ON ALBANESE VARIETIES

BY
SHOJI KOIZUMI

Let V be an r-dimensional algebraic variety, $(P(V), \Phi_0)$, $(A(V), \Psi_0)$ the Picard variety and the Albanese variety attached to V, Φ_0, Ψ_0 being the canonical homomorphisms of $\mathfrak{G}_a^{r-1}$, $\mathfrak{G}_a^0$ onto $P(V)$, $A(V)$ respectively, where $\mathfrak{G}_a^{r-1}$, $\mathfrak{G}_a^0$ denote the groups of divisors and of zero-cycles on V, respectively, which are algebraically equivalent to 0. $(P(V), \Phi_0)$ can be characterized as follows: Let G be any group variety, and Φ any "*algebraic*" group homomorphism of $\mathfrak{G}_a^{r-1}$ into G. Then there exists a (rational) homomorphism $\phi: P(V) \rightarrow G$ such that $\Phi = \phi \circ \Phi_0$. (We shall not dwell here upon the meaning of the word "algebraic"; it has a certain algebraic-geometrical sense, which will be fully explained on another occasion.[1]) Likewise $(A(V), \Psi_0)$ can be characterized in a similiar way: One has only to replace $\mathfrak{G}_a^{r-1}$ by $\mathfrak{G}_a^0$ in the above characterization of $(P(V), \Phi_0)$. Now it is known that (1) the kernel of Φ_0 is the group of divisors on V which are linearly equivalent to 0, (2) the rational mapping F of a total maximal Chow variety W of positive divisors on V onto $P(V)$ induced by Φ_0 is a regular mapping,[2] and (3) if W is a complete total Chow variety, the inverse image of a point of $P(V)$ by F is a Chow variety associated with a complete linear system. We shall prove in the present paper that $(A(V), \Psi_0)$ has properties corresponding to these properties of $(P(V), \Phi_0)$. Namely we shall prove that (1) the kernel of Ψ_0 is the group of zero-cycles on V which are regularly equivalent to 0,[3] (2) n being sufficiently large,[4] the rational mapping F_n of a Chow variety $V(n)$ of positive zero-cycles of degree n onto $A(V)$ induced by Ψ_0 is a regular mapping, and (3) n being again sufficiently large,[5] the inverse image $X_v^{(n)}$ of a point v of $A(V)$ by F_n is a regular variety.[6]

It was proved recently by Y. Taniyama [7] and A. Mattuck [6] that $X_v^{(n)}$ is irreducible. Our result gives additional information on $X_v^{(n)}$. Throughout

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[1] See a forthcoming paper of the author.

[2] Let f be a rational mapping of V into U, and k a common field of definition for V, U, and f. We shall say that f is a *regular mapping* if $k(x)$ is a regular extension of $k(f(x))$, where x is a generic point of V over k.

[3] As to the definition of "*regularly equivalent*", see below, §4.

[4] There is an example of V, such that the canonical mapping of $V = V(1)$ into $A(V)$ is purely inseparable, or separably algebraic, so that we must necessarily consider $V(n)$ for sufficiently large n to obtain an analogy of (2).

[5] T. Matsusaka communicated to me an example of V, such that $X_v^{(1)}$ is irreducible, but is not a regular variety, so that we must again consider $V(n)$ for large n.

[6] A variety V, such that there are no rational mappings from V into abelian varieties other than constant mappings, is called a *regular variety*. (This terminology is of the Italian school.)

the present paper I shall avail myself freely of the definitions and the notations in Weil's book [8].

1. We list here some propositions which we need in the sequel. The proofs are omitted, because all results are essentially contained in Chow's papers [1, 2].

PROPOSITION 1. *Let* $B_1, \cdots, B_m$ *and* A *be abelian varieties, and* f_i $(i = 1, \cdots, m)$ *homomorphisms of* B_i *onto* A, *respectively. The rational mapping* F *of* $B_1 \times \cdots \times B_m$ *into* A, *which is defined by* $F(y^{(1)}, \cdots, y^{(m)}) = \sum_{i=1}^{m} f_i(y^{(i)})$ *for any* $y^{(i)} \in B_i$, *is a homomorphism onto* A. *If the kernel* X_i *of* f_i *is an abelian subvariety for* $i = 1, \cdots, m$, *the kernel* X *of* F *is also an abelian subvariety of* $B_1 \times \cdots \times B_m$.

Let A_u, A be abelian varieties defined, respectively, over fields $k(u)$, k, where $k(u)$ is a regular extension of k. We assume that a homomorphism f_u of A_u onto A is defined over $k(u)$, and that f_u has an abelian subvariety of A_u as its kernel. Denote by $(u_1), (u_2), \cdots, (u_n)$ independent generic specializations of (u) with respect to k, and by (A_i, f_i) the uniquely determined specialization of (A_u, f_u) over $(u) \to (u_i)$ with respect to k. If we define the homomorphism F_n of $A_1 \times \cdots \times A_n$ onto A by $F_n = \sum f_i$, the kernel X_n of F_n is an abelian subvariety which is defined over $K = k(u_1, \cdots, u_n)$. Let $A_0^{(n)}$ be the factor group-variety of $A_1 \times \cdots \times A_n$ by X_n. Then we have the following lemma.

LEMMA. *There is a positive integer* N_0 *such that the abelian variety* $A_0^{(n)}$ *is isomorphic to an abelian variety defined over* k, *by an isomorphism defined over* K, *for any* $n \geqq N_0$.

As direct consequences of this lemma we obtain the following corollaries.

COROLLARY 1. *If* A_u *is an abelian variety defined over a field* $k(u)$ *which is a regular extension of a field* k, *the* k-*image of* A_u *over* $k(u)$ *is, at the same time, the* $k^{p^{-e}}$-*image of* A_u *over* $k^{p^{-e}}(u)$ *for any positive integer* e.

COROLLARY 2. *Denote by* V *a projective variety defined over* k, *which is nonsingular in codimension* 1. *Then the Albanese variety attached to* V *admits a model defined over* k. *Moreover, if* V *contains a* k-*rational simple point, the canonical mapping of* V *to* $A(V)$ *may be also considered as defined over* k.

That $A(V)$ admits a model defined over $k^{p^{-e}}$ for some positive integer e, is a direct consequence of the Chow-Weil theory[7] for fields of definition of abstract varieties. We can descend from $k^{p^{-e}}$ to k owing to the lemma.

2. Let $V(n)$ denote the Chow variety of positive zero-cycles of degree n on a projective variety V. We can naturally define the rational mapping σ_n of an n-ple product $\overbrace{V \times \cdots \times V}^{n}$ onto $V(n)$, which is everywhere defined

[7] Cf. Weil [9] and Chow [1].

on $V \times \cdots \times V$. If A is the Albanese variety and f is the canonical mapping of V into A, A is also the Albanese variety of $V(n)$, and the canonical mapping F_n of $V(n)$ into A is characterized by $F_n \circ \sigma_n(x_1, \cdots, x_n) = \sum_{i=1}^n f(x_i)$, where x_i $(1 \leqq i \leqq n)$ is any point on V.

PROPOSITION 2. *If F_n is a regular mapping, then the mapping F_{n+1} of $V(n+1)$ into A is also regular.*

Proof. We denote by k a common field of definition for V, A, f, and consequently for F_n and F_{n+1}. Let $x_1, \cdots, x_{n+1}$ be independent generic points of V over k; and put

$$y^{(n)} = \sum_{i=1}^n f(x_i), \qquad y^{(n+1)} = \sum_{i=1}^{n+1} f(x_i).$$

Our assumption means that $k(\sigma_n(x_1, \cdots, x_n))$ is a regular extension of $k(y^{(n)})$, and we have to prove that $k(\sigma_{n+1}(x_1, \cdots, x_n, x_{n+1}))$ is regular over $k(y^{(n+1)})$. We have only to show that $k(\sigma_n(x_1, \cdots, x_n), x_{n+1})$ is regular over $k(y^{(n+1)})$. This is a direct consequence of the fact that $k(\sigma_n(x_1, \cdots, x_n), x_{n+1})$ is regular over $k(y^{(n)}, x_{n+1}) = k(y^{(n+1)}, x_{n+1})$ and that $k(y^{(n+1)}, x_{n+1})$ is regular over $k(y^{(n+1)})$.

THEOREM 1.[8] *Let A be the Albanese variety of a variety V. Then there exists a positive integer N_0 such that the canonical mapping F_n of the Chow variety $V(n)$ onto A is a regular mapping for any $n \geqq N_0$.*

Proof. As the property to prove is birationally invariant, we can assume that V is a projective variety nonsingular in codimension 1 and is defined over a field k. We choose a fixed generic point t of V over k and denote by C_u a nonsingular curve which is obtained as an intersection of V and a general plane of a suitable dimension containing t. The curve C_u is defined over $K(u)$ where K means the field $k(t)$ and $K(u)$ is a purely transcendental extension of K. Let $(u^{(1)}), \cdots, (u^{(m)})$ be independent generic specializations of (u) over K, and C_i the uniquely determined specialization of C_u over $(u) \to (u^{(i)})$ with respect to K for $1 \leqq i \leqq m$. $C_1(g) \times \cdots \times C_m(g)$ may be considered as a subvariety of $\overbrace{V(g) \times \cdots \times V(g)}^{m}$,[9] where g means the genus of C_u.[10] If f is the canonical mapping of V into A, the rational mapping F' of $\overbrace{V(g) \times \cdots \times V(g)}^{m}$ onto A is defined by

$$F'(\sigma_g(x_{11}, \cdots, x_{1g}), \cdots, \sigma_g(x_{m1}, \cdots, x_{mg})) = \sum_{i,j} f(x_{ij}).$$

[8] The fundamental inequality in Igusa's paper [3], dim $A \leqq h^{1,0}(V)$ is a direct consequence of this theorem.

[9] $V(g)$ or $C_i(g)$ means the Chow variety of positive zero-cycles of degree g on V or C_i, respectively.

[10] It is obvious that g is at the same time the genus of any C_i $(1 \leqq i \leqq m)$.

We may assume that A, f, and F' are defined over K, and $C_i(g)$ is birationally equivalent to the Jacobian variety J_i of C_i by a birational transformation defined over $K(u^{(i)})$, because C_i contains a rational simple point t over $K(u^{(i)})$. Considering the above fact and the relationship between the Albanese variety of V and the K-image of J_i over $K(u^{(i)})$, we see that the restriction mapping F'_C of F' on[11] $C_1(g) \times \cdots \times C_m(g)$ is a regular mapping for a sufficiently large m.[12] If we prove that F_n is a regular mapping for $n = mg$, the general case will follow from Proposition 2.

Now we shall fix a positive integer m so that F'_C is a regular mapping. If we put $K(u^{(1)}, \cdots, u^{(m)}) = K(\tilde{u})$, and y is a generic point of A over $K(\tilde{u})$, we see that[13]

(1) $F'^{-1}_C(y) = X$ is a variety defined over $K(\tilde{u})(y)$, because F'_C is regular, and that

(2) $F'^{-1}(y) = q \sum_{i=1}^{d} X_i$ is a prime rational cycle over $K(y)$, where $X_1, \cdots, X_d$ denote the conjugate subvarieties over $K(y)$ of V, and q is the order of inseparability of X_i over $K(y)$.

At least one of these subvarieties $X_1, \cdots, X_d$ contains the subvariety X, and since $X_1, \cdots, X_d$ are conjugate over $K(y)$ and consequently over $K(\tilde{u})(y)$, each X_i must contain X.

We shall denote by s the natural mapping $\overbrace{\sigma_g \times \cdots \times \sigma_g}^{m}$ of $\overbrace{V \times \cdots \times V}^{mg}$ onto $\overbrace{V(g) \times \cdots \times V(g)}^{m}$, which is defined everywhere on $V \times \cdots \times V$, and denote by s_C the restriction of s on

$$\overbrace{C_1 \times \cdots \times C_1}^{g} \times \overbrace{C_2 \times \cdots \times C_2}^{g} \times \cdots \times \overbrace{C_m \times \cdots \times C_m}^{g}.$$

Since s and s_C are separably algebraic mappings, simple calculations of cycles show us that

(1′) $s_C^{-1} \circ F'^{-1}_C(y) = (F'_C \circ s_C)^{-1}(y) = s_C^{-1}(X)$ is a prime rational cycle over $K(\tilde{u})(y)$, each component of which carries the coefficient 1, and that

(2′) $s^{-1} \circ F'^{-1}(y) = (F' \circ s)^{-1}(y) = q \sum_{i=1}^{d} s^{-1}(X_i)$.

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[11] It seems true that $C_u(g)$ is simple on $V(g)$, but I have no proof for it. Anyway any rational mapping of $V(g) \times \cdots \times V(g)$ into an abelian variety is defined at any point on $C_1(g) \times \cdots \times C_m(g)$.

[12] Cf. the lemma in the previous section.

[13] If f is a rational mapping of a variety V into another one W, and y is a point on W, $f^{-1}(y)$ is a V-cycle which is defined by the formula

$$f^{-1}(y) \times y = \Gamma_f \cdot (V \times y)$$

when the intersection product in the right side of the formula is defined. (Γ_f means the graph of f.)

Furthermore, we see that

$$(F' \circ s)^{-1}(y) \cdot (\overbrace{C_1 \times \cdots \times C_1}^{g} \times \overbrace{C_2 \times \cdots \times C_2}^{g} \times \cdots \times \overbrace{C_m \times \cdots \times C_m}^{g}) = (F'_C \circ s_C)^{-1}(y).$$

On the other hand since the carrier of each $s^{-1}(X_i)$ contains $s^{-1}(X)$, we can conclude that $d = q = 1$. This completes our proof.

3. Using the notations A, V, f, F_n, $V(n)$, and an integer N_0 with the same meanings as in the previous section, we denote by k a common field of definition for A, V, and f. From now on we shall always assume that V is a projective variety, nonsingular in codimension 1. Theorem 1 implies that if y is a generic point of A over k, $F_n^{-1}(y)$ is a subvariety of $V(n)$. We shall now consider the inverse image of an arbitrary (not necessarily generic) point of A by F_n. Moreover we use the following notations:

q = the dimension of A, $\quad r$ = the dimension of V, $\quad N = (q+1)N_0$.

Suppose that n is an integer such that $n = N_0 + m \geqq N$ (so $m \geqq qN_0$). s is the rational mapping of $V(N_0) \times V(m)$ onto $V(n)$ which is defined by $s \circ (\sigma_{N_0} \times \sigma_m) = \sigma_n$. Put $G = F_n \circ s$. For any mapping H, Γ_H will mean the graph of H. Thus we shall define $X_v^{(l)}$ for a positive integer l and $v \in A$ by $X_v^{(l)} \times v = \Gamma_{F_l} \cap (V(l) \times v)$, and Y by

$$Y \times 0 = \Gamma_G \cap (V(N_0) \times V(m) \times 0)$$

where 0 is the neutral element of A. For the rational mapping $\bar{G} = F_n \circ \sigma$ of $\overbrace{V \times \cdots \times V}^{n}$ onto A we shall define Z by

$$Z \times 0 = \Gamma_{\bar{G}} \cap (V \times \cdots \times V \times 0).$$

PROPOSITION 3. *Under the above notations we have*

(i) *Y is a variety of dimension $nr - q$ defined over k, containing $X_u^{(N_0)} \times X_{-u}^m$, where u is a generic point of A over k.*

(ii) *$X_0^{(n)}$ is a variety of dimension $nr - q$ defined over k. Now a generic point x of $X_0^{(n)}$ over k is expressed by $x = \sigma(x_1, \cdots, x_n)$ where x_i is a point of V.*

(iii) *Any subset $(x_{i1}, \cdots, x_{im})$ of m points of $(x_1, \cdots, x_n)$ gives us a set of independent generic points of V over k.*

(iv) *Let $(1', 2', \cdots, n')$ be a permutation of $(1, 2, \cdots, n)$, and put*

$$\sigma(x_1, \cdots, x_{N_0}) = x^{(1)}, \qquad \sigma(x_{N_0+1}, \cdots, x_n) = x^{(2)},$$

$$\sigma(x_{1'}, \cdots, x_{N_0'}) = x^{(1')}, \qquad \sigma(x_{(N_0+1)'}, \cdots, x_{n'}) = x^{(2')},$$

$$\textstyle\sum_{i=1}^{N_0} f(x_i) = u, \qquad \sum_{i=1}^{N_0} f(x_{i'}) = u'.$$

Then $(x^{(1)}, x^{(2)}, u)$ *is a generic specialization of* $(x^{(1')}, x^{(2')}, u')$ *over* k.

Proof. Let Z^* be a component of the bunch Z. Z^* is a subvariety of $\overbrace{V \times \cdots \times V}^{n}$ defined over the algebraic closure $\bar{k}$ of k. A generic point $(\tilde{x})$ of Z^* over $\bar{k}$ is expressed by

$$(\tilde{x}) = (x_1, \cdots, x_{N_0}; x_{N_0+1}, \cdots, x_{2N_0}; \cdots, x_{qN_0}; x_{qN_0+1}, \cdots, x_n)$$

where $x_i \in V$. At least one of the $(q + 1)$ classes of points $(x_{jN_0+1}, \cdots, x_{(j+1)N_0})$, $j = 0, \cdots, q - 1$, $(x_{qN_0+1}, \cdots, x_n)$ consists of independent generic points of V over k. In fact, if it were not so, we should have $\dim_k(\tilde{x}) \leqq nr - q - 1$ in contradiction to $\dim Z^* \geqq nr - q$.

Now we shall prove the assertion (iii). We assume that $(x_1, \cdots, x_{N_0})$ is a set of independent generic points of V over k and put

$$\textstyle\sum_{i=1}^{N_0} f(x_i) = u, \qquad (x^{(1)}) = \sigma(x_1, \cdots, x_{N_0}), \qquad (x^{(2)}) = \sigma(x_{N_0+1}, \cdots, x_n).$$

We know then that u is a generic point of A over k, and $(x^{(1)})$, $(x^{(2)})$ are respectively points on $X_u^{(N_0)}$, $X_{-u}^{(m)}$ which are varieties by Theorem 1. The inequality

$$\begin{aligned} mr - q \geqq \dim_{k(u)} (x^{(2)}) &\geqq \dim_{k(x^{(1)})} (x_{N_0+1}, \cdots, x_n) \\ &\geqq \dim Z^* - N_0 r \geqq (nr - q) - N_0 r = mr - q \end{aligned}$$

implies that

$$\begin{aligned} \dim_k (x_{N_0+1}, \cdots, x_n) &= \dim_k (x^{(2)}) \\ &= \dim_k (u) + \dim_{k(u)} (x^{(2)}) = mr. \end{aligned}$$

This proves that $(x_{N_0+1}, \cdots, x_n)$ is a set of independent generic points of V over k.

In the same way, we see that if any one of the above-mentioned $(q + 1)$ classes of points, say C, is a set of independent generic points of V over k, then the set of all other points, which remain after removal of C from $(x_1, \cdots, x_n)$, is also a set of independent generic points of V over k. It is also to be noted that the order in which $x_1, \cdots, x_n$ are arranged is irrelevant in the above reasoning.

To complete the proof of (iii), we have to prove that $(x_{i_1}, \cdots, x_{i_m})$ is a set of independent generic points of V over k. We shall denote $(x_{i_1}, \cdots, x_{i_m})$ by C_0 and abbreviate this to "C_0 is independent". Let C_1 be the complementary set of C_0 with respect to $(x_1, \cdots, x_n)$. C_1 consists of N_0 points. If C_1 is independent, then C_0 is independent by what we have just proved. If C_1 is not independent, then C_0 contains some "independent part" (consisting of N_0 or $n - qN_0$ points), say C'. By what we have proved, the complement of C' must be independent. But this complement contains C_1, which is a contradiction.

As to the assertions (i) and (iv), the component Y^* of Y corresponding to

Z^* by $\sigma_{N_0} \times \sigma_m$ contains the subvariety $X_u^{(N_0)} \times X_{-u}^{(m)}$ which is the locus of $(x^{(1)}, x^{(2)})$ over $k(u)$. Furthermore $(x^{(1)}, x^{(2)})$ has Y^* as the locus over k, and any component of Y can be regarded as the locus of $(x^{(1)}, x^{(2)})$ over k. This leads easily to the assertions (i) and (iv), and the assertion (ii) is a direct consequence of (i).

THEOREM 2. *Under the same notations and assumptions as in Proposition* 3, *there exists a positive integer* N *such that* $X_v^{(n)}$ *is a regular variety defined over* k *for any integer* $n \geqq N$ *and any point* v *on* A (*we can choose as* N *the integer* $N = (q+1)N_0$ *in Proposition* 3), *and* $\dim X_v^{(n)} = nr - q$.

Proof. Obviously we have only to prove the theorem for the case $v = 0$. It was already proved in Proposition 3 that $X_0^{(n)}$ is a variety of dimension $(nr - q)$. Now we must prove that the variety $X_0^{(n)}$ is regular. As the regularity of a variety does not depend upon the choice of reference fields, we may assume that k is algebraically closed.

Let B and g be, respectively, the Albanese variety of $X_0^{(n)}$ and a canonical mapping of $X_0^{(n)}$ into B, both of which are defined over k. It is sufficient to prove that g is a constant mapping. Under the same notations as in Proposition 3 we shall denote by $\tilde{g}$ the restriction mapping of $g \circ s$ on a variety $X_u^{(N_0)} \times X_{-u}^{(m)}$. Then $\tilde{g}$ can be expressed by

$$\tilde{g}(x^{(1)}, x^{(2)}) = g_1(x^{(1)}) + g_2(x^{(2)}),$$

where g_1 (or g_2) is a rational mapping of $X_u^{(N_0)}$ (or $X_{-u}^{(m)}$) into B, which is defined over a separably algebraic extension[14] K of $k(u)$. If a complete set of conjugates of g_i over $k(u)$ is given by $(g_i^{(1)}, g_i^{(2)}, \cdots, g_i^{(l)})$ respectively for $i = 1, 2$, we have

$$\tilde{g}(x^{(1)}, x^{(2)}) = g_1^{\nu}(x^{(1)}) + g_2^{\nu}(x^{(2)}) \quad \text{for } \nu = 1, 2, \cdots, l$$

and

$$l\tilde{g}(x^{(1)}, x^{(2)}) = \textstyle\sum_{\nu=1}^{l} g_1^{\nu}(x^{(1)}) + \sum_{\nu=1}^{l} g_2^{\nu}(x^{(2)}) = G_1'(x^{(1)}) + G_2'(x^{(2)}),$$

where G_1' or G_2' is, respectively, a rational mapping of $X_u^{(N_0)}$ to B or of $X_{-u}^{(m)}$ to B, both of which are defined over $k(u)$. Then there exists a rational mapping G_1 (or G_2) of $V(N_0)$ (or $V(m)$) to B, the restriction mapping on $X_u^{(N_0)}$ (or $X_{-u}^{(m)}$) of which is equal to G_1' (or G_2').[15] From the definition of the Albanese variety we know that G_1 and G_2 induce constant mappings, respectively, on $X_u^{(N_0)}$ and $X_{-u}^{(m)}$. This means that $l\tilde{g}$ and also $\tilde{g}$ are constant on $X_u^{(N_0)} \times X_{-u}^{(m)}$, and its value $\tilde{g}(x^{(1)}, x^{(2)}) = g \circ s(x^{(1)}, x^{(2)})$ is rational over $k(u)$. Since $k(u)$ is a subfield of $k(x_1, \cdots, x_{N_0})$, there is a rational mapping h of $\overbrace{V \times \cdots \times V}^{N_0}$ to B, which is defined over k and satisfies

$$g \circ s(x^{(1)}, x^{(2)}) = h(x_1, \cdots, x_{N_0}).$$

[14] That K is separably algebraic over $k(u)$ is not essential for our proof.

[15] G_1 and G_2 are defined over k.

On account of the symmetry of g, we have

$$h(x_1, \cdots, x_{N_0}) = \sum_{i=1}^{N_0} h'(x_i),$$

where h' is a mapping of V to B, and

$$g \circ s(x^{(1)}, x^{(2)}) = g \circ s(x^{(1')}, x^{(2')})$$

where $x^{(1')} = \sigma(x_{1'}, \cdots, x_{N_0'}), x^{(2')} = \sigma(x_{(N_0+1)'}, \cdots, x_{n'})$ for a permutation $(1', 2', \cdots, n')$ of $(1, 2, \cdots, n)$. Combining the above two equalities we have

$$\sum_{i=1}^{N_0} h'(x_i) = \sum_{i=1}^{N_0} h'(x_{i'}).$$

Let $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$ be a set of independent generic points of V over k.[16]

We shall define a mapping $H \circ \sigma$ of $\overbrace{V \times \cdots \times V}^{n}$ to B by

$$H \circ \sigma(\bar{x}_1, \cdots, \bar{x}_n) = N_0 \sum_{i=1}^{n} h'(\bar{x}_i).$$

H may be considered as a mapping of $V(n)$ to B. Then by a simple calculation, we have

$$H \circ \sigma(x_1, \cdots, x_n) = ng \circ \sigma(x_1, \cdots, x_n).$$

Since H must be constant on $X_0^{(n)}$, ng and consequently g are constant on $X_0^{(n)}$. This completes our proof.

4. Now we introduce the notion of regular equivalence between cycles of the same dimension on a variety. This is a special case of algebraic equivalence and a wider notion than of rational or linear equivalences.

Let V be a variety, and X, X' two cycles of the same dimension on V. X, X' will be called directly regularly equivalent if there exist a regular variety W, two points x, x' on W, and a cycle Z on $V \times W$ such that $X \times x = Z \cdot (V \times x)$, $X' \times x' = Z \cdot (V \times x')$. (We should of course require that the intersections $Z \cdot (V \times x)$, $Z \cdot (V \times x')$ can be defined.) Two cycles X, Y on V will be called regularly equivalent if there exist a finite number of cycles $X, X', X'', \cdots, X^{(k)} = Y$, such that X and X', X' and X'', $\cdots$, $X^{(k-1)}$ and $X^{(k)}$ are directly regularly equivalent.

Our Theorem 2 implies that regularly equivalent zero-cycles are always directly regularly equivalent. On the other hand, it is easily seen from the theory of Picard variety (or from the so-called "Seesaw Theorem") that regularly equivalent divisors are linearly equivalent. But the general theory of regular equivalence is not yet made, and we do not even know if regularly equivalent cycles (of other dimensions than 0 and $r - 1$) are always directly regularly equivalent or not.

By using the terminology of regular equivalence, we can formulate our Theorem 2 in the following form:

[16] The following process is not necessary for the case $q > 1$. The proof in case $q > 1$ can be accordingly simplified.

THEOREM 3. *Let A and f be, respectively, the Albanese variety attached to a projective nonsingular variety V and the canonical mapping of V into A. We shall denote by $\mathfrak{G}_a$, $\mathfrak{G}_r$ the group of zero-cycles on V which are of degree* 0 *and the group of zero-cycles on V which are regularly equivalent to* 0, *respectively. Then there exists a group isomorphism between $\mathfrak{G}_a/\mathfrak{G}_r$ and A, given by*

$$\mathfrak{A} = P_1 + \cdots + P_n - Q_1 - \cdots - Q_n \in \mathfrak{G}_a$$
$$\rightarrow \quad f(P_1) + \cdots + f(P_n) - f(Q_1) - \cdots - f(Q_n) \in A.$$

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TOKYO UNIVERSITY OF EDUCATION
TOKYO, JAPAN