ON THE GLOBAL STRUCTURE OF THE TRAJECTORIES OF A POSITIVE QUADRATIC DIFFERENTIAL

BY

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1. A knowledge of the global structure of the trajectories of a positive quadratic differential on a finite oriented Riemann surface is of fundamental importance in the proof of the General Coefficient Theorem [1, 2, 3, 4]. The first steps in this direction were taken by Teichmüller [9] who described the local structure of the trajectories at various critical points, as well as some of the basic types of domains comprising the global structure with, however, little indication of proof. His discussion was limited to some rather special hyperelliptic differentials (i.e., defined on the Riemann sphere). Schaeffer and Spencer [8] gave a fairly complete treatment of the local structure and analyzed the global structure for those differentials treated by Teichmüller, as well as for one other special hyperelliptic differential, by a method whose application is essentially restricted to these particular cases. In particular, it was not decided whether, in the case of a hyperelliptic differential, a trajectory could be everywhere dense in some domain. The first general results on global structure were given in a paper [7] by the author and Spencer, where it was shown that for a hyperelliptic differential the trajectory structure is made up of end, strip, circle, and ring domains [2; pp. 36, 37] together with a finite number of domains in which some of the trajectories having limiting end points at finite critical points of the differential are everywhere dense. It was shown by example that such domains can actually be present. Later the author remarked [1] that the same considerations apply on a general finite oriented Riemann surface, and a complete treatment of this characterization of the global structure is found in [2], where the results are summarized as the Basic Structure Theorem [2; Theorem 3.5]. This result is sufficient for proving the General Coefficient Theorem, but one somewhat unsatisfactory feature remains. This is the lack of knowledge of the structure within those domains where there are everywhere dense trajectories. The only information in this direction is contained in [7; §3]. Now the simplest prototype of everywhere dense structure occurs for everywhere regular quadratic differentials on a closed surface of genus one. If $Q(z)\,dz^2$ is one such differential, then all such are of the form $Ke^{i\theta}Q(z)\,dz^2$ with $0 \leqq \theta < 2\pi$, $K > 0$. For a countable set of values of θ each trajectory is a closed curve; for all other values each trajectory is everywhere dense on the full surface. In this paper we will show that those domains in which the everywhere dense structure occurs decompose into subdomains such that every trajectory in such a subdomain is everywhere dense.

Received June 5, 1959.

[1] This work was sponsored by the Office of Ordnance Research, U.S. Army.

2. For all terms and definitions relating to quadratic differentials, we refer to [2]. Since the trajectories of a quadratic differential form a family F which displays many of the features of the regular curve families studied in the topological theory of functions, it is worth while to recall several concepts associated with these families which extend to the present case [5, 6].

DEFINITION 1. *By a right neighborhood associated with F is meant an open set N such that $\bar{N}$ admits a homeomorphic mapping onto a square in the (u, v)-plane*

$$K: \quad -1 \leqq u \leqq 1, \quad -1 \leqq v \leqq 1$$

such that there is a $(1, 1)$ correspondence between maximal subarcs of $F \mid \bar{N}$ and arcs $u = c$, $-1 \leqq v \leqq 1$, where c ranges over the interval $[-1, 1]$. We term u, v canonical coordinates of the antecedent in N of (u, v) in K.

Evidently every noncritical point lies in a right neighborhood.

DEFINITION 2. *By a transversal of F is meant an open arc or Jordan curve every point of which lies on an open subarc which lies in a right neighborhood and admits the representation $v = \varphi(u)$ in terms of the canonical coordinates. By the principal transversal of a right neighborhood N is meant the open arc in N on which*

$$v = 0, \quad -1 < u < 1.$$

We recall also the following result [5; Lemma 3.1].

LEMMA 1. *Every closed subarc of a trajectory lies in a right neighborhood.*

Finally we have the concept of F-set [6, 2].

DEFINITION 3. *An F-set is a set S such that any trajectory which meets S lies entirely in S.*

We are now ready to prove our key preliminary result.

LEMMA 2. *In an open F-set O in which a finite number of trajectories are collectively dense, there exists a closed transversal of F.*

In a right neighborhood N in O let us denote the set represented by $0 < v < 1$, $-1 < u < 1$ in the canonical coordinates by N^+, the set represented by $-1 < v < 0$, $-1 < u < 1$ by N^-, and the principal transversal of N by λ. Let a trajectory t be sensed and meet λ in a point P. If an open interval of points immediately preceding P on t lies in N^-, we say that t crosses λ at P in the positive sense; in the alternative case we say t crosses λ at P in the negative sense.

Let $t_1, \cdots, t_k$ be the finite number of trajectories in the statement of the lemma with any assigned senses. We show first that for a given right neighborhood N there exist a t_i and an open subarc μ of λ such that t_i crosses μ in the same sense at successive points of intersection (successive in terms of the sense on t_i). Indeed starting with t_1 and a given point of intersection

P_0 of t_1 with λ, regard all points of intersection of t_1 with $\lambda: P_1, P_2, \cdots$ following P_0 in the sense on t_1, and $P_{-1}, P_{-2}, \cdots$ preceding P_0 (following P_0 on t_1 with sense reversed). (These sets may of course be finite or void.) If any P_j, $j > 0$, lies on an interval $P_l P_m$, $0 \leqq l, m < j$, we are done; if not, the P_j, $j \geqq 0$, cannot be dense on λ. A similar argument applies to the P_j, $j \leqq 0$. Thus if t_1 does not have the desired property, there exists a subinterval of λ free of points of t_1. Repeating this argument in succession with $t_2, \cdots, t_k$ (if necessary) and successive subintervals, we must obtain the desired t_i because of the presumed collective density of $t_1, \cdots, t_k$.

We may now suppose the right neighborhood N so restricted that the trajectory $t = t_i$ meets λ in successive points of intersection at which it crosses λ in the same sense. By suitable choice of canonical coordinates, we may suppose that both crossings are in the positive sense, and the successive points of intersection are Q, R (where Q precedes R on t) represented by $(u_1, 0)$, $(u_2, 0)$ with $u_1 < u_2$. Now by Lemma 1 the arc QR on t lies in a right neighborhood N' in which we may take canonical coordinates u', v', so that Q, R are represented by the points $(\tilde{u}', v_1')$, $(\tilde{u}', v_2')$ with $v_2' > v_1'$. In a sufficiently small interval $(\tilde{u}' - \varepsilon, \tilde{u}' + \varepsilon)$, $\varepsilon > 0$, open subarcs of λ will admit in the canonical coordinates of N' in neighborhoods of Q and R the respective representations $v' = \varphi_1(u')$, $v' = \varphi_2(u')$. Let $\tilde{u}' - \varepsilon < \bar{u}' < \tilde{u}'$, and let S be the point represented in N' by $(\bar{u}', \varphi_2(\bar{u}'))$, so in N by $(\bar{u}, 0)$, $u_1 < \bar{u} < u_2$. Let α be the arc joining Q and S represented in N' by the line segment joining $(\tilde{u}', v_1')$ and $(\bar{u}', \varphi_2(\bar{u}'))$. Let β be the arc joining Q and S represented in N by the segment $v = 0$, $u_1 < u < \bar{u}$. Then the union of α and β provides the desired closed transversal.

3. LEMMA 3. *Let $Q(z)\, dz^2$ be a positive quadratic differential on a finite oriented Riemann surface $\mathfrak{R}$ such that there exists in its trajectory structure a domain G in which the trajectories having limiting end points at the finite critical points are everywhere dense. Let U denote the union of trajectories of $Q(z)\, dz^2$ having in each sense a limiting end point at a finite critical point. Then $G - \bar{U}$ consists of a finite number of subdomains of G. Let D be such a subdomain. Then every trajectory in D is everywhere dense in D.*

Of course G may have positive genus. Let τ be a trajectory in D, and K_1 its (point-set) closure. Evidently K_1 is an F-set (a familiar consequence of Lemma 1), and so unless $D \subset K_1$, we have $D - K_1 = K_2$ also an F-set which is open. The finite number of trajectories with limiting end points at finite critical points of $Q(z)\, dz^2$ are collectively dense in K_2. Thus there exists a closed transversal g of F in K_2 by Lemma 2.

As we describe g in either sense, $\mathcal{I} \int (Q(z))^{1/2}\, dz$ varies monotonically. Thus it serves to define a linear measure on g which we call g-length. It is easily seen that g lies in a doubly-connected domain H in K_2, swept out by open arcs on trajectories through points of g, whose boundary components it separates, and which it divides into subdomains H^+ and H^-. A trajectory ray (i.e., one of the open subarcs into which a trajectory is divided by a point on it, sensed to have this point as limiting initial point) with limiting initial point on g and an initial open subarc in H^+ will be said to leave the positive side of g; if it has an initial open subarc in H^-, it will be said to leave the negative side of g.

At most a finite number of trajectory rays leaving the positive side of g have limiting end points at finite critical points and no point of intersection with g. Their limiting initial points divide g into a finite number of open intervals I_j, $j = 1, \cdots, K$. Similarly at most a finite number of trajectory rays leaving the negative side of g have limiting end points at finite critical points and no point of intersection with g. Their limiting initial points divide g into a finite number of open intervals J_j, $j = 1, \cdots, L$. If a trajectory ray r leaving the positive side of g with limiting initial point in an interval I_j had no point of intersection with g, it could not re-enter H; thus apart from the points of its initial open subarc in H, it would be at a positive distance from g in the Q-metric $|Q(z)|^{1/2}|dz|$. Since there is no trajectory ray with limiting initial point in I_j and no point of intersection with g leaving the positive side of g which has a limiting end point at a finite critical point, the same condition would persist through an open interval in I_j. The trajectory rays with limiting initial points in this interval leaving the positive side of g would sweep out an open set with infinite area in the Q-metric. This is impossible, since this set would lie in D which has finite area in the Q-metric. Thus each trajectory ray r with limiting intial point A in I_j leaving the positive side of g has a point of intersection with g. Let B be the first point of intersection of r with g. Let r^* be the trajectory ray with limiting initial point B and sense the reverse of that of r. If r^* leaves the positive side of g, B will lie in an interval I_k; if the negative side, B will lie in an interval J_l. Further if a second trajectory ray r' leaving the positive side of g has limiting initial point A' in I_j, its corresponding point B' will lie in the same interval I_k or J_l as before, and the subintervals AA', BB' will have the same g-length. Thus the interval I_j is mapped onto I_k or J_l, respectively, isometrically in terms of g-length by the correspondence induced from A to B. In particular, we see that in the first case we must have $k \neq j$, since the mid-point of I_j in terms of g-length could not correspond to itself in this manner. We have spoken so far as though there actually are trajectory rays leaving the positive side of g which have a limiting end point at a finite critical point and no point of intersection with g. If there were none, the preceding considerations would show at once that D itself would be a closed surface of genus one on which $Q(z)\,dz^2$ would be regular. Thus under our present hypotheses this cannot occur. Similar arguments apply to intervals J_l and trajectory rays leaving the negative side of g. In particular, the intervals J_l not obtained before as images of intervals I_j are paired by the latter correspondence.

By the preceding construction the intervals I_j, $j = 1, \cdots, K$, and J_j,

$j = 1, \cdots, L$, are grouped in pairs of equal g-length. With each pair is associated a domain Δ swept out by trajectory arcs joining the two intervals. Part of the boundary of the domain Δ consists of the two intervals. We will now describe the remainder of the boundary. Starting with an end point of the one interval we proceed on a trajectory ray until we reach a finite critical point. This is followed on the boundary by another portion of trajectory, which, in case the critical point is a simple pole, may be the preceding reversed in sense. This trajectory cannot continue to unbounded length in the Q-metric without having a limiting end point at a finite critical point or meeting g. This follows from Lemma 1 and the fact that the trajectory arcs in Δ have bounded length. If the former occurs, the boundary continues along another portion of trajectory, and the same assertion applies. After a finite number of steps we must have a point of intersection with g which is evidently an end point of the other interval. Thus Δ is bounded by two arcs on g, arcs on the closure of trajectories with one end point on g the other at a finite critical point, and possible trajectories with limiting end points at finite critical points (at each end). The domains Δ for various pairs of arcs are nonoverlapping. The inner closure of their union is a domain M lying in K_2. Since the points of the arcs on g and of the open arcs on trajectory rays with one limiting end point on g the other at a finite critical point each occur twice as boundary points of these domains, M is bounded by a finite number of trajectories each joining two finite critical points together with their end points. This, however, contradicts the original assumption that both τ and g lay in the domain D.

4. We will now give an improved statement of the Basic Structure Theorem, taking account of the result of Lemma 3. We begin by giving the definition of a fifth type of basic domain.

DEFINITION 4. *A density domain* $\mathfrak{F}$ *(relative to* $Q(z)\,dz^2$*) is a maximal connected open F-set on* $\mathfrak{R}$ *with the properties:*

(i) $\mathfrak{F}$ *contains no point in* H;

(ii) $\mathfrak{F} - C$ *is swept out by trajectories of* $Q(z)\,dz^2$ *each of which is everywhere dense in* $\mathfrak{F}$.

(Here the notations are as in [2]: C is the set of zeros and simple poles of the quadratic differential; H is the set of poles of order at least two.)

The final statement is as follows.

THEOREM 1. *Let* $\mathfrak{R}$ *be a finite oriented Riemann surface, and* $Q(z)\,dz^2$ *a positive quadratic differential on* $\mathfrak{R}$ *where we exclude the following possibilities and all configurations obtained from them by conformal equivalence:*

I. $\mathfrak{R}$ *the z-sphere,* $Q(z)\,dz^2 = dz^2$,

II. $\mathfrak{R}$ *the z-sphere,* $Q(z)\,dz^2 = Ke^{i\alpha}\,dz^2/z^2$, α *real,* K *positive,*

III. $\mathfrak{R}$ *a torus,* $Q(z)\,dz^2$ *regular on* $\mathfrak{R}$.

Let Λ *denote the union of all trajectories of* $Q(z)\,dz^2$ *which have one limiting end point at a point of* C *and a second limiting end point at a point of* $C \cup H$. *Then*

(i) $\mathfrak{R} - \bar{\Lambda}$ *consists of a finite number of end, strip, ring, circle, and density domains;*

(ii) *each such domain is bounded by a finite number of trajectories together with the points at which the latter meet; every boundary component of such a domain contains a point of* C, *except that a boundary component of a circle or ring domain may coincide with a boundary component of* $\mathfrak{R}$; *for a strip domain the two boundary elements arising from points of* H *divide the boundary into two parts on each of which is a point of* C;

(iii) *every pole of* $Q(z)\,dz^2$ *of order* m *greater than two has a neighborhood covered by the inner closure of the union of* $m - 2$ *end domains and a finite number (possibly zero) of strip domains;*

(iv) *every pole of* $Q(z)\,dz^2$ *of order two has a neighborhood covered by the inner closure of the union of a finite number of strip domains or has a neighborhood contained in a circle domain.*

The discussion of §3 can readily be converted into a completely general description of a density domain. It is however not canonical.

5. Theorem 1 has a consequence suggested by the argument of §3 which is of independent interest.

COROLLARY 1. *Let* I, *the real numbers modulo* 1, *be divided into open intervals* $I_1, \cdots, I_k$, $k > 1$, *taken in cyclic order. Let* $J_1, \cdots, J_k$ *be a second such division. Let there exist a reordering* $i_1, \cdots, i_k$ *of* $1, \cdots, k$ *such that* J_{i_j} *has the same length as* I_j. *Let the transformation* T *be defined on* I *apart from the end points of the* I_j *by mapping* I_j *in a linear sense-preserving manner on* J_{i_j}. *For* $r \in I$ *we consider the points* $T^n r$ *for all integral values* n *for which they are defined, denoting their totality by* $\{T^n r\}$. *The set* $\{T^n r\}$ *is finite for at most a finite set* L *of points* $p_1, \cdots, p_N$ *of* I. *If* L *is void, every set* $\{T^n r\}$ *is everywhere dense on* I. *If not,* L *divides* I *into a finite number of intervals which can be grouped into collections such that on the union* U *of such a collection either*

(a) T *is periodic, or*

(b) *for every* $r \in U$, $\{T^n r\}$ *is everywhere dense on* U.

We could of course include the case $k = 1$ which corresponds to excluded possibility III of Theorem 1 and is classical.

To prove our corollary, we represent I as the segment $\xi = 0$, $0 < \eta < 1$ in the ζ-plane ($\zeta = \xi + i\eta$) with the points $(0, 0)$, $(0, 1)$ identified. We consider k rectangles R_j, $j = 1, \cdots, k$, in the z-plane ($z = x + iy$) given by

$$0 \leqq x \leqq 3, \quad 0 \leqq y \leqq l_j$$

where l_j is the length of I_j. We form a Riemann surface $\mathfrak{R}$ by performing the following identifications:

(i) the side $x = 3, 0 \leqq y \leqq l_j$ of R_j is identified linearly and with preservation of direction with $\bar{I}_j$;

(ii) the side $x = 0, 0 \leqq y \leqq l_j$ of R_j is identified linearly and with preservation of direction with $\bar{J}_{i_j}$;

(iii) the segments $2 \leqq x \leqq 3, y = l_j$ on R_j and $2 \leqq x \leqq 3, y = 0$ on R_{j+1} are identified linearly and with preservation of direction, $j = 1, \cdots, k$, where $k + 1$ is to be interpreted as 1;

(iv) if $\begin{pmatrix} 1, & 2, & \cdots, k \\ q_1, & q_2, & \cdots, q_k \end{pmatrix}$ is the permutation inverse to $\begin{pmatrix} 1, & 2, & \cdots, k \\ i_1, & i_2, & \cdots, i_k \end{pmatrix}$, the segments $0 \leqq x \leqq 1, y = l_{q_j}$ on R_{q_j} and $0 \leqq x \leqq 1, y = 0$ on R_{q_j+1} are identified linearly and with preservation of direction, $j = 1, \cdots, k$, the same convention applying as in (iii).

The points of $\mathfrak{R}$ are the points of I, the points of the interiors of the rectangles $R_j, j = 1, \cdots, k$, and the points of the (linear) interiors of the segments in (iii) and (iv). At all points of $\mathfrak{R}$ local uniformizing parameters may be assigned by using the Euclidean geometry, taking account of the identifications if necessary. The same is true for all boundary points of $\mathfrak{R}$ except those arising from the points $(2, l_j) \equiv (2, 0)$ in (iii) and $(1, l_{q_j}) \equiv (1, 0)$ in (iv). At them we use respectively $(2 + il_j - z)^{1/2}$ in R_j, $(2 - z)^{1/2}$ in R_{j+1} and $(z - (1 + il_{q_j}))^{1/2}$ in R_{q_j}, $(z - 1)^{1/2}$ in R_{q_j+1}, the roots in each case chosen to agree with the standard definition of boundary uniformizer (cf. [2; p. 35]). Then it is clear that $\mathfrak{R}$ is a finite oriented Riemann surface. If ζ is the (many-valued) function determined on $\mathfrak{R}$ by defining $\zeta(z)$ on each R_j by the rigid imbedding corresponding to the identifications (i) and (ii), we see that $d\zeta^2$ defines a quadratic differential on $\mathfrak{R}$ (indeed $d\zeta$ itself defines a differential). Evidently $d\zeta^2$ is a regular positive quadratic differential on $\mathfrak{R}$. The trajectories of $d\zeta^2$ in $\mathfrak{R}$ joining pairs of points in C, if such exist, meet I in exactly the points $p_1, \cdots, p_N$. Thus the basic domains associated with the structure of $d\zeta^2$ on $\mathfrak{R}$ meet I in collections of intervals from among those with end points at $p_1, \cdots, p_N$. The only possible basic domains are ring and density domains. Thus on such a set U, either T is periodic, or for every $r \in U$ the set $\{T^n r\}$ is everywhere dense. If there are no trajectories of $d\zeta^2$ in $\mathfrak{R}$ joining pairs of points in C, then $\mathfrak{R}$ is itself a basic domain. Since $d\zeta^2$ has at least one boundary zero on $\mathfrak{R}$, the latter cannot be a ring domain, and thus is a density domain.

There can be little doubt that the Basic Structure Theorem admits a straightforward extension to nonorientable Riemann surfaces which would allow us to admit also sense-reversing linear mappings on the intervals in the preceding corollary.

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