THE p-PERIOD OF A FINITE GROUP

BY

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If π is any finite group, the p-period of π is defined to be the least positive integer q such that the cohomology groups $\hat{H}^i(\pi, A)$ and $\hat{H}^{i+q}(\pi, A)$ have isomorphic p-primary components for all i and all A [1, Ch. XII, Ex. 11]. This is equivalent to the statement that $\hat{H}^q(\pi, Z)$ has an element of order p^ν, the highest power of p dividing the order of π [1, Ch. XII, Ex. 11].

I will say that q is a p-period for π if it is a multiple of *the* p-period. The ordinary period of the cohomology of π is, of course, the least common multiple of all the p-periods. It is known [1, Ch. XII, Ex. 11] that the p-period will be finite if and only if the p-sylow subgroup of π is either cyclic or a generalized quaternion group. The purpose of this paper is to give a simple group-theoretic interpretation of the p-period of π. The methods used here also give a cohomological generalization of Grün's second theorem [3, Ch. V, Th. 6]. This will be presented in the Appendix since it is not needed in proving Theorems 1 and 2.

THEOREM 1. *If the 2-sylow subgroup of π is cyclic, the 2-period is 2. If the 2-sylow subgroup of π is a (generalized) quaternion group, the 2-period is 4.*

THEOREM 2. *Suppose p is odd and the p-sylow subgroup of π is cyclic. Let π_p be a p-sylow subgroup, and let Φ_p be the group of automorphisms of π_p induced by inner automorphisms of π. Then the p-period of π is twice the order of Φ_p.*

The group Φ_p is, of course, isomorphic to $N(\pi_p)/C(\pi_p)$ where N and C denote the normalizer and centralizer, respectively.

Before proving these theorems, I will review some facts about the cohomology of groups. Suppose $h:\rho \to \pi$ is a monomorphism of finite groups. Then h induces a map of cohomology $h^*:\hat{H}^i(\pi, A) \to \hat{H}^i(\rho, A)$. Here A is a π-module and so can be regarded as a ρ-module by means of h. This map h^* is defined as follows. Let W be a Tate complex (or complete resolution in the terminology of [1, Ch. XII, §3]) for π. Then ρ acts on W through h, and W is ρ-free since h is a monomorphism. Thus W is also a Tate complex for ρ. The map h^* is now defined to be the map of cohomology induced by the inclusion $\mathrm{Hom}_\pi(W, A) \subset \mathrm{Hom}_\rho(W, A)$. In case h is an inclusion map, h^* is just the map $i(\rho, \pi)$ of [1, Ch. XII, §8]. Suppose $\pi \subset \Pi$, $x \in \Pi$ and $h:x\pi x^{-1} \to \pi$ is given by $h(y) = x^{-1}yx$. Then $h^*:\hat{H}^*(\pi, A) \to \hat{H}^*(x\pi x^{-1}, A)$ is just the map c_x of [1, Ch. XII, §8].

Let π' be a subgroup of π, and x an element of π. Then there are two obvious monomorphisms $i, f_x : \pi' \cap x\pi' x^{-1} \to \pi'$, namely, $i(y) = y$ and $f_x(y) =$

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$x^{-1}yx$. Recall that an element $\alpha \in \hat{H}^k(\pi', Z)$ is called stable [1, Ch. XII, §9] if $i^*(\alpha) = f_x^*(\alpha)$ for all $x \in \pi$. To see that this agrees with the definition in [1, Ch. XII, §9], we need only observe that $f_x^* = c_x i^*$ by the previous remark about c_x. If π_p is a p-sylow subgroup of π, the p-primary component of $\hat{H}^i(\pi, Z)$ is isomorphic to the subgroup of stable elements of $\hat{H}^i(\pi_p, Z)$ [1, Ch. XII, §9]. It follows that we can determine the p-period if we know the stable elements in $\hat{H}^i(\pi_p, Z)$.

LEMMA 1. *Suppose the p-sylow subgroup π_p of π is abelian. Let Φ_p be the group of automorphisms of π_p induced by inner automorphisms of π. Then an element $\alpha \in \hat{H}^i(\pi_p, Z)$ is stable if and only if it is fixed under the action of Φ_p on $\hat{H}^i(\pi_p, Z)$.*

The action of Φ_p on $\hat{H}^i(\pi_p, Z)$ is, of course, given by $h\cdot\alpha = (h^{-1})^*\alpha$.

Proof. Suppose first that α is stable. Let $x \in \pi$ be such that $x\pi_p x^{-1} = \pi_p$. The maps $i, f_x : \pi_p = \pi_p \cap x\pi_p x^{-1} \to \pi_p$ are just $i(y) = y$ and $f_x(y) = x^{-1}yx$. Thus f_x is an operation of Φ_p. Since α is stable, $f_x^*(\alpha) = i^*(\alpha) = \alpha$. Therefore α is fixed under Φ_p.

Now assume α is fixed under Φ_p. Let $x \in \pi$, and let $i, f_x : \pi_p \cap x\pi_p x^{-1} \to \pi_p$ be as above. We must show that $f_x^*(\alpha) = i^*(\alpha)$. Let C be the centralizer of $\pi_p \cap x\pi_p x^{-1}$ in π. Clearly $\pi_p, x\pi_p x^{-1} \subset C$ since π_p is abelian. Since π_p is a p-sylow subgroup of π, it is also a p-sylow subgroup of C. Therefore there is a $t \in C$ such that $tx\pi_p x^{-1}t^{-1} = \pi_p$. Now, since $t \in C$, we see that if $y \in \pi_p \cap x\pi_p x^{-1}$, we have $f_x(y) = f_x(t^{-1}yt) = x^{-1}t^{-1}ytx = f_{tx}(y)$. In other words, $f_x = f_{tx}\cdot i$ where $f_{tx} \in \Phi_p$. Since α is fixed under Φ_p, it follows that $f_x^*(\alpha) = i^*(\alpha)$.

LEMMA 2. *Suppose π has a cyclic p-sylow subgroup π_p. Then q is a multiple of the p-period of π if and only if q is even and every element of $\hat{H}^q(\pi_p, Z)$ is fixed under Φ_p.*

Proof. If q is a p-period for π, it must be a p-period for π_p [1, Ch. XII' Prop. 11.3]. Since the p-period of π_p is 2 [1, Ch. XII, §7], q must be even.

Assume now that q is even. Let p^ν be the order of π_p. Then q will be a p-period for π if and only if $\hat{H}^q(\pi, Z)$ has an element of order p^ν [1, Ch. XII, Ex. 11]. But, $\hat{H}^q(\pi, Z)$ is isomorphic to the subgroup of stable elements in $\hat{H}^q(\pi_p, Z)$. Since q is even and π_p is cyclic, $\hat{H}^q(\pi_p, Z)$ is cyclic of order p^ν. Therefore q is a p-period for π if and only if all elements of $\hat{H}^q(\pi_p, Z)$ are stable. By Lemma 1, this is true if and only if all elements of $\hat{H}^q(\pi_p, Z)$ are fixed under Φ_p.

We can now prove half of Theorem 1. Suppose π_2 is cyclic. The group Φ_2 is the quotient $N(\pi_2)/C(\pi_2)$. Since $\pi_2 \subset C(\pi_2)$ and π_2 is a 2-sylow subgroup, Φ_2 must have odd order. However, the group of all automorphisms of π_2 has order $\phi(2^\nu) = 2^{\nu-1}$. Thus Φ_2 is trivial. Therefore every element of $\hat{H}^2(\pi_2, Z)$ is fixed under Φ_2. By Lemma 2, the 2-period of π is 2.

In order to prove Theorem 2, we must make a small computation.

LEMMA 3. *Let ρ be a cyclic group of order n. Then any automorphism $f:\rho \to \rho$ has the form $f(x) = x^r$ where r is prime to n. If $\alpha \in \hat{H}^{2i}(\rho, Z)$, then $f^*(\alpha) = r^i\alpha$.*

Proof. The first statement is well known [3, Ch. IV, §3]. The assertion $f^*(\alpha) = r^i\alpha$ is easily proved by a direct computation. Since this is purely mechanical, it will be omitted (cf. [2, §8]).

Remark 1. Since ρ and $\hat{H}^{2i}(\rho, Z)$ are both cyclic of the same order [1, Ch. XII, §7], we can restate the lemma by saying that f^* acts on $\hat{H}^{2i}(\rho, Z)$ in the same way that f^i acts on ρ.

It is now easy to prove Theorem 2. Since π_p is a cyclic p-group with p odd, its automorphism group is cyclic [3, Ch. IV, §3]. Consequently Φ_p is also cyclic. Let $f \in \Phi_p$ be a generator. Then an element of $\hat{H}^q(\pi_p, Z)$ is fixed under Φ_p if and only if it is fixed under f^*. By Lemma 2 we can assume $q = 2i$ is even. By Remark 1, $\hat{H}^{2i}(\pi_p, Z)$ is fixed under f^* if and only if π_p is fixed under f^i, i.e., if and only if i is divisible by the order of Φ_p. Theorem 2 now follows immediately from Lemma 2.

The only case left to consider is that in which π_2 is a (generalized) quaternion group. In this case the 2-period cannot be 2 because the 2-period of π_2 is 4 [1, Ch. XII, §7]. Therefore, to prove Theorem 1, it will be sufficient to show that all elements of $H^4(\pi_2, Z)$ are stable.

Suppose that x is any element of π. Let $\rho = \pi_2 \cap x\pi_2 x^{-1}$. As before $i, f_x : \rho \to \pi_2$ by $i(y) = y$, $f_x(y) = x^{-1}yx$. We must show that $i^* = f_x^*$ on $\hat{H}^4(\pi_2, Z)$.

A (generalized) quaternion group of order $2t$ has a presentation

$$\Gamma = \{a, b : a^t = b^2, aba = b\} \qquad [1, \text{Ch. XII, §7}].$$

LEMMA 4. *Let ϕ be an automorphism of the (generalized) quaternion group Γ defined by $\phi(a) = a$, $\phi(b) = a^ib$. Then ϕ^* is the identity map on $\hat{H}^4(\Gamma, Z)$.*

Proof. It is enough to prove this for the automorphism $a \to a$, $b \to ab$ since ϕ is obtained by iterating this automorphism i times. For this automorphism, the lemma is proved by a mechanical computation using the explicit complex of [1, Ch. XII, §7]. (Cf. [2, §8].)

LEMMA 5. *Let Γ be the ordinary quaternion group (of order 8). If h is any automorphism of Γ, then h^* is the identity on $\hat{H}^4(\Gamma, Z)$.*

Proof. By enumerating all automorphisms of Γ, it is easy to check that the automorphism group of Γ is generated by automorphisms of the type considered in Lemma 4 for various choices of the generators a, b.

LEMMA 6. *Let ρ be the ordinary quaternion group. Let Γ be any (generalized) quaternion group. Then all monomorphisms $f:\rho \to \Gamma$ induce the same map $\hat{H}^4(\Gamma, Z) \to \hat{H}^4(\rho, Z)$.*

Proof. Let Γ have the presentation used in Lemma 4. The only elements of order 4 in Γ are $a^{t/2}$ and a^ib for all i. Therefore $f(\rho)$ must contain $a^{t/2}$

and some a^ib. By applying an automorphism to Γ of the type considered in Lemma 4, we can assume that $f(\rho)$ contains $a^{t/2}$ and b. Any two such maps clearly differ by an automorphism of ρ. Thus the result follows from Lemmas 4 and 5.

LEMMA 7. *Let ρ be cyclic of order at most 4. Let Γ be any (generalized) quaternion group. Then any two monomorphisms $f:\rho \to \Gamma$ induce the same map $\hat{H}^4(\Gamma, Z) \to \hat{H}^4(\rho, Z)$.*

Proof. Γ has only one element of order 2, namely a^t. Thus the result is trivial if ρ has order 1 or 2. If ρ has order 4, $f(\rho)$ is contained in an ordinary quaternion subgroup Γ' of Γ. Any two monomorphisms $\rho \to \Gamma'$ differ by an automorphism of Γ'. Therefore the result follows from Lemmas 5 and 6.

Now let us return to the 2-sylow subgroup π_2 of π and the maps $i, f_x : \rho \to \pi_2$ where $\rho = \pi_2 \cap x\pi_2 x^{-1}$. If ρ is an ordinary quaternion group or is cyclic of order at most 4, it follows from Lemmas 6 and 7 that f_x^* and i^* agree on $\hat{H}^4(\pi_2, Z)$. Therefore we have only to consider the case where ρ is cyclic of order greater than 4 or is a properly generalized quaternion group (i.e., of order at least 16). Since such a group ρ is not contained in an ordinary quaternion group, π_2 must be a properly generalized quaternion group.

For any group Γ, let $\Gamma_{(8)}$ be the subgroup of Γ generated by all elements having order at least 8. This is a characteristic subgroup and is, in fact, stable under monomorphisms. If Γ is a properly generalized quaternion group with the presentation $\{a, b : a^t = b^2, aba = b\}$, then $\Gamma_{(8)}$ is the cyclic subgroup generated by a. The only reason for introducing the notation $\Gamma_{(8)}$ is to give a natural way of picking out this subgroup.

Note that an inner automorphism of Γ will send a into either a or a^{-1}. Choose a generator z for $\rho_{(8)}$, and let H be the subgroup of π consisting of all elements u such that uzu^{-1} is either z or z^{-1}. By the remark just made about inner automorphisms of Γ, we see that π_2 and $x\pi_2 x^{-1} \subset H$. Since they are 2-sylow subgroups, there is a $u \in H$ such that $ux\pi_2 x^{-1}u^{-1} = \pi_2$. We can assume that $uzu^{-1} = z$ because, if $uzu^{-1} = z^{-1}$, we can replace u by bu where $\pi_2 = \{a, b : a^t = b^2, aba = b\}$ is a presentation for π_2.

We have now factored f_x as $f_x = Ag$ where $A : \pi_2 \to \pi_2$ by $A(y) = x^{-1}u^{-1}yux$ and $g : \rho \to \pi_2$ by $g(y) = uyu^{-1}$. By the choice of u, we have $g(z) = z$. Thus, if ρ is cyclic, we have $g = i$, the inclusion map $\rho \to \pi_2$ (because ρ has order divisible by 8, and so $\rho = \rho_{(8)}$). If ρ is a generalized quaternion group, it is generated by z and some a^ib where $\{a, b : a^t = b^2, aba = b\}$ is a presentation of π_2. Suppose $g(a^ib) = a^jb$. Let ϕ be the automorphism of π_2 given by $\phi(a) = a$, $\phi(b) = a^{j-i}b$. Then $g = \phi i$. This shows that, whether ρ is cyclic or not, we can write $f_x = A\phi i$ where ϕ is an automorphism of π_2 of the type considered in Lemma 4. Now, $f_x^* = i^*\phi^*A^*$. By Lemma 4, $\phi^* = 1$. In order to show that all elements of $\hat{H}^4(\pi_2, Z)$ are stable, it remains to show that $A^* = 1$.

For each element $v \in N(\pi_2)$, let $A_v : \pi_2 \to \pi_2$ be the map $A_v(y) = vyv^{-1}$.

The map $v \to (A_v^{-1})^*$ defines a homomorphism α of $N(\pi_2)$ into the group of automorphisms of $\hat{H}^4(\pi_2, Z)$. If $v \in \pi_2$, A_v is an inner automorphism of π_2, and so $\alpha(v) = 1$ [1, Ch. XII, §8, (5)]. Therefore $\pi_2 \subset \ker \alpha$. Since π_2 is a 2-sylow subgroup of π, the image of α must have odd order. But $\hat{H}^4(\pi_2, Z)$ is a cyclic 2-group, so its automorphism group is again a 2-group [3, Ch. IV, §3]. Therefore α must be trivial. Since $A^* = \alpha(ux)$, it follows that $A^* = 1$. This proves Theorem 1.

Appendix

Lemma 1 may be considered a generalization of a well-known theorem of Burnside [3, Ch. V, Th. 4]. For suppose π_p is in the center of $N(\pi_p)$. Then the group Φ_p is the trivial group. Consequently, by Lemma 1, all elements of $\hat{H}^i(\pi_p, Z)$ are stable, and so $i^*: \hat{H}^i(\pi, Z) \to \hat{H}^i(\pi_p, Z)$ is an isomorphism of the p-primary component of $\hat{H}^i(\pi, Z)$ onto $\hat{H}^i(\pi_p, Z)$. Now, for $i = -2$, this map is just the classical transfer $\pi \to \pi_2$ [1, Ch. XII, Ex. 10]. Since it is onto, π splits over π_2.

It is natural to inquire whether any of the other classical theorems on transfer generalize to theorems on cohomology. I will show here that the second theorem of Grün [3, Ch. V, Th. 6] does have such a generalization.

The statement that an element $\alpha \in \hat{H}^i(\pi_p, A)$ is stable depends of course on the group π in which π_p is contained. When it is necessary to emphasize this, I will say that α is stable with respect to π.

THEOREM 3. *Suppose π is p-normal* [3, Ch. V, §2]. *Let π_p be a p-sylow subgroup of π, and let N be the normalizer of the center of π_p. Then for any π-module A, an element $\alpha \in \hat{H}^i(\pi_p, A)$ is stable with respect to π if and only if it is stable with respect to N.*

For the proof, we need the following lemma which is extracted from the usual proof of Grün's second theorem [3, Ch. V, Th. 6].

LEMMA 8. *Let π be p-normal. Let $\mathfrak{z}_1$, $\mathfrak{z}_2$ be p-centers* [3, Ch. V, §2] *of π. If $\mathfrak{z}_1$ and $\mathfrak{z}_2$ are contained in a subgroup H of π, they are conjugate in H.*

Proof. If H is a p-group, it is contained in a p-sylow subgroup P of π. Since π is p-normal, $\mathfrak{z}_1$ and $\mathfrak{z}_2$ must both be equal to the center of P, and so $\mathfrak{z}_1 = \mathfrak{z}_2$ in this case.

Now, if H is any subgroup of π, imbed $\mathfrak{z}_1$ and $\mathfrak{z}_2$ in p-sylow subgroups P_1 and P_2 of H. An inner automorphism of H will take P_2 into P_1. By the remark just made about p-groups, this inner automorphism takes $\mathfrak{z}_2$ into $\mathfrak{z}_1$.

Proof of Theorem 3. If α is stable with respect to π, it is clearly stable with respect to N.

Assume now that α is stable with respect to N. Let $x \in \pi$. Let

$$\rho = \pi_p \cap x\pi_p x^{-1}.$$

Let i, $f_x : \rho \to \pi_p$ by $i(y) = y$ and $f_x(y) = x^{-1}yx$. We must show that $i^*(\alpha) = f_x^*(\alpha)$.

Let $\mathfrak{z}$ be the center of π_p. Then $x\mathfrak{z}x^{-1}$ is the center of $x\pi_p x^{-1}$. Both of these are in the centralizer $C(\rho)$ of ρ. By Lemma 8, there is an element $t \in C(\rho)$ such that $tx\mathfrak{z}x^{-1}t^{-1} = \mathfrak{z}$. In other words, $tx \in N$. Since $t \in C(\rho)$, the maps $i, f_x : \rho \to \pi_p$ can be obtained by composing the inclusion map $\rho \to \pi_p \cap tx\pi_p x^{-1}t^{-1}$ with the maps $i', f'_{tx} : \pi_p \cap tx\pi_p x^{-1}t^{-1} \to \pi_p$ defined by $i'(y) = y$ and $f'_{tx}(y) = x^{-1}t^{-1}ytx$. Now $i'^*(\alpha) = f'^*_{tx}(\alpha)$ since $tx \in N$ and α is stable with respect to N. Therefore $i^*(\alpha) = f^*_x(\alpha)$. This shows that α is stable with respect to π.

COROLLARY. *Let π be p-normal, and let N be the normalizer of a p-center of π. Let A be any π-module. Then the inclusion and transfer maps both are isomorphisms between the p-primary components of $\hat{H}^i(\pi, A)$ and $\hat{H}^i(N, A)$.*

This follows immediately from Theorem 3 and the results of [1, Ch. XII, §9].

If we set $i = -2$ and $A = Z$ in this corollary, we get the classical second theorem of Grün.

We can also recover Lemma 1 from Theorem 3 for, if π_p is abelian, it is normal in N. Therefore, an element of $\hat{H}^i(\pi_p, A)$ is stable with respect to N if and only if it is fixed under the action of N on π_p by $y \to x^{-1}yx$ for each $x \in N$.

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