 59 (1954), pp. 145-158.
3. P. E. CONNER, *Orbits of uniform dimension*, Michigan Math. J., vol. 6 (1959), pp. 25-32.
4. F. FRANKL AND L. PONTRJAGIN, *Ein Knotensatz mit Anwendung auf die Dimensionstheorie*, Math. Ann., vol. 102 (1930), pp. 785-789.
5. E. E. MOISE, *Affine structures in 3-manifolds, V. The triangulation theorem and Hauptvermutung*, Ann. of Math. (2), vol. 56 (1952), pp. 96-114.
6. D. MONTGOMERY, H. SAMELSON, AND C. T. YANG, *Groups on E^n with $(n-2)$-dimensional orbits*, Proc. Amer. Math. Soc., vol. 7 (1956), pp. 719-728.
7. D. MONTGOMERY AND C. T. YANG, *Orbits of highest dimension*, Trans. Amer. Math. Soc., vol. 87 (1958), pp. 284-293.
8. D. MONTGOMERY, *Groups on R^n or S^n*, Michigan Math. J., vol. 6 (1959), pp. 123-130.
9. H. SEIFERT AND W. THRELFALL, *Lehrbuch der Topologie*, Leipzig und Berlin, 1934.
10. C. T. YANG, *On singular orbits*, to appear.

THE INSTITUTE FOR ADVANCED STUDY
PRINCETON, NEW JERSEY
UNIVERSITY OF PENNSYLVANIA
PHILADELPHIA, PENNSYLVANIA