STRUCTURE OF CLEFT RINGS II

BY

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I. INTRODUCTION AND PRELIMINARIES

1A. Introduction

Let R be a ring with the minimum condition on its set of left ideals. A cleaving for R is a direct decomposition, as an additive group,

$$R = S \oplus N$$

where S is a semisimple subring and N is the radical of R. Any algebra over a field K such that R/N is a separable algebra of finite rank over K affords an example of such a ring by virtue of the Wedderburn Principal Theorem.

This paper is a sequel to [8] appearing in this journal. Here we develop the concepts of structural modules, structures of modules, and structures of rings which were introduced in [8]. Certain relations between structural modules and the lattices of submodules of a module are developed in Part II with the view of application in Parts III and IV. In Part III, particular submodules of a structural module are identified as modules which are isomorphic to those formed by the endomorphism fields of an irreducible R-module in one case and to the cohomology modules $H^1(R, \operatorname{Hom}_K(F_i, F_j))$ in another case.

The structures of rings were used in [8] to give conditions which characterized when there exists an extension $I: R \to R'$ of an isomorphism $I_0: S \to S'$ of the semisimple components of two cleft rings R and R'. Such a condition was expressed in terms of the conformality of the structures of R and R'. In Part III, we give a condition which is equivalent to comformality, but which is simpler in statement. This condition demands that there exist an isomorphism of the structural modules which satisfies a certain commutativity relation with the coboundary operator.

In the final part, there is presented an application of these results to graded rings. A *grading* of a cleft ring R is a direct decomposition

$$R = S \oplus M \oplus M^2 \oplus \cdots \oplus M^r$$

where S is a semisimple subring, M is an (S, S)-submodule, M^q is the (S, S)-module generated by products of q elements of M and $N = \oplus_{q=1}^{r} M^q$. Here we show that there exists an extension to an automorphism of R of any isomorphisms of the semisimple component of one grading to the semisimple component of a second grading; moreover, the automorphism may be specified

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to map the components of the first grading onto the corresponding components of the second grading. It is also shown that any automorphism of a semisimple component of a cleft ring R may be extended to an automorphism of R leaving the (S, S)-submodules of R invariant. This result is also extended to a class of semiprimary rings whose radical satisfies $\bigcap_{q=1}^{\infty} N^q = 0$, which are complete in the N-adic topology and for which R/N^q satisfies the minimum condition on the set of left ideals.

1B. Summary of previous results

Here we review the basic ideas of [8] in order to establish our notation and to provide an outline of the theory which we previously developed. All modules introduced will be left modules unless it is otherwise specified; furthermore, they will be assumed to possess a finite composition series. Since S is a semisimple ring with minimum condition, $S = \oplus_{i=1}^{k} S_i$ where S_i is a simple ideal with identity e_i. Let $F_1, F_2, \cdots, F_k$ be a set of R- and S-irreducible modules such that $S_i F_i \neq 0$. Let K_i be the endomorphism sfield of F_i; we assume that the elements of K_i also act on the left as operators of F_i.

Let $R_{ji} = e_j R e_i$; these are (S_j, S_i)-modules[2] and are called the Cartan submodules of R. We have that $R = \oplus_{j,i=1}^{k} R_{ji}$. Also R is the direct sum of indecomposable left ideals $R\varepsilon$ where ε is a primitive idempotent of R. Then $R\varepsilon/N\varepsilon$ is an irreducible left R-module, and $N\varepsilon$ is a maximal submodule. Two such ideals $R\varepsilon$ and $R\varepsilon'$ are isomorphic if and only if the modules $R\varepsilon/N\varepsilon$ and $R\varepsilon'/N\varepsilon'$ are isomorphic. We will let U_i, $i = 1, 2, \cdots, k$, be a set of modules such that U_i is isomorphic to an indecomposable left-ideal component of R and U_i/NU_i is isomorphic to F_i. These will be called the *principal indecomposable modules of R*.

Because of Proposition 1.1 of [8], R may be regarded as the direct sum of ideals each of which is an algebra over some field. Then we reduce our considerations to the case that R is an algebra of possibly infinite dimension over a field K.

A representation module of an (S_j, S_i)-module M is the (K_j, K_i)-module $\mathrm{Hom}_{(S_j,S_i)}(M, \mathrm{Hom}_K(F_i, F_j))$. The structural modules H_{ji}, $i, j = 1, 2, \cdots, k$, are defined as

$$H_{ji} = \mathrm{Hom}_{(S,S)}(R, \mathrm{Hom}_K(F_i, F_j)) = \mathrm{Hom}_{(S_j,S_i)}(R_{ji}, \mathrm{Hom}_K(F_i, F_j)).$$

The identification may be made since $\mathrm{Hom}_{(S,S)}(R_{ml}, \mathrm{Hom}_K(F_i, F_j)) = 0$ unless $j = m$ and $i = l$; this is because $\gamma_m \alpha \gamma_l = 0$, and hence $\gamma_m \psi(\alpha)\gamma_l = 0$ unless $j = m$ and $i = l$ when $\psi \in \mathrm{Hom}_{(S,S)}(R_{ji}, \mathrm{Hom}_K(F_i, F_j))$, $\alpha \in R_{ji}$, $\gamma_m \in S_m$, and $\gamma_l \in S_l$.

A structural element $\psi[f^*, f]$ of a module X is an element of H_{ji} which is defined for $f^* \in \mathrm{Hom}_S(X, F_j)$ and $f \in \mathrm{Hom}_S(F_i, X)$ by $\psi[f^*, f](\alpha) = f^*\alpha_L f$ where α_L is the left multiplication on X determined by $\alpha \in R$. We noted in

[2] By an (S_j, S_i)-module X, we mean a double module; that is, X is a left S_j-module and a right S_i-module such that $(\alpha x)\beta = \alpha(x\beta)$ for $\alpha \in S_j$ and $\beta \in S_i$.

[8] that $\mathrm{Hom}_S(X, F_j)$ can be identified with the dual module $\mathrm{Hom}_S{}^*(F_j, X)$ of $\mathrm{Hom}_S(F_j, X)$. A structure $|\psi|$ of X is a set of bilinear mappings

$$\psi : \mathrm{Hom}_S{}^*(F_j, X) \times \mathrm{Hom}_S(F_i, X) \to H_{ji}$$

defined for $i, j = 1, 2, \cdots, k$ by $(f^*, f) \to \psi[f^*, f]$. A structure $\Sigma(R, S)$ of a ring R is a set of structures $|\psi_i|$ of the principal indecomposable modules U_i, $i = 1, 2, \cdots, k$.

Let $R = S \oplus N$ and $R = S' \oplus N$ be two cleavings for a ring R. Let $I_0 : S \to S'$ be an isomorphism. Let $I_i : S_i \to S_i'$, $i = 1, 2, \cdots, k$, be the isomorphism of the simple ideal component S_i of S onto the simple ideal component S_i' which is induced by I_0. An I_i-isomorphism φ of an S_i-module A onto an S_i'-module, for example, is understood to be an isomorphism of the additive groups such that $\varphi(\alpha x) = \alpha^{I_i}\varphi(x)$ when $\alpha \in S_i$ and $x \in A$. In the case of double (S_j, S_i)-modules, we speak of (I_j, I_i)-isomorphisms.

The isomorphism I_i then induces an I_i-isomorphism ω_i of the irreducible module F_i associated with S_i onto an irreducible module F_i' which is similarly associated with S_i'. This in turn induces an isomorphism, which we again denote by I_i, of the endomorphism ring K_i of F_i onto the endomorphism ring K_i' of F_i'. Let H_{ji}', $i, j, = 1, 2, \cdots, k$, be the structural modules determined from the cleaving $R = S' \oplus N$. The principal theorem for double modules of [8] asserts that there exists an (I_0, I_0)-isomorphism of R considered as an (S, S)-module onto R considered as an (S', S')-module if and only if for all $i, j = 1, 2, \cdots, k$ there exist (I_j, I_i)-isomorphisms[3] $\theta : H_{ji} \to H_{ji}'$.

In order that I be a ring isomorphism, certain other conditions must be satisfied by the isomorphisms θ inducing I. Let $|\psi_\xi|$ and $|\psi_\xi'|$ be the structures of the principal indecomposable module U_ξ of R relative to the cleavings $R = S \oplus N$ and $R = S' \oplus N$, respectively. Then the principal theorem of [8] asserts that a necessary and sufficient condition for I to be an isomorphism is that there exists for ξ, $i = 1, 2, \cdots, k$, I_i-isomorphisms φ and φ^* where φ^* is contragredient to φ and

$$\varphi : \mathrm{Hom}_S(F_i, U_\xi) \to \mathrm{Hom}_{S'}(F_i', U_\xi'),$$

$$\varphi^* : \mathrm{Hom}_S{}^*(F_j, U_\xi) \to \mathrm{Hom}_{S'}{}^*(F_j', U_\xi')$$

such that

$$\theta\psi_\xi[f^*, f] = \psi_\xi'[\varphi^* f^*, \varphi f]$$

where $f^* \in \mathrm{Hom}_S{}^*(F_j, U_\xi)$ and $f \in \mathrm{Hom}_S(F_i, U_\xi)$. When such conditions are satisfied, it is said that the structures $\Sigma(R, S)$ and $\Sigma(R, S')$ are *conformal.*

1C. Extensions and cocycles

In this section, we review the theory of extensions for the purpose of establishing our notation (cf. [2; p. 289] or [5]). An extension (X, π, φ) of an

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[3] Actually, we should write θ_{ji}, but the notation is more convenient when the subscripts are suppressed.

R-module B by an R-module A is an exact sequence formed with an R-module X and R-homomorphisms π and φ such that

$$(1.1) \qquad 0 \to B \xrightarrow{\varphi} X \xrightarrow{\pi} A \to 0.$$

Since B, X, and A are also S-modules, the sequence (1.1) splits as an exact sequence of S-modules and S-homomorphisms. Thus there exists an exact sequence

$$(1.2) \qquad 0 \leftarrow B \xleftarrow{\varphi^{-1}} X \xleftarrow{\pi^{-1}} A \leftarrow 0$$

of S-modules and S-homomorphisms such that $\pi\pi^{-1} = 1_A$ is the identity isomorphism of A and $\varphi^{-1}\varphi$ is the identity isomorphism 1_B of B. Sequence (1.2) will be called a *splitting sequence* to the sequence (1.1) or to the extension (X, π, φ).

The homomorphism π^{-1} is not uniquely determined; however, given π^{-1}, there is only one homomorphism φ^{-1} such that (1.2) is exact and $\varphi^{-1}\varphi = 1_B$. It then follows that $X = \pi^{-1}A \oplus B$ when it is considered as a sum of S-modules. We will call the homomorphisms π^{-1} *cross-sections* of the extensions (X, π, φ).

Let $\rho(\alpha):A \to B$ be the K-homomorphism determined by[4]

$$(1.3) \qquad \rho(\alpha) = \varphi^{-1}(\alpha_L \pi^{-1} - \pi^{-1}\alpha_L) = \varphi^{-1}\alpha_L \pi^{-1}$$

for $\alpha \in R$ where α_L is the left multiplication determined by α on A and on X. If φ is the inclusion mapping, we adopt the convention of writing for $\alpha \in R$

$$(1.3a) \qquad \rho(\alpha) = \alpha_L \pi^{-1} - \pi^{-1}\alpha_L .$$

Now $\rho:\alpha \to \rho(\alpha)$ is a 1-cocycle because for $\alpha, \beta \in R$

$$(1.4) \qquad \rho(\alpha\beta) = \alpha\rho(\beta) + \rho(\alpha)\beta$$

where we set $\alpha_L\, \rho(\beta) = \alpha\rho(\beta)$ and $\rho(\alpha)\beta_L = \rho(\alpha)\beta$. Furthermore, $\rho(S) = 0$ so that $\rho(\lambda\alpha\mu) = \lambda\rho(\alpha)\mu$ where $\lambda, \mu \in S$ and $\alpha \in R$. Such 1-cocycles will be called the *cocycles of the extension* (X, π, φ) or *S-cocycles*. They form a subgroup $Z_S^1\ (R, \operatorname{Hom}_K(A, B))$ of the additive group of 1-cocycles. The S-cocycles ρ for which $\rho(\alpha) = \alpha\lambda - \lambda\alpha$ where $\lambda \in \operatorname{Hom}_K(A, B)$ and $\alpha \in R$ are the coboundaries. Because $\rho(S) = 0$, λ actually is in $\operatorname{Hom}_S(A, B)$. These coboundaries are the cocycles which are derived from the split extensions. They form a subgroup $B_S^1 = B_S^1(R, \operatorname{Hom}_K(A, B))$ of Z_S^1. It is not difficult to verify that the factor group Z_S^1/B_S^1 is isomorphic to the cohomology group $H^1(R, \operatorname{Hom}_K(A, B))$. This fact may also be derived from the theory of relative homology (cf. [6]).

It follows from the theory of extensions that two cross-sections of the same extension determine cohomologous cocycles. Furthermore, Hochschild has

[4] While $\varphi^{-1}\pi^{-1} = 0$ because of the splitting sequence (1.2), we prefer to use the form $\varphi^{-1}(\alpha_L \pi^{-1} - \pi^{-1}\alpha_L)$ for a cocycle because of its relation to the conventional formula (1.3a).

shown that there is an isomorphism between the cohomology group $H^2(R, \mathrm{Hom}_K(A, B))$ and the group of extensions under the Baer multiplication. In particular, to every cocycle there corresponds an extension.

In what follows, we will consider A often to be an irreducible R-module with endomorphism sfield K_A. Then A is a left K_A-module and $\mathrm{Hom}_K(A, B)$ is a right K_A-module. Then it follows that Z_S^1, B_S^1, and H_S^1 are right K_A-modules.

II. Composition Forms and Structures of Modules

2A. Composition forms

A *composition form* $\mathcal{C}$ of a module X given by a composition series

$$X = X_1 \supset X_2 \supset \cdots \supset X_t \supset X_{t+1} = 0 \tag{2.1}$$

is a composite concept consisting of a set of extensions

$$0 \to X_{\mu+1} \xrightarrow{\varphi_\mu} X_\mu \xrightarrow{\pi_\mu} F_{i_\mu} \to 0 \tag{2.2}$$

for $\mu = 1, 2, \cdots, t$ and corresponding splitting sequences given by cross-sections $\overset{-1}{\pi_\mu}$

$$0 \leftarrow X_{\mu+1} \xleftarrow{\overset{-1}{\varphi_\mu}} X_\mu \xleftarrow{\overset{-1}{\pi_\mu}} F_{i_\mu} \leftarrow 0 \tag{2.3}$$

where φ_μ is the inclusion mapping and $\overset{-1}{\varphi_\mu}$ is, therefore, the projection of X^μ onto $X_{\mu+1}$ with kernel $\overset{-1}{\pi_\mu} F_{i_\mu}$. We denote this composition form by $\mathcal{C}(\pi_\mu, \overset{-1}{\pi_\mu})$. The cocycles χ_μ defined by the sequences (2.3) will be called the *cocycles of the form* $\mathcal{C}(\pi_\mu, \overset{-1}{\pi_\mu})$. Because $\overset{-1}{\varphi_\mu}$ is the identity on $X_{\mu+1}$, we have

$$\chi_\mu(\alpha) = \alpha_L \overset{-1}{\pi_\mu} - \overset{-1}{\pi_\mu} \alpha_L . \tag{2.4}$$

Proposition 2.1. *Given a composition form $\mathcal{C}(\pi_\mu, \overset{-1}{\pi_\mu})$ with a composition series* (2.1), *extensions* (2.2), *and splitting sequences* (2.3), *there exists a direct family of homomorphisms $\{f_\mu^*, f_\mu \mid 1 \leqq \mu \leqq t\}$ representing X as the S-direct sum of the modules $F_1, F_2, \cdots, F_k$ such that*

$$f_\mu^* = \pi_\mu p_\mu \quad \textit{and} \quad f_\mu = i_\mu \overset{-1}{\pi_\mu} \tag{2.5}$$

for $1 \leqq \mu \leqq t$ and where $p_\mu : X \to X_\mu$ is a projection with kernel

$$\oplus_{\xi=1}^{\mu-1} f_\xi F_{i_\xi}$$

and $i_\mu : X_\mu \to X$ is the inclusion mapping.

The direct family[5] $\{f_\mu^*, f_\mu\}$ thus determined will be called the *direct family of the composition form* $\mathcal{C}$; sometimes we distinguish $\mathcal{C}$ by setting $\mathcal{C} = \mathcal{C}(f_\mu^*, f_\mu)$.

Proof. Clearly f_μ^* and f_μ defined in (2.5) are S-epimorphisms and S-monomorphisms, respectively. We wish to show that they form a direct family.

[5] Direct families are discussed in §1C of [8].

First, $p_\mu f_\nu = 0$ if $\nu < \mu$, and $p_\mu f_\nu = f$ if $\nu \geqq \mu$. But if $\nu > \mu$, $f_\nu F_{i_\mu} \subseteqq X_{\mu+1}$ so that $\pi_\mu f_\nu = 0$. Hence $f_\mu^* f_\mu = 0$ when $\mu \neq \nu$. On the other hand, $f_\mu^* f_\mu = \pi_\mu p_\mu i_\mu \pi_\mu^{-1} = \pi_\mu \pi_\mu^{-1} = 1$.

Next we prove that $\sum_{\mu=1}^t f_\mu f_\mu^* = 1$. Let $A_\mu = f_\mu F_{i_\mu}$, and let $B_s = \oplus_{\mu=1}^s A_\mu$. We argue by induction that if $x \in B_s$,

$$(2.6) \qquad \sum_{\mu=1}^t f_\mu f_\mu^* x = \sum_{\mu=1}^s f_\mu f_\mu^* x = x.$$

First, if $s = 1$, then $f_\mu^* x = 0$ when $\mu > 1$. Also the restriction of f_1^* to A_1 is an isomorphism. But since $f_1^* f_1 = 1$, $f_1 f_1^*$ is the identity on A_1; that is, $f_1 f_1^* x = x$ for $x \in A_1 = B_1$. Suppose now that (2.6) holds with s replaced by $s - 1$. Let $x \in B_s$. Then $y = x - f_s f_s^* x$ is in B_s and $f_s^* y = 0$. Hence $y \in B_{s-1}$ and $\sum_{\mu=1}^s f_\mu f_\mu^* y = y$. From this, follows (2.6).

The structural elements $\psi[f_\mu^*, f_\nu]$, $\mu, \nu = 1, 2, \cdots, t$, determined by the direct family $\{f_\mu^*, f_\nu\}$ of a composition form $\mathfrak{C}$ are called the *structural elements* of the composition form $\mathfrak{C}$. The following proposition summarizes their important properties.

PROPOSITION 2.2. *Let $\psi[f_\mu^*, f_\nu]$ be the structural elements of a composition form of a module X. Let χ_μ, $\mu = 1, 2, \cdots, t$, be the cocycles of $\mathfrak{C}(f_\mu^*, f_\nu)$.*

(i) *If $\mu < \nu$, then $\psi[f_\mu^*, f_\nu] = 0$.*

(ii) *If $\mu > \nu$, then $\psi[f_\mu^*, f_\nu] = f_\mu^* \chi_\nu$.*

(iii) *If $\mu = \nu$, then $\psi[f_\mu^*, f_\nu] = \psi[f_\mu^*, f_\mu] = \iota$*

where ι is the mapping of R onto the ring of K_{i_μ}-endomorphisms of F_{i_μ} given by $\alpha \to \alpha_L$.

Proof. (i) We have $Rf_\nu F_{i_\nu} = R\pi_\nu^{-1} F_{i_\nu} = \pi_\nu^{-1} RF_{i_\nu} + \chi_\nu(R) F_{i_\nu} \subseteqq X_\nu$. But $X_\nu \subseteqq X_{\mu+1}$, the kernel of π_μ; so $f_\mu^* X_\nu \subseteqq \pi_\mu p_\mu X_{\mu+1} = \pi_\mu X_{\mu+1} = 0$. Therefore, $\psi[f_\mu^*, f_\nu] = 0$.

(ii) $f_\mu^* \chi_\nu(\alpha) = f_\mu^* i_\nu \chi_\nu(\alpha) = f_\mu^* i_\nu(\alpha_L \pi_\nu^{-1} - \pi_\nu^{-1} \alpha_L) = f_\mu^* \alpha_L f_\nu = \psi[f_\mu^*, f_\nu](\alpha)$.

(iii) Since $\alpha f_\mu F_{i_\mu} = \alpha i_\mu \pi_\mu^{-1} F_{i_\mu} = i_\mu \alpha \pi_\mu^{-1} F_{i_\mu} \subseteqq X_\mu$,

$$f_\mu^* \alpha_L f_\mu = \pi_\mu p_\mu \alpha_L f_\mu = \pi_\mu \alpha_L f_\mu = \alpha_L \pi_\mu f_\mu = \alpha_L f_\mu^* f_\mu = \alpha_L .$$

2B. Principal indecomposable modules

A *principal indecomposable R-module* is a module which is isomorphic to an indecomposable left ideal of R. It may be also characterized as an indecomposable projective R-module (cf. [1] or [2]). We recall that NU_i is the unique maximal submodule of U_i and that we have chosen U_i so that the exact sequence (2.7) may be formed:

$$(2.7) \qquad 0 \to NU_i \to U_i \xrightarrow{\lambda_i} F_i \to 0$$

with the inclusion mapping $NU_i \to U_i$. Let λ_i^{-1} be a cross-section for (2.7) which determines the splitting sequence

$$(2.8) \qquad 0 \leftarrow NU_i \xleftarrow{\varphi_i^{-1}} U_i \xleftarrow{\lambda_i^{-1}} F_i \leftarrow 0.$$

Let ρ_i be the cocycle defined from (2.8); we will call such a cocycle a *principal cocycle*, and we will call a set $\{\rho_i \mid i = 1, 2, \cdots, k\}$ of principal cocycles which is derived as above from each of the distinct principal indecomposable modules a *complete set of principal cocycles.* The corresponding cross-sections will be called *principal cross-sections.*

Now let there be given a composition series for an R-module X

$$X = X_1 \supset X_2 \supset \cdots \supset X_{t+1} = 0 \tag{2.9}$$

and extensions defined for $\mu = 1, 2, \cdots, t$

$$0 \to X_{\mu+1} \to X_\mu \xrightarrow{\pi_\mu} F_{i_\mu} \to 0. \tag{2.10}$$

Because U_i is projective, we may form the following commutative diagram with R-homomorphisms:

$$\begin{array}{ccccccccc} 0 & \to & NU_{i_\mu} & \to & U_{i_\mu} & \xrightarrow{\lambda_{i_\mu}} & F_{i_\mu} & \to & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \theta_\mu} & & \| & & \\ 0 & \to & X_{\mu+1} & \to & X_\mu & \xrightarrow{\pi_\mu} & F_{i_\mu} & \to & 0. \end{array} \tag{2.11}$$

Then $\pi_\mu^{-1} = \theta\lambda_{i_\mu}^{-1}$ is a cross-section for the lower sequence in (2.11) and gives a splitting sequence for (2.10). The composition form $\mathcal{C}(\pi_\mu, \pi_\mu^{-1})$ which is thus obtained for X will be said to be formed with the complete set of principal cocycles ρ_i, $i = 1, 2, \cdots, k$, and the homomorphisms θ_μ, $\mu = 1, 2, \cdots, t$. One may verify that the cocycles of this composition form are $\theta_\mu \rho_i$.

2C. Structures of modules

Certain submodules of a module X frequently occur in our investigation; because of this, we will formalize our method of handling them. Also we will study their relationship to the structures of X.

Let $f \in \mathrm{Hom}_S(F_i, X)$. Then set $A(f) = fF_i$. This is an irreducible S-submodule of X. Let $X(f) = RA(f)$; then $X(f)$ is an R-submodule of X.

PROPOSITION 2.3. *Let A be an irreducible S-module. Then $RA = A \oplus NA$. Furthermore, RA is an epimorph of a principal indecomposable submodule U, and NA is its unique maximal submodule.*

Proof. We have that $RA = (S + N)A = A + NA$. Either $A \cap NA = 0$ or $A \subseteqq NA$. Should the latter case hold, then $A \subseteqq RA = NA = N^2A = \cdots = N^{r+1}A = 0$ if $r + 1$ is the index of the radical N. Hence as $A \neq 0$, $RA = A \oplus NA$.

Let U be a principal indecomposable left R-module such that U/NU is isomorphic to A. Then there exists an epimorphism $\lambda: U \to A$ with kernel NU. Also there exists an R-epimorphism $\pi: RA \to A$ with kernel NA. Because U is projective, there exists a homomorphism $\varphi: U \to RA$ such that $\pi\varphi = \lambda$. We wish to show that φ is an epimorphism.

It follows from Proposition 3.5 of [8] that $U = B \oplus NB$ where B is a suitably chosen irreducible S-submodule of U; furthermore, $NB = NU$. Let $C = \varphi B$. Then $\pi C = \pi\varphi B = A$. Hence $C \cap NA = 0$. Let x be an S-generator for A, and y the element of C such that $\pi x = \pi y$. Since $Rx = RSx = RA$, there exists $\alpha \in R$ such that $\alpha x = y$. But $\pi(\alpha x) = \pi x$. Hence $\alpha = 1 + \eta$ where $\eta \in N$. Since η is quasi-regular, there exists $\beta \in R$ such that $\beta\alpha = 1$. Hence $\beta y = x$. This means that $RC = Ry = R\beta y = Rx = RA$. But $\varphi U = \varphi RB = RC$. Hence φ is an epimorphism.

The kernel V of φ is contained in the unique maximal submodule NU of U. Hence U/V and thus RA have unique maximal submodules. Thus NA is the unique maximal submodule of A. This concludes the proof.

In particular, we have that

$$X(f) = RA(f) = A(f) \oplus NX(f) = A(f) \oplus NA(f). \tag{2.12}$$

To each element f^* in $\mathrm{Hom}_S^*(F_i, X)$, there corresponds a maximal R-submodule $X(f^*)$ such that $f^*X(f^*) = 0$. It is easy to see that $X(f^*)$ is unique.

We define the *degree of a homomorphism* $f \in \mathrm{Hom}_S(F_i, X)$ to be the nonnegative l such that $fF_i = A(f) \subseteqq N^lX$ but $A(f) \cap N^{l+1} = 0$. Hence $X(f) \subseteqq N^lX$, but $X(f) \cap N^{l+1}X \neq X(f)$. We define the *degree* of $f^* \in \mathrm{Hom}_S^*(F_i, X)$ to be the nonnegative integer l such that $X(f^*) \supseteqq N^{l+1}X$ but $f^*N^lX \neq 0$.

LEMMA 2.4. *Let* $|\psi|$ *be the structure of a module* X. *Let* $f^* \in \mathrm{Hom}_S^*(F_j, X)$ *and* $f \in \mathrm{Hom}_S(F_i, X)$. *Then* $\psi[f^*, f] = 0$ *if* $\deg f^* < \deg f$, *or if* $\deg f^* = \deg f$ *and* $f^*f = 0$.

Proof. When $\deg f^* < \deg f = l$, $X(f) \subseteqq N^lX \subseteqq X(f^*)$. Also when $\deg f^* = \deg f = l$, $NX(f) \subseteqq N^{l+1}X \subseteqq X(f^*)$ and, if $f^*f = 0$, $A(f) \subseteqq X(f^*)$. Thus, in both cases, $X(f) = A(f) \oplus NX(f) \subseteqq X(f^*)$; that is, $f^*X(f) = f^*RA(f) = f^*RfF_i = 0$. Hence $\psi[f^*, f](R) = f^*Rf = 0$.

Let X again be an R-module with a composition series

$$X = X_1 \supset X_2 \supset \cdots \supset X_t \supset X_{t+1} = 0 \tag{2.13}$$

which is a refinement of the upper Loewy series

$$X \supset NX \supset N^2X \supset \cdots \supset N^rX \supset N^{r+1}X = 0. \tag{2.14}$$

A composition form $\mathfrak{C}$ given with such a series as (2.13) will be called a *refined composition form.*

LEMMA 2.5. *Let* $\mathfrak{C}$ *be a refined composition form which is given by the composition series* (2.13). *Let* $\{f_\mu^*, f_\mu \mid \mu = 1, 2, \cdots, t\}$ *be the direct family of* $\mathfrak{C}$. *Then*

$$\deg f_\mu^* = \deg f_\mu,$$

and if $\mu < \nu$,

$$\deg f_\mu \leqq \deg f_\nu \quad \text{and} \quad \deg f_\mu^* \leqq \deg f_\nu^*.$$

Conversely, if $\deg f_\mu < \deg f_\nu$ *or* $\deg f_\mu^* < \deg f_\nu^*$, *then* $\mu < \nu$.

Proof. From Proposition 2.1, it follows that $f_\mu{}^*X_{\mu+1} = \pi_\mu p_\mu X_{\mu+1} = 0$. But $A(f_\mu) = f_\mu F_{i_\mu} = i_\mu \pi_\mu^{-1} F_{i_\mu} \subseteqq X_\mu$. Since $f_\mu{}^*A(f_\mu) \neq 0$,

$$X_\mu = A(f_\mu) \oplus X_{\mu+1}.$$

Because (2.13) is a refinement of (2.14), there exists a positive integer l such that $N^l X \supseteqq X_\mu \supset X_{\mu+1} \supseteqq N^{l+1}X$. Thus $f_\mu{}^*N^{l+1}X = 0$ while $f_\mu{}^*N^l X \neq 0$, and $A(f_\mu) \subseteqq N^l X$ while $A(f_\mu) \cap N^{l+1}X = 0$. Hence $l = \deg f_\mu{}^* = \deg f_\mu$.

If $\nu > \mu$, then $A(f_\nu) \subseteqq X_\nu \subseteqq X_\mu \subseteqq N^l X$. But if $\deg f_\nu = m$, then m is the largest integer such that $A(f_\nu) \subseteqq N^m X$. Hence $l \leqq m$; that is, $\deg f_\mu \leqq \deg f_\nu$. From the first result, it follows that $\deg f_\mu{}^* \leqq \deg f_\nu{}^*$. To establish the stated converse, merely observe that we have shown that if $\deg f_\mu > \deg f_\nu$ or $\deg f_\mu{}^* > \deg f_\nu{}^*$, then $\mu \geqq \nu$. The result then follows by an obvious change of notation since clearly $\mu \neq \nu$.

III. Homological Interpretation of Structural Modules

3A. Submodules of the structural modules

If M is a (two-sided) ideal of R, then it is a (S, S)-module. From the theory of functors, it is known that $\mathrm{Hom}_{(S,S)}(R/M, \mathrm{Hom}_K(F_i, F_j))$ may be regarded as a (K_j, K_i)-submodule of the (K_j, K_i)-module

$$H_{ji} = \mathrm{Hom}_{(S,S)}(R, \mathrm{Hom}_K(F_i, F_j)).$$

In particular, we define

$$H_{ji}^q = \mathrm{Hom}_{(S,S)}(R/N^{q+1}, \mathrm{Hom}_K(F_i, F_j)). \tag{3.1}$$

Then the module H_{ji}^q may be regarded as the submodule of elements of H_{ji} which vanish on N^{q+1}. We have

$$0 \subset H_{ji}^0 \subset H_{ji}^1 \subset \cdots \subset H_{ji}^{r+1} = H_{ji} \tag{3.2}$$

where $r + 1$ is the index of the radical of R.

The natural isomorphism of R/N onto S induces an isomorphism of $\mathrm{Hom}_{(S,S)}(S, \mathrm{Hom}_K(F_i, F_j))$ onto H_{ji}^0; we will use this isomorphism to identify these two modules.

The module $H_{ji}^q = \mathrm{Hom}_{(S,S)}(R/N^{q+1}, \mathrm{Hom}_K(F_i, F_j))$ may be interpreted as the representation module[6] of the ring $R_q = R/N^{q+1}$ with radical $N_q = N/N^{q+1}$. Since $S \cap N^{q+1} = 0$, we may and will identify S with the semisimple subring $(S + N^{q+1})/N^{q+1}$ of R_q to obtain the splitting

$$R_q = S \oplus N_q. \tag{3.3}$$

Let $T_{ji} = \mathrm{Hom}_{(S,S)}(R/S, \mathrm{Hom}_K(F_i, F_j))$. This is the module of elements ψ of H_{ji} such that $\psi(S) = 0$. Clearly, it is isomorphic to

[6] Cf. §1B or Part III of [8].

$\mathrm{Hom}_{(S,S)}(N, \mathrm{Hom}_K(F_i, F_j))$. Let

$$T_{ji}^q = \mathrm{Hom}_{(S,S)}(R/(S + N^{q+1}), \mathrm{Hom}_K(F_i, F_j)).$$

Since this is the submodule of H_{ji} consisting of the elements $\psi \in H_{ji}$ such that $\psi(S) = 0$ and $\psi(N^{q+1}) = 0$, we will identify T_{ji}^q with $\mathrm{Hom}_{(S,S)}(R_q/S, \mathrm{Hom}_K(F_i, F_j))$. Clearly T_{ji}^q is isomorphic to $\mathrm{Hom}_{(S,S)}(N_q, \mathrm{Hom}_K(F_i, F_j))$. Because of the cleavings of R_q, we obtain the direct decompositions

$$H_{ji} = H_{ji}^0 \oplus T_{ji} \quad \text{and} \quad H_{ji}^q = H_{ji}^0 \oplus T_{ji}^q.$$

In particular, $H_{ji}^1 = H_{ji}^0 \oplus T_{ji}^1$.

LEMMA 3.1. *Every element $\psi \in T_{ji}$ can be represented as a structural element $\psi_i[f^*, f_1]$ belonging to a refined composition form of the principal indecomposable module U_i. Here f_1 may be taken to be a generating element*[7] *for U_i.*

Proof. For convenience, set $U_i = X$. Let f_1 be a generating homomorphism for $U_i = X$. Then $A(f_1) \oplus NX = X$. It follows from Proposition 3.6 of [8] that there exists $f^* \in \mathrm{Hom}_S{}^*(F_i, X)$ such that $\psi = \psi_i[f^*, f_1]$. Furthermore, as $\psi(S) = 0$, $f^*Sf_1 = 0$; hence $f^*f_1 = 0$. Then there exists $l > 1$ such that $f^*N^lX \neq 0$ and $f^*N^{l+1}X = 0$. Let, say, (2.13) be a composition series for X refining (2.14). Then for some ξ, $f^*X_\xi \neq 0$ and $f^*X_{\xi+1} = 0$. Since $X_2 = NX$, $\xi > 1$. Another way of stating this is to say that $f^*X_\mu = f^*X_{\mu+1}$ for $\mu \neq \xi$ and $\xi \neq 1$.

Choose a direct family of monomorphisms[8] $\{f_\mu \mid \mu = 1, 2, \cdots, t\}$ representing X as the S-direct sum of the modules $F_1, F_2, \cdots, F_k$ in the following manner. Let f_1 be the generating element for X chosen in the preceding paragraph. Let f_ξ be such that $f^*f_\xi = 1$. Then $X = A(f_\xi) \oplus X_{\xi+1}$. Choose f_μ, $\mu \neq 1$ and $\mu \neq \xi$, so that $f^*f_\mu = 0$ and $X_\mu = A(f_\mu) \oplus X_{\mu+1}$; this can be done because $f^*X_\mu = f^*X_{\mu+1}$. Let $\{f_\mu{}^*, f_\mu\}$ be the corresponding direct family of homomorphisms. Then the restriction π_μ of $f_\mu{}^*$ to X_μ is an S-homomorphism with kernel $X_{\mu+1}$. Then π_μ is an R-homomorphism, and we may use π_μ, $\mu = 1, 2, \cdots, t$, to form the extensions of a composition form. Here $f_\mu{}^* = \pi_\mu p_\mu$ in the terminology of Proposition 2.1. Let $\pi_\mu^{-1} = p_\mu f_\mu$. Then form the composition form $\mathfrak{C}(\pi_\mu, \pi_\mu^{-1})$; $\{f_\mu{}^*, f_\mu\}$ will be a direct family for C. As $f_\xi{}^*f_\mu = f^*f_\mu$, $\mu = 1, 2, \cdots, t$, $f_\xi{}^* = f^*$. Hence $\psi = \psi_i[f^*, f_1] = \psi_i[f_\xi{}^*, f_1]$.

3B. Cohomology of structural modules

Interpretations of the modules H_{ji}^0 and T_{ji}^k are the objective of this section. For this purpose, we introduce the coboundary operator δ which is a (K_j, K_i)-isomorphism into the (K_j, K_i)-module $C_S^2(R, \mathrm{Hom}_K(F_i, F_j))$ of those

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[7] Cf. [8; §3C].

[8] Cf. §1C of [8].

2-cochains which are also (S, S)-homomorphisms. The defining equation for δ is[9]

$$\delta\psi(\alpha, \beta) = \psi(\alpha\beta) - \alpha\psi(\beta) - \psi(\alpha)\beta. \tag{3.4}$$

PROPOSITION 3.2. *The (K_j, K_i)-module H_{ii}^0 is isomorphic to $K_i = \mathrm{Hom}_S(F_i, F_i)$, and if $\psi \in H_{ii}^0$, $\psi(\alpha) = \sigma\alpha_L$ for some $\sigma \in K_i$. Furthermore, $H_{ji}^0 = 0$ when $j \neq i$.*

Proof. Let f_1 be the element of $\mathrm{Hom}_S(F_i, U_i)$ which is the S-cross-section λ_i^{-1} of the extension

$$0 \to NU_i \to U_i \xrightarrow{\lambda_i} F_i \to 0. \tag{3.5}$$

Then f_1 can be seen to be a generating element of U_i in the sense of [8]. From Proposition 3.6 of [8], it follows that $f^* \to \psi_i[f^*, f_1]$ is a K_j-isomorphism of $\mathrm{Hom}_S{}^*(F_j, U_i)$ onto H_{ji}. If $\psi_i[f^*, f_1] \in H_{ji}^0 = \mathrm{Hom}_{(S,S)}(R/N, \mathrm{Hom}_K(F_i, F_j))$, then $f^*Nf_1 = 0$. This means that $f^*NU_i = 0$, and thus f^* must be an R-homomorphism. But because U_i has a unique maximal submodule NU_i such that U_i/NU_i is isomorphic to F_i, $f^* = 0$ unless $i = j$. Furthermore, if $i = j$, then $f^* = \sigma\lambda_i$ where $\sigma \in K_i$. Hence $\psi_i[f^*, f_1] = \sigma\lambda_i\, \alpha_L\, \lambda_i^{-1} = \sigma\alpha_L$. Thus $\psi(\alpha)$ is nothing more than the mapping $x \to \sigma\alpha_L\, x = \alpha_L\, \sigma x$ of F_i. It is easily seen that the mapping $\psi \to \sigma$ is a (K_i, K_i)-isomorphism of H_{ji}^0 onto $\mathrm{Hom}_S(F_i, F_i) = K_i$.

PROPOSITION 3.3. *Let $\psi[f_\xi^*, f_\eta]$ be a structural element of a refined composition form for a module X. Then for $\alpha, \beta \in R$*

$$\delta\psi[f_\xi^*, f_\eta](\alpha, \beta) = \textstyle\sum_{\xi<\mu<\eta} \psi[f_\xi^*, f_\mu](\alpha)\psi[f_\mu^*, f_\eta](\beta) \tag{3.6}$$

where the summands in (3.6) *are nonzero only if* $\deg f_\xi^* > \deg f_\mu$ *and* $\deg f_\mu^* > \deg f_\eta$.

Proof. Because $\sum_{\mu=1}^t f_\mu f_\mu^* = 1$, we have that

$$\psi[f_\xi^*, f_\eta](\alpha\beta) = \textstyle\sum_{\mu=1}^t \psi[f_\xi^*, f_\mu](\alpha)\psi[f_\mu^*, f_\eta](\beta). \tag{3.7}$$

From Lemma 2.4 it follows that $\psi[f_\xi^*, f_\mu] \neq 0$ only when $\deg f_\xi^* \geqq \deg f_\mu$, and $\psi[f_\mu^*, f_\eta] \neq 0$ only when $\deg f_\mu^* \geqq \deg f_\eta$. Lemma 2.5 implies that the summands of (3.7) are nonzero only when $\deg f_\xi^* = \deg f_\mu$, $\deg f_\mu^* = \deg f_\eta$, or $\xi > \mu > \eta$. Furthermore, we may obtain from Lemma 2.4 that when $\deg f_\xi^* = \deg f_\mu$, $\psi[f_\xi^*, f_\mu] \neq 0$ only when $f_\xi^* f_\mu \neq 0$; this happens only when $\xi = \mu$. Then $\psi[f_\xi^*, f_\mu](\alpha) = \alpha_L$. Likewise when $\deg f_\mu^* = \deg f_\eta$, $\psi[f_\mu^*, f_\eta] \neq 0$ only when $\mu = \eta$, and then $\psi[f_\mu^*, f_\eta](\beta) = \beta_L$. Hence we have from (3.7)

$$\psi[f_\xi^*, f_\eta](\alpha\beta) = \textstyle\sum_{\xi\leqq\mu\leqq\eta} \psi[f_\xi^*, f_\mu](\alpha)\psi[f_\mu^*, f_\eta](\beta). \tag{3.8}$$

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[9] Actually, this is the negative of the coboundary operator usually used in the theory of associative algebras (cf. [5]).

From the preceding remarks and (3.8) follows (3.6). In (3.6) neither $\mu = \xi$ nor $\mu = \eta$. Thus the last remark in the proposition is a direct consequence of Lemma 2.4.

PROPOSITION 3.4. *The (K_j, K_i)-module T^1_{ji} is the (K_j, K_i)-module of S-cocycles $Z^1_S(R, \mathrm{Hom}_K(F_i, F_j))$, which, in turn, is (K_j, K_i)-isomorphic to the cohomology module $H^1(R, \mathrm{Hom}_K(F_i, F_j))$.*

Proof. If $\psi \in T^1_{ji} \subseteqq T_{ji}$, then ψ may be represented as the structural element $\psi = \psi_i[f^*, f_1]$ of a refined composition form for U_i by virtue of Lemma 3.1. As f_1 is a generating element, $Nf_1 F_i = NU_i$ and $N^2 f_1 F_i = N^2 U_i$. Since $\psi(N) \neq 0$ and $\psi(N^2) = 0$, it follows that $\deg f^* - \deg f_1 = 1$. Hence from Proposition 3.3, $\delta\psi = \delta\psi_i[f^*, f_i] = 0$ as all the summands in (3.6) vanish. Thus $T^1_{ji} \subseteqq Z^1_S$.

On the other hand, as we mentioned in §1C, there exists an extension

$$0 \to F_j \to X \to F_i \to 0 \tag{3.9}$$

with a given element $\psi \in Z^1_S(R, \mathrm{Hom}_K(F_i, F_j))$ as the cocycle that is derived from a cross-section. Furthermore, ψ may be represented as a structural element of a composition form of the module X which defines the extension (3.9). This, of course, is a structural element of the module X and, therefore, belongs to H_{ji}. Since $N^2X = 0$, $\psi(N^2) = 0$. Since ψ is an S-cocycle, $\psi(S) = 0$. Thus $\psi \in H^1_{ji}$. This shows that $T^1_{ji} = Z^1_S(R, \mathrm{Hom}_K(F_i, F_j))$.

Now we claim that the module of coboundaries $B^1_S(R, \mathrm{Hom}_K(F_i, F_j))$ is zero. First, we observe that if $\psi = \delta\lambda$ where $\lambda \in \mathrm{Hom}_K(F_i, F_j)$, and if $\psi(S) = 0$, then $\gamma\lambda - \lambda\gamma = 0$ for all $\gamma \in S$. Hence $\lambda \in \mathrm{Hom}_S(F_i, F_j)$. Thus, if $i \neq j$, $\lambda = 0$; hence $\psi = 0$ in this case. If $i = j$, $\lambda \in K_i = \mathrm{Hom}_S(F_i, F_i)$. When $\alpha \in R$, $\alpha = \gamma + \eta$ where $\gamma \in S$ and $\eta \in N$. But then $\psi(\alpha) = \psi(\eta)$ and $\psi(\gamma) = 0$. Hence $\psi(\alpha) = \eta\lambda - \lambda\eta$. However, $\eta F_i = 0$. Hence $\psi(\alpha) = 0$ for all $\alpha \in R$. Thus $\psi = 0$. From this and the remarks of §1C, it follows that $Z^1_S(R, \mathrm{Hom}_K(F_i, F_j))$ is isomorphic to $H^1(R, \mathrm{Hom}_K(F_i, F_j))$.

3C. Reformulation of the principal theorem

In this section, we will simplify the statement of the main theorem of [8] (Theorem 3) quoted in §1B of this paper. The relatively complex notion of conformality is replaced by a commutativity condition involving the coboundary operator. Nevertheless, as we will see in Part IV, the concept of conformality is still useful.

Let $R = S \oplus N$ and $R' = S' \oplus N'$ be cleavings for cleft rings R and R'. Suppose that $I_0 : S \to S'$ is an isomorphism. Then let $\omega_i : F_i \to F'_i$, $i = 1, 2, \cdots, k$, be the I_0-isomorphisms of the irreducible S-modules onto the irreducible S'-modules. Then, in turn, there are induced isomorphisms $I_i : K_i \to K'_i$, $i = 1, 2, \cdots, k$, of the endomorphism sfields of F_i onto the endomorphism sfields of F'_i.

The principal theorem for double modules [8; Theorem 2] yields the following condition for I_0 to be extendable to an (S, S)-isomorphism I of R onto R'. This is that there exists an (I_j, I_i)-isomorphism θ of the corresponding structural modules

$$\theta : H_{ji} \to H'_{ji}, \qquad i, j = 1, 2, \cdots, k.$$

Then θ satisfies the following equation for $\alpha' \in R'$:

$$\theta\psi(\alpha') = \omega_j \psi(\alpha'^J)\omega_i^{-1} \tag{3.10}$$

where $J = I^{-1}$.

Now we develop conditions for I_0 to be extendable to an isomorphism. First let $C^2(R, \mathrm{Hom}_K(F_i, F_j))$ be the (K_j, K_i)-module of 2-cochains. We extend θ given in (3.10) to $C^2(R, \mathrm{Hom}_K(F_i, F_j))$ by setting for $\alpha, \beta \in R$

$$\theta\psi(\alpha, \beta) = \omega_j \psi(\alpha^J, \beta^J)\omega_i^{-1}. \tag{3.11}$$

Then we have the following theorem.

THEOREM 1. *A necessary and sufficient condition that there exist an isomorphism $I : R \to R'$ which extends I_0 is that there exist an (I_j, I_i)-isomorphism*

$$\theta : T_{ji} \to T'_{ji}$$

such that $\theta\delta = \delta\theta$ where δ is the coboundary operator.

Proof. If I is an extension of I_0 which is a ring isomorphism, set $J = I^{-1}$. Then if $\alpha', \beta' \in R'$, we have for $\psi \in T_{ji}$

$$\begin{aligned}\theta\delta\psi(\alpha', \beta') &= \omega_j \delta\psi(\alpha'^J, \beta'^J)\omega_i^{-1} \\ &= \omega_j(\psi(\alpha'^J, \beta'^J) - \alpha'^J\psi(\beta'^J) - \psi(\alpha'^J)\beta'^J)\omega_i^{-1} \\ &= \omega_j(\psi(\alpha'\beta')^J)\omega_i^{-1} - \alpha'\omega_j\psi(\beta'^J)\omega_i^{-1} - \omega_j\psi(\alpha'^J)\omega_i^{-1}\beta' \\ &= \delta\theta\psi(\alpha', \beta').\end{aligned}$$

On the other hand, should θ exist satisfying the hypothesis of the theorem, we proceed by first extending θ to H_{ji} by setting for $\psi \in H^0_{ii}$, $\theta\psi(\alpha') = \sigma^{I_i}\alpha'_L$ if $\psi(\alpha'^J) = \sigma\alpha'^J_L$ where J is induced by θ. Then since $\sigma^{I_i} = \omega_i\sigma\omega_i^{-1}$, we have that $\theta\psi(\alpha'\beta') = \omega_i\psi((\alpha'\beta')^J)\omega_i^{-1}$. On the other hand, as α'_L and β'_L act on irreducible modules, $\alpha'_L\beta'_L = (\alpha'_0)_L(\beta'_0)_L$ where $\alpha' = \alpha'_0 + \eta$ with $\alpha'_0 \in S$ and $\eta \in N$, and where $\beta' = \beta'_0 + \zeta'$ with $\beta_0 \in S$ and $\zeta' \in N$. Since the restriction J to S' is a ring isomorphism, we have that $(\alpha'_L\beta'_L)^J = ((\alpha'_0\beta'_0)^J)_L = (\alpha'^J_0\beta'^J_0)_L = \alpha^J_L\beta^J_L$. Hence $\omega_i(\psi(\alpha'\beta')^J)\omega_i^{-1} = \omega_i\psi(\alpha'^J\beta'^J)\omega_i^{-1}$, when $\psi \in H^0_{ii}$.

For $\psi \in T_{ji}$, we have that $\theta\delta\psi = \delta\theta\psi$. Then for $\alpha', \beta' \in R'$

$$\begin{aligned}\omega_j(\psi(\alpha'\beta')^J)\omega_i^{-1} = \theta\psi(\alpha'\beta') &= \delta\theta\psi(\alpha', \beta') + \alpha'\theta\psi(\beta') + \theta\psi(\alpha')\beta' \\ &= \theta\delta\psi(\alpha', \beta') + \alpha'\omega_j\psi(\beta'^J)\omega_i^{-1} + \omega_j\psi(\alpha'^J)\omega_i^{-1}\beta' \\ &= \omega_j\psi(\alpha'^J\beta'^J)\omega_i^{-1}.\end{aligned}$$

Here we make use of (3.11). Hence $\psi((\alpha'\beta')^J) = \psi(\alpha'^J\beta'^J)$ for all $\psi \in H_{ji}$

where $i, j = 1, 2, \cdots, k$. As we mentioned in the introduction, H_{ji} is a representation module for the (S_j, S_i)-module $R_{ji} = e_j Re_i$. Hence if $\alpha \in R$ and $\psi(\alpha) = 0$ for all $\psi \in H_{ji}$, the components[10] $e_j \alpha e_i$ of α in R_{ji} are zero. Consequently, if $\psi(\alpha) = 0$ for all $\psi \in H_{ji}$ and $i, j = 1, 2, \cdots, k$, $\alpha = 0$. Thus in our case

$$(\alpha'\beta')^J = \alpha'^J\beta'^J.$$

This means that J and, consequently, I are ring isomorphisms. This proves the theorem.

IV. Extensions of Isomorphisms. Graded Rings

4A. Extensions of automorphisms

As an application of the theory we have presented, we have the following theorem for cleft rings with minimum condition.

Theorem 2. *Any automorphism I_0 of a semisimple component S of a cleft ring R may be extended to an automorphism I of R.*

Proof. Let I_i be the restriction of I_0 to the simple component S_i of S. If $\alpha \in S_i$, denote by α_L the left multiplication by α on F_i. Then there exists a semilinear transformation $\omega_i : F_i \rightarrow F_i$ such that $\omega_i \alpha_L \omega_i^{-1} = (\alpha^{I_i})_L$. Again designate by I_i the automorphism of K_i belonging to ω_i. Define on F_i a new module multiplication $\alpha \cdot x$ for $\alpha \in R$ and $x \in F_i$ given by $\alpha \cdot x = \alpha\omega_i x$. Denote this module by F_i'. When it is specified that x is in F_i', we will write αx instead of $\alpha \cdot x$. Under this convention $\omega_i : F_i \rightarrow F_i'$ is an isomorphism of S-modules.

Let H_{ji} be the structural module $\mathrm{Hom}_{(S,S)}(R, \mathrm{Hom}_K(F_i, F_j))$, and H_{ji}' the structural module $\mathrm{Hom}_{(S,S)}(R, \mathrm{Hom}_K(F_i', F_j'))$. Define $\theta : H_{ji} \rightarrow H_{ji}'$, $i, j = 1, 2, \cdots, k$ by $\theta\psi = \omega_j \psi \omega_i^{-1}$ for $\psi \in H_{ji}$. Clearly θ is an (I_j, I_i)-isomorphism for each pair (i, j). Then θ induces an (I_0, I_0)-isomorphism J of R onto itself when considered as an (S, S)-module by Theorem 2 of [8]. Let $I = J^{-1}$; we will show that I is an extension of I_0 and that it is a ring automorphism. From Theorem 2 of [8], we have for $\psi \in H_{ji}$

$$\theta\psi(\alpha^I) = \omega_j \psi(\alpha)\omega_i^{-1}; \qquad \theta\psi(\alpha) = \omega_j \psi(\alpha^J)\omega_i^{-1}. \tag{4.1}$$

Using (4.1) and Proposition 3.2, we have for $\psi \in H_{ii}^0$

$$\theta\psi(\alpha^I) = \omega_i \psi(\alpha)\omega_i^{-1} = \omega_i \sigma\alpha_L \omega_i^{-1} = \sigma^{I_i}\omega_i \alpha_L \omega_i^{-1} \tag{4.2}$$

when $\alpha \in R$. Let $\alpha_{L'}$ denote left multiplication by α on F_i'; then for some $\tau \in K_i$, $\theta\psi(\alpha^I) = \tau(\alpha^I)_{L'}$ by virtue of Proposition 3.2. Setting $\alpha = 1$ and comparing with (4.2), we obtain that $\tau = \sigma^{I_i}$. Then again from (4.2), $(\alpha^I)_{L'} = \omega_i \alpha_L \omega_i^{-1}$. But if $\alpha \in S$, $\omega_i \alpha_L \omega_i^{-1} = (\alpha^{I_i})_{L'} = (\alpha^{I_0})_{L'}$. Thus if $\alpha \in S$, $\alpha^I = \alpha^{I_0}$, and I is an extension of I_0.

In order to show that I is a ring automorphism, we will show that the structures of R are conformal. To that end, let U_i, $i = 1, 2, \cdots, k$, be the principal indecomposable modules of R and define

$$\varphi\colon \mathrm{Hom}_S(F_\xi, U_i) \to \mathrm{Hom}_S(F'_\xi, U_i)$$
$$\varphi^*\colon \mathrm{Hom}_S{}^*(F_\xi, U_i) \to \mathrm{Hom}_S{}^*(F'_\xi, U_i)$$

by setting $\varphi f = f\omega_\xi^{-1}$ and $\varphi^* f^* = \omega_\xi f^*$ for $f \in \mathrm{Hom}_S(F_\xi, U_i)$ and $f^* \in \mathrm{Hom}_S{}^*(F_\xi, U_i)$. One may verify that φ and φ^* are contragredient. Next let $| \psi_i |$ be a principal structure of R associated with U_i. If $f^* \in \mathrm{Hom}_S{}^*(F_\xi, U_i)$ and $f \in \mathrm{Hom}_S(F_\eta, U_i)$, we have that

$$\theta\psi_i[f^*, f] = \psi_i[\varphi^* f^*, \varphi f]$$

because $\theta\psi_i[f^*, f](\alpha^I) = \omega_\xi \psi_i[f^*, f](\alpha)\omega_\eta^{-1} = \omega_\xi f^* \alpha_L f \omega_\eta^{-1}$. Hence we have established the conformality of the structures, and the result follows from Theorem 3 of [8].

We must also show that I leaves the (S, S)-modules M of R invariant. First observe that θ maps the (K_j, K_i)-submodule

$$H_{ji}(M) = \mathrm{Hom}_{(S,S)}(R/M, \mathrm{Hom}_K(F_i, F_j))$$

of H_{ji} onto the (K_j, K_i)-submodule

$$H'_{ji}(M) = \mathrm{Hom}_{(S,S)}(R/M, \mathrm{Hom}_K(F'_i, F'_j))$$

of H'_{ji}. Then if $\alpha \in M$, and for all $\psi \in H_{ji}(M)$, $\psi(\alpha) = 0$, and hence $\omega_j \psi(\alpha)\omega_i^{-1} = \theta\psi(\alpha^I) = 0$. This means that $\psi'(\alpha^I) = 0$ for all $\psi' \in H'_{ji}(M)$. As this is true for $i, j = 1, 2, \cdots, k$, this means that[10] $e_j \alpha^I e_i$ is in $e_j M e_i$ for $i, j = 1, 2, \cdots, k$. Hence $M^I \subseteqq M$. Similarly $M^J \subseteqq M$. Consequently, $M = M^I$.

4B. Extensions of isomorphisms of graded rings

A grading of a cleft ring R is defined in the Introduction (§1A). Let

$$R = S \oplus M \oplus M^2 \oplus \cdots \oplus M^r, \tag{4.3}$$

$$R = S' \oplus M' \oplus M'^2 \oplus \cdots \oplus M'^r \tag{4.4}$$

be two gradings for R. We study the relation between these gradings in the following theorem. Because M^q, M'^q and N^q/N^{q+1} are isomorphic as (S, S)-modules or (S', S')-modules, as the case may be, the same number of components appear in (4.3) and (4.4).

THEOREM 3. *Let* (4.3) *and* (4.4) *be gradings for R. Let $I_0\colon S \to S'$ be an isomorphism. Then I_0 may be extended to an automorphism I of R which maps M^q onto M'^q, $q = 1, 2, \cdots, r$.*

Proof. To prove this theorem, we may assume that I_0 induces the identity automorphism on R/N since, by Theorem 2, there always exists an automorphism I' of R which leaves S invariant and which induces the same

[10] For example, refer to the proof of Theorem 3 of [8].

automorphism $\bar{I}_0$ as I_0 on R/N. Therefore, we will take the irreducible R-modules $F_1, F_2, \cdots, F_k$ for the irreducible S'-modules in forming the structural modules $H'_{ji} = \mathrm{Hom}_{(S',S')}(R, \mathrm{Hom}_K(F_i, F_j))$. Then we have that $(\alpha^{I_0})_L = \alpha_L$ when $\alpha \in S$ and β_L represents the left multiplication induced on F_i by an element $\beta \in R$. The isomorphisms $\omega_i : F_i \rightarrow F'_i$ induced by the restriction I_i of I_0 to the simple component S_i are identities. Thus we must find, first of all, (K_j, K_i)-isomorphisms $\theta : H_{ji} \rightarrow H'_{ji}$, $i, j = 1, 2, \cdots k$.

To do this, we first observe that (4.3) and (4.4) induce a decomposition of the structural modules H_{ji}. Indeed, let $\hat{R}^q = \oplus_{p \neq q} M^p$ where $M^0 = S$. Let $\hat{R}'^q = \oplus_{p \neq q} M'^p$. Then set

$$\hat{H}^q_{ji} = \mathrm{Hom}_{(S,S)}(R/\hat{R}^q, \mathrm{Hom}_K(F_i, F_j)),$$

$$\hat{H}'^q_{ji} = \mathrm{Hom}_{(S',S')}(R/\hat{R}'^q, \mathrm{Hom}_K(F_i, F_j)).$$

Note that $H^0_{ji} = \hat{H}^0_{ji}$ and $H'^0_{ji} = \hat{H}'^0_{ji}$, and that $T^1_{ji} = \hat{H}^1_{ji}$ and $T'^1_{ji} = \hat{H}'^1_{ji}$. Furthermore, because of (4.3) and (4.4), we have

$$H_{ji} = \oplus^r_{q=0} \hat{H}^q_{ji}, \qquad H'_{ji} = \oplus^r_{q=0} \hat{H}'^q_{ji},$$
$$H^p_{ji} = \oplus^p_{q=0} \hat{H}^q_{ji}, \qquad H'^p_{ji} = \oplus^p_{q=0} \hat{H}'^q_{ji}. \tag{4.5}$$

To prove Theorem 3, we establish two refined composition forms $\mathfrak{C}_i$ and $\mathfrak{C}'_i$ on each principal indecomposable module U_i, $i = 1, 2, \cdots, k$, which are defined from the cleavings of R that are given by the gradings (4.3) and (4.4) and which are related in a particular manner. First of all, let ε be a primitive idempotent of the simple component S_i of S. Then U_i is isomorphic to $R\varepsilon$. But the gradings (4.3) and (4.4) give the direct decompositions $R\varepsilon = \oplus^r_{q=0} M^q\varepsilon = \oplus^r_{q=0} M'^q\varepsilon$. It will be convenient to set $U_i = X$ in order that the notation of this section should correspond with that of the previous sections. Let $\hat{X}^p$, $p = 1, 2, \cdots, r$, be the S-submodules, and $\hat{X}'^p$, $p = 1, 2, \cdots, r$, the S'-submodules of X corresponding to the components $M^p\varepsilon$ and $M'^p\varepsilon$ of $N\varepsilon$, respectively. Then $N^qX = \oplus^r_{p=q} \hat{X}^p = \oplus^r_{p=q} \hat{X}'^p$. Let

$$X = X_1 \supset X_2 \supset \cdots \supset X_t \supset X_{t+1} = 0 \tag{4.6}$$

be a composition series for X which is a refinement of the upper Loewy series for X.

Let q be chosen so that $N^qX \supseteqq X_\mu \supset X_{\mu+1} \supseteqq N^{q+1}X$. Then by the modular law, $X_\mu = (X_\mu \cap \hat{X}^q) \oplus N^{q+1}X$. Because a similar result holds for $X_{\mu+1}$, we may conclude that $X_\mu = A_\mu \oplus X_{\mu+1}$ where $A_\mu \subseteqq \hat{X}^q$ and is an irreducible S-module. Similarly, $X_\mu = A'_\mu \oplus X_{\mu+1}$ where $A'_\mu \subseteqq \hat{X}'^q$ and is an irreducible S'-module. Then

$$X_\mu = \oplus^t_{\xi=\mu} A_\xi = \oplus^t_{\xi=\mu} A'_\xi. \tag{4.7}$$

We may and will further require that $A_\mu \oplus N^{q+1}X = A'_\mu \oplus N^{q+1}X$; that is, we choose A_μ and A'_μ from the same cosets of the completely reducible module $N^qX/N^{q+1}X$.

Let $\{f_\mu{}^*, f_\mu\}$ and $\{g_\mu{}^*, g_\mu\}$ be the direct families of S-homomorphisms and of S'-homomorphisms which, respectively, give the direct decompositions of (4.7) when $\mu = 1$. The restriction π_μ of $f_\mu{}^*$ to X_μ is an S-epimorphism of X_μ onto F_{i_μ}; since the kernel of π_μ is the R-module $X_{\mu+1}$, π_μ is an R-epimorphism. Likewise, the restriction π_μ' of $g_\mu{}^*$ to X_μ is an R-epimorphism of X_μ onto F_{i_μ}. But the kernels of π_μ and π_μ' coincide. Hence we may replace $g_\mu{}^*$ and g_μ by $\sigma g_\mu{}^*$ and $g_\mu \sigma^{-1}$, respectively, with $\sigma \in K_{i_\mu}$, if necessary, so that the restrictions of $f_\mu{}^*$ and $g_\mu{}^*$ coincide on X_μ. Let $\mathfrak{C} = \mathfrak{C}_i$ and $C' = C_i'$ be the composition forms defined on X with the extensions

$$(4.8) \qquad 0 \to X_{\mu+1} \to X_\mu \xrightarrow{\pi_\mu} F_{i_\mu} \to 0$$

and respective cross-sections $\pi_\mu^{-1} = p_\mu f_\mu$, where $p_\mu : X \to X_\mu$ is the projection with kernel $\oplus_{\xi=1}^{\mu-1} A_\xi$ in the first case, and $\pi_\mu'^{-1} = p_\mu' g_\mu$, where $p_\mu' : X \to X_\mu$ is the projection with kernel $\oplus_{\xi=1}^{\mu-1} A_\xi'$ in the second case. Then $\{f_\mu{}^*, f_\mu\}$ is the direct family of $\mathfrak{C}$, and $\{g_\mu{}^*, g_\mu\}$ is the direct family of $\mathfrak{C}'$. Let ρ_μ and ρ_μ' be the cocycles formed from the extensions (4.8) with the respective cross-sections π_μ^{-1} and $\pi_\mu'^{-1}$. Of course, ρ_μ and ρ_μ' are cohomologous.

Let $|\psi|$ and $|\psi'|$ be the structures of the module X determined from the cleavings given by (4.3) and (4.4), respectively. It is clear from the grading of R that $M^q\hat{X}^p = \hat{X}^{q+p}$. Hence $f_\mu{}^*M^q\hat{X}^p = f_\mu{}^*\hat{X}^{q+p} = 0$ unless $\deg f_\mu{}^* = q + p$. Let $\deg f_\nu = p$. Then $A_\nu = Rf_\nu F_{i_\nu} \subseteqq \hat{X}^p$ and $f_\mu{}^*M^qRf_\nu F_i = 0$ unless $\deg f_\mu{}^* - \deg f_\nu = q$. That is, $\psi[f_\mu{}^*, f_\nu](M^q) = 0$ unless $\deg f_\mu{}^* - \deg f_\nu = q$. Thus $\psi[f_\mu{}^*, f_\nu]$ vanishes on $\hat{R}^q$ and $\psi[f_\mu{}^*, f_\nu] \in \hat{H}_{ji}^q$. Similarly, under the same circumstances, $\psi'[g_\mu{}^*, g_\nu] \in \hat{H}_{ji}'^q$.

We will now define (K_j, K_i)-isomorphisms $\theta_q : H_{ji}^q \to H_{ji}'^q$ inductively for $q \geqq 0$ so that θ_{q+1} is an extension of θ_q. We will further show that when $\deg f_\mu{}^* - \deg f_\nu = q$,

$$(4.9) \qquad \theta_q \psi[f_\mu{}^*, f_\nu] = \psi'[g_\mu{}^*, g_\nu].$$

We first treat the case that $q = 0$. Then if $j \neq i$, $H_{ji}^0 = H_{ji}'^0 = 0$. By Proposition 3.2, the elements of H_{ii}^0 are given by the form $\psi(\alpha) = \sigma\alpha_L$ where $\sigma \in K_i$ and α_L is a left multiplication on F_i. The same is true for the elements of $H_{ii}'^0$. Hence $H_{ii}^0 = H_{ii}'^0$. Therefore, define θ_0 to be the identity on H_{ji}^0. If $\deg f_\mu{}^* - \deg f_\nu = 0$, $\psi[f_\mu{}^*, f_\nu] = 0$ unless $\mu = \nu$. But if $\mu = \nu$, then $\psi[f_\mu{}^*, f_\mu](\alpha) = f_\mu{}^*\alpha_L f_\mu$. But $A(f_\mu) = A_\mu \subseteqq X_\mu$. Hence $f_\mu{}^*\alpha_L f_\mu = \pi_\mu \alpha_L f_\mu = \alpha_L \pi_\mu f_\mu = \alpha_L f_\mu{}^* f_\mu = \alpha_L$. Similarly, $\psi'[g_\mu{}^*, g_\mu](\alpha) = \alpha_L$. Hence

$$\theta_0 \psi[f_\mu{}^*, f_\nu] = \psi'[g_\mu{}^*, g_\nu],$$

which verifies (4.9) in the case that $q = 0$.

We also treat the case that $q = 1$ in (4.9) before we establish the induction. As both $\hat{H}_{ji}^1 = T_{ji}^1$ and $\hat{H}_{ji}'^1 = T_{ji}'^1$, we have from Proposition 3.4 that $\hat{H}_{ji}^1$ and $\hat{H}_{ji}'^1$ are both submodules of the cocycle module $Z^1(R, \operatorname{Hom}_K(F_i, F_j))$ which are isomorphic to the cohomology module $H^1(R, \operatorname{Hom}_K(F_i, F_j))$ under the natural homomorphism onto $H^1(R, \operatorname{Hom}_K(F_i, F_j))$. Therefore, we define

θ_1 by setting $\theta_1 \psi$ to be the unique element of H'^1_{ji} which is cohomologous to $\psi \in H^1_{ji}$. Clearly, this defines an extension θ_1 of θ_0 to H^1_{ji}.

We next observe that if $\deg f_\mu^* = q$, then the restriction π_μ^* of f_μ^* to N^qX coincides with the restriction π'^*_μ of g_μ^* to N^qX, and that this is an R-homomorphism. Indeed, π_μ^* induces an S-homomorphism of the completely reducible module $N^qX/N^{q+1}X$ onto an irreducible module. Thus π_μ^* induces an R-homomorphism. Since $N^{q+1}X$ is an R-module, π_μ^* is an R-homomorphism. Since π'^*_μ induces the same S-homomorphism of $N^qX/N^{q+1}X$ as does π_μ^*, we have that $\pi_\mu^* = \pi'^*_\mu$.

Next we assert that $\rho_\nu(R)F_i = NX(f_\nu)$. Indeed, $\rho_\nu(S) = 0$; so $\rho_\nu(R) = \rho_\nu(N)$. For $\eta \in N$ and $x \in F_i$, we have that $\eta x = 0$. Hence $\rho_\nu(\eta)x = \eta\pi_\nu^{-1}x$. Thus $\rho_\nu(R)F_{i_\nu} = N\pi_\nu^{-1}F_{i_\nu}$. Since $\pi^{-1}F_{i_\nu} = A_\nu = A(f_\nu)$,

$$\rho_\nu(R)F_{i_\nu} = NA(f_\nu) = NX(f_\nu).$$

Let $\deg f_\nu = q$; then $N^qX \supseteqq X(f_\nu)$; so $N^{q+1}X \supseteqq NX(f_\nu) = \rho_\nu(R)F_{i_\nu}$. But if $f_\mu^* \in \mathrm{Hom}_S^*(F_{i_\mu}, X)$ and $\deg f_\mu^* = q + 1$, then $\psi[f_\mu^*, f_\nu] = f_\mu^*\rho_\nu = \pi_\mu^*\rho_\nu$. Likewise, $\psi'[g_\mu^*, g_\nu] = \pi'^*_\mu\rho'_\nu$. Since $\pi_\mu^* = \pi'^*_\mu$ and ρ_ν and ρ'_ν are cohomologous, $\theta_1\psi[f_\mu^*, f_\nu] = \psi'[g_\mu^*, g_\nu]$. This verifies (4.9) for the case where $q = 1$.

Now suppose that θ_q has been defined on each of the modules H^q_{ji}, $i, j = 1, 2, \cdots, k$, so that (4.9) is satisfied. We wish to define θ_{q+1}. First, using Proposition 2.3 note that f_1 and g_1 are generating homomorphisms for $X = U_i$. Thus the elements $\psi[f_\mu^*, f_1]$, $\mu = 1, 2, \cdots, t$, for which $f_\mu^* \in \mathrm{Hom}_S^*(F_j, X)$ form a basis for H_{ji}. Because of the decomposition (4.5), those elements $\psi[f_\mu^*, f_1]$, $\mu = 1, 2, \cdots, t$, for which $f_\mu^* \in \mathrm{Hom}_S^*(F_j, X)$ and $\deg f_\mu^* = q + 1$ form a basis for $\hat{H}^{q+1}_{ji}$. Similarly, those elements $\psi'[g_\mu^*, g_1]$, $\mu = 1, 2, \cdots, t$, for which $g_\mu^* \in \mathrm{Hom}_{S'}^*(F_j, X)$ and $\deg g_\mu^* = q + 1$ form a basis for $\hat{H}'^{q+1}_{ji}$. We define θ_{q+1} to be the extension of θ_q given by the K_j-isomorphism obtained by setting $\theta_{q+1}\psi[f_\mu^*, f_1] = \psi'[g_\mu^*, g_1]$ for this basis of $\hat{H}^{q+1}_{ji}$.

Let $\psi \in \hat{H}^{q+1}_{ji}$ so that $\psi = \psi[f^*, f_1]$ where $f^* = \sum \sigma_\mu f_\mu^*$ is a K_j-linear combination of elements of degree $q + 1$ that belong to $\mathrm{Hom}_S^*(F_j, X)$. Then, as in Proposition 3.3,

$$(4.10) \quad \delta\psi(\alpha, \beta) = \delta\psi[f^*, f_1](\alpha, \beta) = \textstyle\sum_\xi \psi[f^*, f_\xi](\alpha)\psi[f_\xi^*, f_1](\beta),$$

where the summation extends over certain indices described in Proposition 3.3. Here $\psi[f^*, f_\xi] = \sum \sigma_\mu \psi[f_\mu^*, f_\xi]$ is a K_j-combination of elements in H^u_{ji} with $u = \deg f_\mu^* - \deg f_\xi$ while $\psi[f_\xi^*, f_1] \in H^v_{ji}$ where $v = \deg f_\xi^* - \deg f_1 = \deg f_\xi^*$. Hence $u \leqq q$ and $v \leqq q$. This means that $\delta\psi(\alpha, \beta) = 0$ if $\alpha \in N^{q+1}$ or $\beta \in N^{q+1}$.

On the other hand, we have defined θ_q on

$$H^q_{ji} = \mathrm{Hom}_{(S,S)}(R/N^{q+1}, \mathrm{Hom}_K(F_i, F_j)).$$

Then by Theorem 2 of [8], θ_q induces an (I_j^{-1}, I_0^{-1})-isomorphism J_q of R/N^{q+1} taken as an (S', S')-module onto R/N^{q+1} taken as an (S, S)-module such that

$\theta_q \psi(\bar{\alpha}) = \psi(\bar{\alpha}^{J_q})$ where $\bar{\alpha} \in R/N^{q+1}$. Thus we note that if $\alpha - \alpha' \in N^{q+1}$, then $\psi(\alpha) = \psi(\alpha')$; hence we may set for $\alpha \in \oplus_{p=0}^{q} M'^p$, α^{J_q} to be the unique element of $\oplus_{p=0}^{q} M^p$ in the coset $\bar{\alpha}^{J_q}$ where $\bar{\alpha}$ contains α. Thus $\theta_q \psi(\alpha) = \psi(\alpha^{J_q})$ for $\alpha \in \oplus_{p=0}^{q} M^p$. We may now define a (K_j, K_i)-homomorphism of δT_{ji}^q, which we again denote by θ_q, by setting $\theta_q \psi_0(\alpha, \beta) = \psi_0(\alpha^{J_q}, \beta^{J_q})$ for $\psi_0 \in \delta T_{ji}^q$. But then by (4.10), for $\alpha, \beta \in \oplus_{p=0}^{q} M'^p$,

$$\begin{aligned}\theta_q \delta\psi(\alpha, \beta) &= \textstyle\sum_\xi \psi[f^*, f_\xi](\alpha^J)\psi[f_\xi^*, f_1](\beta^J)\\ &= \textstyle\sum_\xi \theta_q \psi[f^*, f_\xi](\alpha)\theta_q \psi[f_\xi^*, f_1](\beta)\\ &= \textstyle\sum_\xi \psi'[g^*, g_\xi](\alpha)\psi'[g_\xi^*, g_1](\beta)\\ &= \delta\psi'[g^*, g_1](\alpha, \beta) = \delta\theta_{q+1} \psi[f^*, f_1](\alpha, \beta).\end{aligned}$$

Thus we have obtained

$$\theta_q \delta\psi(\alpha, \beta) = \delta\theta_{q+1} \psi(\alpha, \beta).$$

Now δ is a (K_j, K_i)-isomorphism of T_{ji}^{q+1}. The kernel of δ is $\hat{H}_{ji}^1 = T_{ji}^1$. Thus on $\oplus_{p=2}^{q+1} \hat{H}_{ji}^p$, $\theta_{q+1} = \delta^{-1}\theta_q \delta$, and hence the restriction of θ_{q+1} to this submodule is a (K_j, K_i)-isomorphism. The restriction of θ_{q+1} to H_{ji}^1 is θ_1, which we have shown to be a (K_j, K_i)-isomorphism. Hence θ_{q+1} is a (K_j, K_i)-isomorphism.

Now let $\deg f_\mu^* - \deg f_\nu = q + 1$; then we have seen that $\psi[f_\mu^*, f_\nu] \in \hat{H}_{ji}^{q+1}$ and $\psi'[g_\mu^*, g_\nu] \in \hat{H}_{ji}'^{q+1}$. But by Proposition 3.3,

$$\begin{aligned}\delta\psi[f_\mu^*, f_\nu](\alpha, \beta) &= \textstyle\sum_\xi \psi[f_\mu^*, f_\xi](\alpha)\psi[f_\xi^*, f_\nu](\beta),\\ \delta\psi'[g_\mu^*, g_\nu](\alpha, \beta) &= \textstyle\sum_\xi \psi'[g_\mu^*, g_\xi](\alpha)\psi'[g_\xi^*, g_\nu](\beta).\end{aligned}$$

By the argument of the preceding paragraphs, we then obtain that

$$\begin{aligned}\theta_q \delta\psi[f_\mu^*, f_\nu](\alpha, \beta) &= \textstyle\sum_\xi \psi[f_\mu^*, f_\xi](\alpha^{J_q})\psi[f_\xi^*, f_\nu](\beta^{J_q})\\ &= \textstyle\sum_\xi \psi'[g_\mu^*, g_\xi](\alpha)\psi'[g_\xi^*, g_\nu](\beta)\\ &= \delta\psi[g_\mu^*, g_\nu](\alpha, \beta).\end{aligned}$$

Since $\theta_q \delta = \delta\theta_{q+1}$ and δ is an isomorphism of $\hat{H}_{ji}^{q+1}$, we have that $\theta_{q+1} \psi[f_\mu^*, f_\nu] = \psi'[g_\mu^*, g_\nu]$. This establishes (4.9) for the case $q + 1$.

To conclude the proof of Theorem 3, we define $\theta: H_{ji} \rightarrow H_{ji}'$ to be the (K_j, K_i)-isomorphism θ such that $\delta\theta = \theta\delta$. This is obtained from the above argument by taking $q = r$. From Theorem 1, it follows that θ induces an automorphism of R. From (4.1) we obtain that if $\alpha \in S$ and $\psi \in T_{ji}$, $\theta\psi(\alpha^I) = 0$. Because $\theta T_{ji} = T_{ji}'$, we have that $\alpha^I \in S'$ so that $S^I = S'$. Furthermore, the restriction of I to S is the isomorphism induced by the restriction θ_0 to $H_{ji}^0 = \mathrm{Hom}_{(S,S)}(R/N, \mathrm{Hom}_K(F_i, F_j))$. Since $\theta_0 = 1$, the restriction of I to S is I_0.

Because of the grading (4.3), the set $\mathrm{Hom}_{(S,S)}(R/M, \mathrm{Hom}_K(F_i, F_j))$

of elements of H_{ji} which vanish on M is $\oplus_{p\neq 1} \hat{H}^p_{ji}$. Then it follows that

$$\theta \operatorname{Hom}_{(S,S)}(R/M, \operatorname{Hom}_K(F_i, F_j)) = \operatorname{Hom}_{(S',S')}(R/M', \operatorname{Hom}_K(F_i, F_j)).$$

As we have argued in the proof of Theorem 2, this implies that $M^I = M'$. We have thus proved the theorem.

4C. Complete graded rings

Let R be a semiprimary ring; that is, let R be a ring with radical N such that R/N is a semisimple ring with the minimum condition on its left ideals. We assume, furthermore, that $\bigcap_{q=1}^{\infty} N^q = 0$ and that R/N^q possesses the minimum condition on its left ideals. The sets N^q, $q = 0, 1, 2, \cdots$, form a subbase for the neighborhoods of zero for a topology in which R becomes a topological ring. In [9], for example, it is shown[11] that when R is complete in this topology, R is the inverse limit

$$R = \varprojlim R/N^q. \tag{4.11}$$

Here we use the natural homomorphism $\pi_{pq}: R/N^q \to R/N^p$ for $1 \leqq p \leqq q$ to define (4.11). We say that a complete semiprimary ring is a *complete graded ring* if there exists a semisimple subring S and an (S, S)-submodule M such that for $r \geqq 1$

$$R = S \oplus M \oplus M^2 \oplus \cdots \oplus M^r \oplus N^{r+1}. \tag{4.12}$$

A set of decompositions (4.12) will be called a *grading* of R. If R is not complete, but $\bigcap_{q=1}^{\infty} N^q = 0$, then it is known that $\bar{R} = \varprojlim R/N^q$ is complete, and we may apply our considerations to $\bar{R}$.

THEOREM 4. *Let R be a complete semiprimary ring with gradings*

$$R = S \oplus M \oplus M^2 \oplus \cdots \oplus M^r \oplus N^{r+1}, \qquad r \geqq 1, \tag{4.13}$$

$$R = S' \oplus M' \oplus M'^2 \oplus \cdots \oplus M'^r \oplus N^{r+1}, \qquad r \geqq 1. \tag{4.14}$$

Then an isomorphism $I_0: S \to S'$ may be extended to an automorphism I of R which maps M^r onto M'^r.

Proof. We will show that there exists a map of the inverse limit $\varprojlim R/N^q$ onto itself which is given by the automorphisms $I^q: R/N^q \to R/N^q$ such that $\pi_{pq} I^q = I^p \pi_{pq}$. Then these mappings will induce an automorphism of the inverse limit by virtue of [4; p. 219]. By further requiring that I^q, $q = 1, 2, \cdots$, extend I_0, we will obtain an extension of I_0 to R.

Let H_{ji} and H'_{ji}, $i, j = 1, 2, \cdots, k$, be the structural modules for R relative to the cleavings given by (4.13) and (4.14), respectively. Set $R_q = R/N^{q+1}$ as in §3A. It follows that each ring R_q is a graded ring with

[11] Although the theory is developed for topological groups, the results extend immediately to topological rings.

gradings

$$R_q = S_q \oplus M_q \oplus M_q^2 \oplus \cdots \oplus M_q^q, \tag{4.15}$$

$$R_q = S'_q \oplus M'_q \oplus M'^2_q \oplus \cdots \oplus M'^q_q, \tag{4.16}$$

where $S_q = (S + N^{q+1})/N^{q+1}$, $M_q = (M + N^{q+1})/N^{q+1}$, etc. Furthermore, H^q_{ji} and H'^q_{ji}, $i, j = 1, 2, \cdots, k$ are the structural modules of the ring R_q. As in §3A, we identify S_q with S.

It was established in the proof of Theorem 3 that there exist ring automorphisms $I_q = J_q^{-1}$ of R_q which extend the isomorphism I_0 and which are induced by (K_j, K_i)-isomorphisms $\theta_q : H^q_{ji} \to H'^q_{ji}$. The restriction of θ_q to H^p_{ji} for $p \leqq q$ is θ_p. On the other hand, π_{pq} induces the injection $\lambda_{pq} : H^p_{ji} \to H^q_{ji}$. Hence, $\theta_q \lambda_{qp} = \lambda_{qp} \theta_p$. But this means that for $\alpha \in R_q$ and $\psi \in H^p_{ji}$

$$\theta_q \lambda_{qp} \psi(\alpha) = \lambda_{qp} \psi(\alpha^{I_q}) = \psi(\pi_{pq}(\alpha^{I_q})),$$

$$\lambda_{qp} \theta\psi(\alpha) = \theta_p \psi(\pi_{pq} \alpha) = \psi((\pi_{pq} \alpha)^{I_p}).$$

Hence $\pi_{pq} I^q = I^p \pi_{pq}$.

Furthermore, we defined θ_q so that $\theta_q H^p_{ji} = H'^p_{ji}$, $p \leqq q$. This means that $(M_q^r)^{I_q} = M'^r_q$. But $M^r = \varprojlim M_q^r$ inasmuch as $\pi_{pq} M_q^r = M_q^r$ when $p \leqq q$. Thus there is an automorphism I of R extending I_0 such that $M^I = M$. Then $(M^r)^I = M'^r$. This proves the theorem.

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