MARKOFF CHAINS AND MARTIN BOUNDARIES[1]

BY

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In the latter half of [5] Doob extends to Markoff chains many results he had previously obtained for Brownian motions. Roughly, his argument rests on the theory of martingales and on properties of the Martin boundary established by R. S. Martin and M. Brelot using classical methods, the bridge between the two groundworks being the equivalence of resolutivity of the boundary and almost certain convergence of an appropriate Markoff chain.

This paper presents another argument, using only the basic properties of martingales and Markoff chains, in which the main convergence theorem of Doob is proved at the beginning by reversing the sense of time in a Markoff chain. I first intended to write a note giving the simple proof of this convergence by explicit calculation; the subject is so attractive, however, that I decided upon a brief complete exposition, including some material omitted from §17 of [7] about which I shall say a word.

In view of the symmetry of past and future in the notion of Markoff chain, the lack of such symmetry in defining Markoff chains with stationary transitions must puzzle many a probabilist. Now, a slight and momentarily ugly alteration of the latter definition yields the notion of random chain with approximately stationary transitions, a notion symmetric in past and future This symmetry is used in §2 to establish the convergence mentioned above and in §5 to reduce problems concerning the entrance boundary to ones concerning the exit boundary. The chains themselves are studied in §1 and in the first part of §5 in order to furnish the proper background for [5] and [7].

Doob's convergence theorem, established directly, leads to the proofs in §3 and §4 of the Poisson-Martin representation of excessive functions, the behavior of excessive functions near the Martin boundary, and the resolutivity of the Martin boundary. Of course, it is only the arrangement of material that distinguishes these sections.

Some remarks are deferred until §6, the last section, since most of them merely explain the departures from the language and definitions of Doob and Brelot.

Doob's argument and ours both hold for Brownian motions or, more generally, for the processes discussed in the third part of [7].

1. Random chains

The space of states is a countable set R which is provided with the discrete topology as a topological space and with the field of all its subsets as a meas-

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urable space. Except for a moment in §4, functions and measures on this space are to be positive (nonnegative), perhaps infinite, and the convention is recalled only in a few important definitions. We sometimes write an integral, rather than a sum, in order to distinguish measures from functions.

For each r in R let $P(r, ds)$ be a positive measure on R satisfying

$$P(r, R) \leqq 1, \qquad r \in R. \tag{1.1}$$

The transition function P defines a transformation $f \rightarrow Pf$ of functions,

$$Pf.(r) \equiv \int_R P(r, ds) f(s), \qquad r \in R, \tag{1.2}$$

as well as a transformation $\nu \rightarrow \nu P$ of measures,

$$\nu P.(C) \equiv \int_R \nu(dr) P(r, C), \qquad C \subset R. \tag{1.3}$$

A positive function h is excessive (relative to P) if it dominates Ph, concordant if it is finite and coincides with Ph. A positive measure ζ is excessive if it dominates ζP, concordant if it is finite on finite sets and coincides with ζP.

Denote by $P_0(r, ds)$ the unit mass at the point r, and define recursively

$$P_{n+1}(r, C) \equiv \int_R P_n(r, ds) P(s, C), \qquad r \in R, \quad C \subset R, \tag{1.4}$$

so that P_1 is just P. The kernel $G(r, ds)$ for a potential theory is taken to be

$$G(r, ds) \equiv \sum_{n \geqq 0} P_n(r, ds). \tag{1.5}$$

Both P_n and G define transformations of functions or measures by formulas like (1.2) or (1.3). Clearly, Gf and νG are excessive if f is a positive function and ν a positive measure.

As we shall see in §5, an excessive measure determines the initial behavior of a Markoff chain having P for transition function, whereas an excessive function determines the final behavior of the chain. In order for such a statement to have the proper scope one needs an enlargement of the notion of Markoff chain with given transition function, which we proceed to explain.

Let $(\Omega, \mathcal{B}, \mathcal{P})$ be a measure space; that is to say, $\mathcal{B}$ is a Borel field of subsets of Ω, which itself belongs to $\mathcal{B}$, and $\mathcal{P}$ is a positive measure on $\mathcal{B}$. Let α and β be measurable functions on Ω; the values of α are to be integers or $-\infty$, those of β are to be integers or $+\infty$, and the inequality $\alpha \leqq \beta$ is to hold. Let $x(n, \omega)$ be defined as a point of R for almost all ω and for all integers n satisfying $\alpha(\omega) \leqq n \leqq \beta(\omega)$. In order to speak of functions in the ordinary sense, we extend the definition to all integers, taking $x(n, \omega)$ to be a for $n < \alpha(\omega)$ and b for $n > \beta(\omega)$, with a and b distinct objects not belonging to R; after the extension the function $x(n, \cdot)$ denoted later by $x(n)$, is to be measurable over $\mathcal{B}$ for each n. The triple (x, α, β), where x stands for the function $x(\cdot, \cdot)$, is said to be a *random chain on* R, defined over Ω, if the following statement is true:

(1.6) *Let* $\Lambda_{n,r}$ *be the set in* Ω *where* $\alpha \leqq n \leqq \beta$ *and* $x(n) = r$. *Then* $\mathcal{P}\{\Lambda_{n,r}\}$ *is finite, for every integer* n *and every* r *in* R.

The measures of the sets where $x(n)$ takes on the value a or b may of course be infinite. Under (1.6) the space Ω is the union of countably many sets of finite measure.

A random chain (x, α, β) is said to be a Markoff chain if the past and future are independent once the present is fixed. To be precise, let k, m, n be integers satisfying $k < m < n$, let $r_k, \cdots, r_n$ be points of R, and let Λ, Λ', Λ'' be the sets in Ω defined by the conditions

$$\begin{aligned}
&\Lambda: \quad \alpha \leqq m \leqq \beta, && x(m) = r_m, \\
(1.7) \qquad &\Lambda': \quad \alpha \leqq k, \quad m \leqq \beta, && x(j) = r_j \quad \text{for} \quad k \leqq j \leqq m, \\
&\Lambda'': \quad \alpha \leqq m, \quad n \leqq \beta, && x(j) = r_j \quad \text{for} \quad m \leqq j \leqq n.
\end{aligned}$$

Then the relation

$$(1.8) \qquad \frac{\mathcal{P}\{\Lambda' \cap \Lambda''\}}{\mathcal{P}\{\Lambda\}} = \frac{\mathcal{P}\{\Lambda'\}}{\mathcal{P}\{\Lambda\}} \cdot \frac{\mathcal{P}\{\Lambda''\}}{\mathcal{P}\{\Lambda\}}$$

is to hold, provided $\mathcal{P}\{\Lambda\}$ is not zero.

A Markoff chain (x, α, β) is said to have P for stationary transition function, or simply to be a P-chain, if

$$(1.9) \qquad \mathcal{P}\{\Lambda_{n,r} \cap \Lambda_{n+1,s}\} = \mathcal{P}\{\Lambda_{n,r}\}P(r, s)$$

for all r, s in R and all integers n, the sets $\Lambda_{n,r}$ being defined as in (1.6). This definition differs from the usual one only in permitting Ω to have arbitrary mass and in providing the chain with an initial time α as well as a terminal time β.

Consider now a random chain (x, α, β) and a function σ measurable over $\mathcal{B}$ having $-\infty$, $+\infty$, or integers for values. On the set Ω' where σ is finite and satisfies the condition $\alpha \leqq \sigma \leqq \beta$, a triple $(y, 0, \gamma)$ is defined by the formulas

$$(1.10) \qquad \gamma(\omega) = \beta(\omega) - \sigma(\omega), \qquad y(n, \omega) = x(\sigma(\omega) + n, \omega), \qquad \omega \in \Omega'.$$

The random time σ is said to reduce (x, α, β) to a P-chain if $(y, 0, \gamma)$ is a P-chain defined over Ω'. The random chain (x, α, β) itself is said to be an approximate P-chain, or to have P for approximate stationary transition function, if there is a sequence of random times α_n with these properties: The values of each α_n are $+\infty$ or integers; the α_n decrease to α almost certainly (that is to say, except on a set null for $\mathcal{P}$); and each α_n reduces (x, α, β) to a P-chain.

A narrower definition is preferable in some circumstances. An approximate P-chain is said to be strongly approximate if it is a Markoff chain and if the random times α_n reducing it to a P-chain may be taken as stopping times for the chain. We recall that σ is a stopping time for the random chain (x, α, β) if for each integer n the set where $\sigma \leqq n$ belongs to the Borel subfield of $\mathcal{B}$ generated by sets of the form $\Lambda_{k,r}$ with $k \leqq n$ and r in R.

An approximate P-chain differs trivially from a true P-chain if the initial time α is finite, because α itself reduces the chain to a P-chain provided condition (1.6) remains valid. For most choices of P, however, there are approximate P-chains defined over spaces of finite mass which cannot be turned into true P-chains by any random shift in time.

We shall now investigate the effect of reversing the sense of time. If (x, α, β) is a random chain, then the triple (x', α', β'), defined over the same probability space by

$$\alpha' = -\beta, \qquad \beta' = -\alpha, \qquad x'(n, \omega) = x(-n, \omega), \tag{1.11}$$

is also a random chain, which we shall term the reversed chain. The chain (x, α, β) determines a measure η on R by the formula

$$\eta(C) \equiv \int_\Omega \sum_{\alpha \leqq n \leqq \beta} \chi(x(n))\mathcal{P}(d\omega), \tag{1.12}$$

with χ the characteristic function of the set C; the reversed chain clearly determines the same measure. In discussing either chain, we may suppose R to be replaced by the set of points s for which $\eta(s)$ is strictly positive.

THEOREM 1.1. *Let (x, α, β) be an approximate P-chain for which the measure η in (1.12) is finite on finite sets. Then the reversed chain (x', α', β') is an approximate Q-chain, with $Q(s, dr)$ defined as*

$$Q(s, r) = \eta(r)P(r, s)/\eta(s) \tag{1.13}$$

for $\eta(s) > 0$ and otherwise arbitrary. Moreover, the reversed chain is strongly approximate if the original chain is so.

Given a subset D of R, take $\tau(\omega)$ to be $-\infty$ if $x(n, \omega)$ lies outside D for all n, otherwise to be the supremum of the n for which $x(n, \omega)$ belongs to D; we speak of τ as the time chain (x, α, β) leaves D. It never exceeds β, and it is almost certainly finite or $-\infty$ if $\eta(D)$ is finite, in particular if D is finite. We shall prove that $-\tau$ reduces the reversed chain to a Q-chain.

The quantity $L(s)$ will occur in the computation. It vanishes for s outside D, and for s in D it is the probability that $y(n) \in R - D$ for $0 < n \leqq \gamma$, with $(y, 0, \gamma)$ a P-chain defined over a probability space of unit mass and having $y(0)$ identically s.

Consider first a P-chain of the form $(x, 0, \beta)$, and let ν be the distribution measure of $x(0)$. The measure η can then be written $\sum \nu P_n$. Given points $r_0, \cdots, r_k$ in R, let Λ be the set in Ω defined by the conditions

$$\Lambda: \quad k \leqq \tau < \infty, \qquad x(\tau - j) = r_j \quad \text{for} \quad 0 \leqq j \leqq k, \tag{1.14}$$

and let Λ_m be the part of Λ where $\tau = m$. Then Λ is the union of the Λ_m, and

$$(1.15) \qquad \mathcal{P}\{\Lambda_m\} = \nu P_{m-k}(r_k)P(r_k, r_{k-1}) \cdots P(r_1, r_0)L(r_0).$$

Summed on m these equations yield

$$(1.16) \qquad \mathcal{P}\{\Lambda\} = \eta(r_k)P(r_k, r_{k-1}) \cdots P(r_1, r_0)L(r_0).$$

The last equation holds in fact for an arbitrary approximate P-chain (x, α, β). To see this, consider the random times α_n giving the chain its structure of approximate P-chain, let $(x_n, 0, \beta_n)$ be the associated true P-chains defined as in (1.10), and let τ_n and η_n be defined in terms of $(x_n, 0, \beta_n)$. Clearly, η_n increases to η. Also, for almost all ω and for n great enough, $\alpha_n(\omega) + \tau_n(\omega)$ coincides with $\tau(\omega)$ and $x_n(\tau_n(\omega) - j, \omega)$ coincides with $x(\tau(\omega) - j, \omega)$, provided of course $\tau(\omega)$ is finite. Thus we obtain (1.16) for (x, α, β) by writing the equation for $(x_n, 0, \beta_n)$ and passing to the limit.

Equation (1.16) evidently proves that $-\tau$ reduces the reversed chain (x', α', β') to a Q-chain. Moreover, $-\tau$ is a stopping time for the reversed chain. One now obtains the second sentence of the theorem on letting D swell to R through an increasing sequence of finite sets, and the third on remarking that the definition of Markoff chain is symmetric in past and future.

We have in fact proved a little more than the theorem asserts. First, regarding (x, α, β) as obtained from (x', α', β') by reversing the sense of time, we see that the random times α_n giving (x, α, β) its structure of approximate P-chain may be taken as stopping times for the chain. Next, $Q(s, R)$ cannot exceed 1 if $\eta(s)$ is strictly positive, because Q serves as approximate stationary transition function for the reversed chain; thus, taking $Q(s, dr)$ to be an arbitrary measure of unit mass or less whenever $\eta(s)$ vanishes, we may suppose Q to satisfy the analogue of (1.1). Finally, the measure η is excessive relative to P; this statement follows from the preceding observation, or directly from the definitions without the hypothesis that η is finite on finite sets.

It is easy to see that a positive function on R is excessive relative to P if and only if the sequence $f(x(n))$ is a supermartingale for every choice of (x, α, β) as a P-chain. Here supermartingale means lower semimartingale in the sense of [3], but with infinite measure space and infinite expectations permitted, while $f(x(n))$ stands for $+\infty$ if $n < \alpha$ and for zero if $n > \beta$. The $f(x(n))$ may not form a supermartingale if (x, α, β) is only an approximate P-chain, but the martingale convergence theorems remain valid, as the next two propositions show.

PROPOSITION 1.2. *Let f be a finite excessive function and (x, α, β) an approximate P-chain. Then $f(x(n, \omega))$ almost certainly converges as n increases to $\beta(\omega)$.*

Here and later, convergence means that the sequence reaches its final value if $\beta(\omega)$ is finite, and converges in the usual sense to a finite limit if $\beta(\omega)$ is infinite.

The assertion follows from a standard proposition in martingale theory. Let σ reduce the chain to a true P-chain $(y, 0, \gamma)$, and let $(y', 0, \gamma')$ be the latter chain restricted to the set in the measure space where $y(0)$ belongs to some given finite set C. Thus $(y', 0, \gamma')$ is a P-chain defined over a measure space Ω' of finite mass, which we may suppose to be 1. Over Ω' define random variables ϕ_n by taking $\phi_n(\omega)$ to be $f(y'(n, \omega))$ if $n \leqq \gamma'(\omega)$, or zero if $n > \gamma'(\omega)$. The ϕ_n then form a positive lower semimartingale, in the usual sense of the words, so that $\phi_n(\omega)$ almost certainly converges to a finite limit as n increases. We obtain the proposition on letting C swell to R and then letting σ decrease to α.

PROPOSITION 1.3. *Let $f \equiv Gg$, with g a positive function, and let (x, α, β) be an approximate P-chain. If $\int_R g\, d\eta$ is finite, with η defined by* (1.12), *then $f(x(n, \omega))$ almost certainly converges as n decreases to $\alpha(\omega)$.*

We have already noted that Gg is excessive, since g is positive. The proposition can be extended to more general excessive functions, but the condition of finiteness then becomes more complicated.

In the proof, again let σ reduce (x, α, β) to a true P-chain $(y, 0, \gamma)$, defined on the subset Ω^* of the measure space, and let ν be the distribution measure of $y(0)$ on R. Then

$$\int_{\Omega^*} f(y(0))\mathcal{P}(d\omega) = \int_R f\, d\nu = \int_R Gg\, d\nu \leqq \int_R g\, d\eta,$$

because η dominates the measure νG, as one sees from the definitions. Denote by ϕ the number of downcrossings of the interval $[c, d]$ by the supermartingale $f(y(n))$. On making the same reductions as in the preceding proof and then applying a standard martingale argument, we obtain

$$(1.17) \qquad \int_{\Omega^*} \phi(\omega)\mathcal{P}(d\omega) \leqq \frac{1}{d-c}\int_{\Omega^*} f(y(0))\mathcal{P}(d\omega) \leqq \frac{1}{d-c}\int_R g\, d\eta.$$

The inequality on which this is based is a little sharper than the one in [3; page 316], because we are dealing with a positive supermartingale, but the proof given in [3] requires only minor changes. Now let σ decrease to α. In the limit we obtain the inequality with Ω^* replaced by the full measure space Ω and ϕ interpreted as the number of downcrossings of $[c, d]$ by the random sequence $f(x(n))$, with $\alpha \leqq n \leqq \beta$. Thus, almost certainly the number of downcrossings is finite for every rational interval $[c, d]$, so that the sequence almost certainly has a limit as n decreases to α. The finiteness of the limit also follows from (1.17), written with Ω replacing Ω^*. To see this, denote by Ω_∞ the set on which the upper limit of $f(x(n))$ is infinite; the value of ϕ at a point of Ω_∞ is at least 1 for c and d sufficiently great; so, taking d to be $2c$ and letting c increase, we see from (1.17) that the measure of Ω_∞ must vanish.

As an illustration of later arguments, we establish the inequality

$$G(r, s) \leqq G(s, s), \tag{1.18}$$

which holds for all points of R. Consider a true P-chain $(x, 0, \beta)$ starting at r, that is to say, $x(0)$ is identically r and the underlying measure space Ω has unit mass. Take $\tau(\omega)$ to be the least n for which $x(n, \omega) = s$, or to be $+\infty$ if there is no such n. As remarked after Theorem 1.1, the random time τ reduces the chain to a P-chain $(y, 0, \gamma)$, defined over the set Ω' where τ is finite and having $y(0)$ identically s. Now, by the definition of G and τ,

$$\begin{aligned} G(r, s) &= \int_\Omega \sum_{0 \leqq n \leqq \beta} \chi(x(n))\mathcal{P}(d\omega) \\ &= \int_{\Omega'} \sum_{\tau \leqq n \leqq \beta} \chi(x(n))\mathcal{P}(d\omega) \\ &= \int_{\Omega'} \sum_{0 \leqq n \leqq \gamma} \chi(y(n))\mathcal{P}(d\omega) = P\{\Omega'\}G(s, s), \end{aligned} \tag{1.19}$$

where χ is the characteristic function of the set reduced to s. Thus, (1.18) merely states that the mass of Ω' cannot exceed one.

The preceding results will be completed in §5, in which we prove the existence of approximate P-chains associated with excessive measures.

In the remainder of the paper the field $\mathcal{B}$ of the underlying measure space $(\Omega, \mathcal{B}, \mathcal{P})$ will be assumed complete for the measure $\mathcal{P}$, and occasionally the space will be restricted further by requiring the existence of conditional probability distributions. A random quantity is a function defined up to a null set on Ω and satisfying some obvious condition of measurability. We shall often employ the notation and language of probability theory, usually arranging matters so that the underlying measure space has unit mass. It should be noted, however, that a conditional expectation $\mathcal{E}\{\phi \mid \mathfrak{F}\}$ has a sensible interpretation even on a space of infinite mass, provided the space is the union of countably many sets of finite measure which belong to $\mathfrak{F}$ and on which ϕ is integrable; this being so, there is seldom a real need for the reduction to a space of unit mass. We shall also use, without particular mention, the strong Markoff property; it is treated fully in [1], and it is nearly trivial in our applications, since both space and time are countable.

2. The Martin exit boundary

Although we continue to regard R and P as basic, we shall consider with profit certain related spaces and transition functions first introduced by Brelot and Feller. Given a positive function h, defined on R and excessive relative to P, denote by R^h the set where h is finite and strictly positive, and denote by P^h the function

$$P^h(r, C) \equiv \frac{1}{h(r)} \int_C P(r, ds)h(s), \qquad r \in R^h, \quad C \subset R^h. \tag{2.1}$$

Clearly, P^h is a transition function on R^h since it satisfies the analogue of (1.1). Also, R^h and P^h reduce to R and P if h is a strictly positive constant.

We speak of h-excessive functions or measures, meaning those defined on R^h and excessive relative to P^h. Similarly, an h-chain is a Markoff chain having R^h for state space and P^h for transition function. The kernels P_n^h and G^h are defined as in (1.4) and (1.5); they may also be expressed as

$$P_n^h(r, ds) = \frac{1}{h(r)} P_n(r, ds)h(s), \tag{2.2}$$

$$G^h(r, ds) = \frac{1}{h(r)} G(r, ds)h(s), \tag{2.3}$$

with r and s restricted to R^h.

A quantity defined in terms of h, R^h, P^h usually bears h as superscript. The superscript is often omitted from a quantity when h is a strictly positive constant, for then 1, R, P may be taken as the reference triple; such quantities are regarded as absolute, and it is sometimes convenient to express relative quantities in terms of them, as in the preceding display. Of course, these conventions only simplify the notation; any triple k, R^k, P^k may be taken as the absolute in the discussion, the function h occurring in the formulas then being k-excessive and P^h being defined accordingly.

From now on we suppose $G(r, s)$ to be finite for all r and s. (This assumption is made only to simplify the exposition; one can treat persistent states either by the method of [5] or by altering P slightly.) Then $G^h(r, s)$ is finite for all r and s in R^h, according to (2.3), and there are no persistent states for h-chains; consequently, almost all paths of an approximate h-chain meet a given finite set only finitely many times.

We also fix a reference measure γ on R satisfying

$$\gamma(R) < \infty, \qquad \gamma(r) > 0 \quad \text{for } r \text{ in } R, \tag{2.4}$$

and from now on we consider only excessive functions that are integrable for γ, hence finite everywhere. If h is such a function, γ^h denotes the finite strictly positive measure $h\,d\gamma$ on R^h. Note that $P(r, s)$ vanishes, hence $G(r, s)$ also, if s lies in R^h and r outside R^h.

The h-excessive function $G^h(\cdot, s)$ is integrable for γ^h for every s in R^h according to (1.18). The measure $\gamma^h G^h$, denoted ζ^h from now on, is therefore strictly positive and finite on finite subsets of R^h. We use this measure to introduce the functions

$$K^h(r, s) \equiv \frac{G^h(r, s)}{\zeta^h(s)} = \frac{1}{h(r)} K(r, s), \tag{2.5}$$

which are defined on $R^h \times R^h$ and which satisfy

$$\int_{R^h} \gamma^h(dr) K^h(r, s) = 1. \tag{2.6}$$

Clearly, the function $K^h(r, \cdot)$ is bounded by $1/\gamma^h(r)$. A probabilistic interpretation of ζ^h and K^h will be given later in this section.

We shall now complete the space R^h, following Martin and Doob. First choose a metric d_1 on R under which the completion of R is the Alexandroff compactification of R, and denote by d_1^h the restriction of d_1 to R^h. Next choose a strictly positive function δ on R so that $\sum \delta(r)$ is finite, and set

$$(2.7)\qquad \begin{aligned} d_2^h(s, t) &\equiv \int_{R^h} | K^h(r, s) - K^h(r, t) | \delta(r)\gamma^h(dr) \\ &= \int_R | K(r, s) - K(r, t) | \delta(r)\gamma(dr) \end{aligned}$$

for s and t in R^h. The equality of the integrals is a consequence of relation (2.5) and the vanishing of $K(r, s)$ for s in R^h and r outside R^h; it ensures that d_2^h is just the restriction of d_2 to R^h. The function d_2^h is a finite metric on R^h, by (2.6).

A sequence (s_n) is a Cauchy sequence for d_1^h if and only if it tends to infinity or else is ultimately constant. It is a Cauchy sequence for d_2^h if and only if the $K^h(r, s_n)$ form a Cauchy sequence of real numbers for each fixed r in R^h. This assertion is true only because the function δ appears in the integral; fortunately, the alternate description of Cauchy sequences shows the choice of δ to be irrelevant.

Now take $^*R^h$ to be the completion of R^h under the metric $d_1^h + d_2^h$. Clearly, $^*R^h$ is compact because of the description of convergence in the second metric; the topology of $^*R^h$ induces the discrete topology on R^h because of the presence of the first metric; and R^h is open dense in $^*R^h$. The Martin exit boundary (relative to γ^h) is by definition the compact set $^*R^h - R^h$, denoted by B^h.

The definitions have been phrased so that R^h may be considered a subspace of the metric space R. Therefore $^*R^h$ and B^h may be considered subspaces of $^*R$ and B, which are obtained on taking h to be a strictly positive constant.

Let ξ be a point of B^h. The formula

$$(2.8)\qquad K^h(r, \xi) \equiv \lim_{s \to \xi} K^h(r, s), \qquad s \in R^h,$$

defines a function $K^h(\cdot, \xi)$ which is h-excessive, as one sees by a passage to the limit, and which satisfies

$$(2.9)\qquad \int_{R^h} \gamma^h(r) K^h(r, \xi) \leqq 1.$$

As now defined, $K^h(\cdot, \cdot)$ is continuous on the product space $R^h \times {}^*R^h$, and also

$$(2.10)\qquad K^h(r, \xi) = \frac{1}{h(r)} K(r, \xi), \qquad r \in R^h, \quad \xi \in {}^*R^h,$$

where $K(\cdot, \xi)$, defined on R, is obtained by taking h to be a strictly positive constant. Note that $K(\cdot, \xi)$ vanishes on $R - R^h$ if $\xi \in {}^*R^h$.

The equations

$$G^h - P^hG^h = G^h - G^hP^h = P_0^h, \tag{2.11}$$

where the products denote composition of transformations, show that $G^h(\cdot, s)$ determines the point s, and $G^h(r, \cdot)$ the point r. Therefore, a point s of R^h is determined by $K^h(\cdot, s)$, and of course by definition a point ξ of B^h is determined by $K^h(\cdot, \xi)$. On the other hand, a point of R^h and a point of B^h may give rise to the same function.

The process of completing R^h introduces points irrelevant for h-chains, and we must now single out the pertinent subset of $^*R^h$.

A finite h-excessive function g is said to be extreme if the equation $g = g_1 + g_2$, with g_1 and g_2 both h-excessive, implies that g_1 and g_2 are constant multiples of g. This notion is really independent of h, provided h is finite, in the sense that g is extreme if and only if the excessive function defined as gh on R^h and as zero on $R - R^h$ is extreme relative to the basic triple $1, R, P$. Note also that g is h-concordant if and only if gh, extended trivially, is concordant.

Denote by B_e^h the set of points ξ in B^h for which the function $K^h(\cdot, \xi)$ is h-concordant, is extreme as an h-excessive function, and satisfies

$$\int_{R^h} \gamma^h(dr)K^h(r, \xi) \equiv \int_R \gamma(dr)K(r, \xi) = 1. \tag{2.12}$$

(The integrals coincide, by (2.5) and a subsequent remark.) According to the preceding paragraph, B_e^h is precisely $B_e \cap B^h$, and moreover it is a Borel subset of B. The importance of B_e^h is shown by the following theorem.

THEOREM 2.1. *Let h be excessive and integrable for γ, and let (x, α, β) be an approximate h-chain. Then, almost certainly, either β is finite and $x(\beta)$ a point of R^h, or else β is infinite and $x(n)$ converges in the topology of $^*R^h$ to a point of B_e^h as $n \to \infty$.*

By definition $x(\beta)$ almost certainly is a point of R^h if β is finite; and $x(n)$ almost certainly approaches the boundary B^h if β is infinite, since there are no persistent states. Thus, in view of the definition of the topology, we have only to establish convergence of the sequence of functions $K^h(\cdot, x(n))$ as n increases to β and to determine the nature of the limit function. Also, by an argument familiar from §1, it is enough to prove the theorem for one true h-chain $(x, 0, \beta)$ such that $\mathcal{P}\{x(0) = r\}$ is strictly positive for each r in R^h.

We choose an h-chain $(x, 0, \beta)$ having γ^h for initial distribution measure. Then not only is $\gamma^h(r)$ strictly positive for all r in R^h, but the chain is related to the measure ζ^h by a formula like (1.12), so that computations become especially simple. The reversed chain has

$$Q(s, dr) \equiv \zeta^h(dr)\, \frac{P^h(r, s)}{\zeta^h(s)} \tag{2.13}$$

for approximate stationary transition function, according to Theorem 1.1, and the analogue of (1.4) gives

$$Q_n(s, dr) \equiv \zeta^h(dr) \frac{P_n^h(r, s)}{\zeta^h(s)}. \tag{2.14}$$

Hence the kernel

$$H(s, dr) \equiv \zeta^h(dr) \frac{G^h(r, s)}{\zeta^h(s)} = \zeta^h(dr) K^h(r, s) \tag{2.15}$$

bears the same relation to Q that G^h bears to P^h. Therefore, by Proposition 1.3 applied to the reversed chain, $H(x(n), r)$ almost certainly converges as n increases to β. The similar convergence of $K^h(r, x(n))$ follows immediately, by (2.15) and the strict positivity of $\zeta^h(r)$.

For the moment let (x, α, β) be again an arbitrary approximate h-chain. The preceding paragraphs imply that $K^h(r, x(n))$ almost certainly converges as n increases to β. Thus the formula

$$x(\beta) \equiv \lim_{n\to\beta} x(n) \tag{2.16}$$

defines a random point of $*R^h$, since the limit almost certainly exists in the topology of $*R^h$ if β is infinite, and the limit is attained if β is finite; this definition agrees with the original meaning of $x(\beta)$ for β finite. One sees, by a passage to the limit, that $x(\beta)$ is measurable relative to the topological Borel field of $*R^h$ and the field of the measure space underlying the chain, provided one allows as we do an exceptional set of measure zero in defining a random quantity. We shall sometimes speak of $x(\beta)$ as the final state of the chain. Clearly, $x(\beta)$ almost certainly belongs to B^h if β is infinite; but we have not proved it then lies in B_e^h. This last step in proving the theorem will be carried out in the middle of the next section; meanwhile we shall develop the consequences of what has already been established.

We revert to the h-chain $(x, 0, \beta)$ having γ^h for initial distribution measure. Denote by μ^h the distribution measure of $x(\beta)$ on $*R^h$,

$$\mu^h(C) = \mathcal{P}\{x(\beta) \in C\}, \qquad C \subset *R^h, \tag{2.17}$$

and by μ_R^h or μ_B^h its restriction to R^h or B^h. Often, thinking of $*R^h$ as a subspace of $*R$, we shall consider these measures defined on $*R$, supposing them extended by the null measure. The measure μ, obtained on taking h to be a strictly positive constant, will be given a special role in order to simplify the notation; the reader should note that in many formulas the pair 1, h may be replaced by a more general pair.

Sometimes we shall use ξ for a point of $*R$ not necessarily confined to B, but we shall continue using r and s for points of R.

The measure $K^h(r, \xi)\mu^h(d\xi)$, denoted μ_r^h temporarily, has a meaning similar to that of μ^h. Indeed, taking τ to be the time the h-chain $(x, 0, \beta)$ with initial measure γ^h leaves the finite set D, we have

$$\begin{aligned}
\mathcal{P}\{\tau \geqq 0, x(\tau) = s \mid x(0)\} &= G^h(x(0), s)L(s) \\
&= K^h(x(0), s)\zeta^h(s)L(s) \\
&= K^h(x(0), s)\mathcal{P}\{\tau \geqq 0, x(\tau) = s\},
\end{aligned} \tag{2.18}$$

by using (1.16) twice, first for $(x, 0, \beta)$ with $x(0)$ held fast, and then for the unconditioned chain. As D swells to R^h the random point $x(\tau)$ almost certainly approaches $x(\beta)$, so that its distribution measure approaches μ^h in the sense of weak convergence of measures on the compactum $^*R^h$. Thus, passing to the limit gives

$$\mathcal{P}\{x(\beta) \in C \mid x(0)\} = \int_C K^h(x(0), \xi)\mu^h(d\xi), \qquad C \subset {}^*R^h, \tag{2.19}$$

for every Borel set C in $^*R^h$. This amounts to saying μ_r^h is the distribution measure of the final state of a true h-chain that starts at the point r and is defined over a measure space of unit mass.

In order to remain in the framework of probability theory one may replace h by a multiple of itself to make γ^h of unit mass. The measure μ^h then has unit mass also. Without this replacement, μ^h has the same mass as γ^h, but the measures $K^h(r, \xi)\mu^h(d\xi)$ continue to have unit mass.

There are more explicit expressions of μ_r^h. Taking D to be R^h in (1.16) gives

$$\mathcal{P}\{\beta < \infty, x(\beta) = s \mid x(0)\} = G^h(x(0), s)[1 - P^h(s, R^h)], \tag{2.20}$$

since $L(s)$ then reduces to the factor in square brackets. After replacing $x(0)$ by r and using (2.3), we obtain

$$h(r)\mu_r^h(s) = K(r, s)\mu^h(s) = G(r, s)[h(s) - Ph.(s)], \tag{2.21}$$

a formula for the restriction of μ_r^h to R. The second equation is valid for all points of R, because the two members vanish unless r and s lie in R^h.

The restriction to B^h of course cannot be expressed similarly. However,

$$P_n^h(r, ds) \to K^h(r, \xi)\mu_B^h(d\xi), \qquad n \to \infty, \tag{2.22}$$

in the sense of weak convergence of measures on $^*R^h$. The verification is trivial, since $P_n^h(r, C)$ is just the probability that an h-chain starting at r lasts n steps at least and finds itself in C at the moment n. We shall ordinarily use (2.22) in the form

$$P_n(r, ds)h(s) \to K(r, \xi)\mu_B^h(d\xi), \qquad n \to \infty, \tag{2.23}$$

which is valid for all r in R.

In deriving (2.19) one can avoid using the topology of $^*R^h$ by turning to the underlying measure space, as we do in §4 in a similar situation.

3. The Martin representation

A few more properties of the measures μ^h are needed in completing the proof of Theorem 2.1. In (2.19) replace C by the whole space $^*R^h$, write r for $x(0)$, and multiply through by $h(r)$. The result is

$$h(r) = \int_{^*R^h} K(r, \xi)\mu^h(d\xi), \qquad r \in R, \tag{3.1}$$

according to (2.10). The equation derived in this way for r in R^h turns out to be valid on all R, because both members vanish outside R^h. Comparing (2.9) with the equation obtained from (3.1) by integrating with respect to $\gamma(dr)$, we find that (2.12) holds for all ξ in $^*R^h$, excepting a set null for μ^h.

We shall sometimes write $\mu(h)$ for μ^h, as in the next proposition.

PROPOSITION 3.1. *Let h, h_1, h_2 be excessive and integrable for γ, and let λ be a positive number. Then*

$$\mu(h_1 + h_2) = \mu(h_1) + \mu(h_2), \tag{3.2}$$

$$\mu(\lambda h) = \lambda\mu(h), \tag{3.3}$$

$$\mu_B(h_1) \geqq \mu_B(h_2) \quad \textit{if} \quad h_1 \geqq h_2 . \tag{3.4}$$

The statements follow at once from relations (2.21) and (2.23), integrated with respect to $\gamma(dr)$, and from the first paragraph of this section. A more instructive proof goes the following way. Choose an h_i-chain $(x_i, 0, \beta_i)$ defined over a space Ω_i and having $h_i\, d\gamma$ for initial distribution measure. Take Ω to be the set-theoretic sum of Ω_1 and Ω_2, take $\mathcal{B}$ to be the field generated by the sets in $\mathcal{B}_1$ or in $\mathcal{B}_2$, and take $\mathcal{P}$ to be the obvious measure determined by $\mathcal{P}_1$ and $\mathcal{P}_2$. Next, define the triple $(x, 0, \beta)$ over Ω by setting

$$\beta(\omega) = \beta_i(\omega), \qquad x(n, \omega) = x_i(n, \omega), \qquad \omega \in \Omega_i .$$

This triple is an $(h_1 + h_2)$-chain with $(h_1 + h_2)\, d\gamma$ for initial measure, as an easy computation shows. Equation (3.2) follows immediately from this construction and the meaning of μ^h. The other two relations are proved similarly, the second trivially and the third by reduction to (3.2) for concordant functions.

We shall now complete the proof of Theorem 2.1. Consider again the h-chain $(x, 0, \beta)$ defined over Ω and having γ^h for initial measure. By the very definition of h-chain, the function

$$\mathcal{P}\{\beta = \infty \mid x(0) = r\} \equiv \int_B K^h(r, \xi)\mu_B^h(d\xi) \tag{3.5}$$

is h-concordant as a function of r on R^h. Consequently $K^h(\cdot, \xi)$ is h-concordant for every ξ in B, excepting a set null for μ_B^h.

In proving that $K^h(\cdot, \xi)$ is extreme, we shall suppose the chain $(x, 0, \beta)$ to have a conditional distribution relative to $x(\beta)$, so that one may speak of the random chain $(x, 0, \beta)$ with $x(\beta)$ held fast. (The assumption can always be satisfied by properly choosing the underlying measure space Ω, and we need to consider only one sufficiently general chain.) Let C be any Borel set in $^*R^h$, and let Λ be the set in Ω where

$$(3.6) \qquad \beta \geqq n, \qquad x(i) = r_i \quad \text{for} \quad 0 \leqq i \leqq n, \qquad x(\beta) \in C.$$

Using the Markoff property and (2.19) we find

$$(3.7) \qquad \begin{aligned} \mathcal{P}\{\Lambda\} &= \gamma^h(r_0)P^h(r_0, r_1) \cdots P^h(r_{n-1}, r_n) \int_C K^h(r_n, \xi)\mu^h(d\xi) \\ &= \gamma(r_0)P(r_0, r_1) \cdots P(r_{n-1}, r_n) \int_C K(r_n, \xi)\mu^h(d\xi), \end{aligned}$$

the second equality being justified by (2.1) and (2.10). Consequently, the conditional probability of Λ given $x(\beta)$ has the expression

$$(3.8) \qquad \gamma(r_0)P(r_0, r_1) \cdots P(r_{n-1}, r_n)K(r_n, x(\beta)),$$

or, with k written momentarily for $K(\cdot, x(\beta))$ and (2.1) used several times, the equivalent expression

$$(3.9) \qquad \gamma^k(r_0)P^k(r_0, r_1) \cdots P^k(r_{n-1}, r_n).$$

Thus the chain $(x, 0, \beta)$ with $x(\beta)$ held fast is almost certainly a $K(\cdot, x(\beta))$-chain. This fact, together with the almost certain convergence of $x(n)$ to $x(\beta)$, implies that the measure associated with the excessive function $K(\cdot, x(\beta))$ almost certainly is the unit mass placed at $x(\beta)$.

Consider now any excessive function k integrable for γ, and suppose μ^k to be located at a single point ξ of $^*R$. If k is the sum of two excessive functions u and v, then μ^u and μ^v must be concentrated at ξ, according to (3.2), so that u and v are multiples of k, which must therefore be extreme. In particular, reverting to the chain $(x, 0, \beta)$, we see that $K(\cdot, x(\beta))$ almost certainly is extreme as an excessive function; consequently $K^h(\cdot, x(\beta))$ almost certainly is extreme as an h-excessive function, by the remark made immediately after the definition of this notion.

The preceding three paragraphs and the first paragraph of the section show that $x(\beta)$ almost certainly belongs to $R^h \cup B_e^h$. The proof of Theorem 2.1 is now complete.

We have in fact proved a little more than the theorem, for $K(\cdot, \xi)$ was shown to be extreme for every ξ in $^*R$, excepting a set which is null for every measure μ^h. Let us take h to be $K(\cdot, s)$ for some s in R. On comparing (2.11) with (2.21), taking r to be s, we find that $\mu^h(s)$ is strictly positive. So $K(\cdot, s)$ is an extreme excessive function for each s in R.

The functions $K(\cdot, \xi)$ corresponding to points of $R \cup B_e$ are therefore extreme excessive functions satisfying (2.12), and to distinct points correspond distinct functions because of (2.11), the subsequent remarks, and the exclusion from B_e of points yielding functions that are not concordant.

THEOREM 3.2. *The excessive functions h integrable for γ stand in one-one correspondence with the positive bounded Borel measures ν on $R \cup B_e$, the correspondence being*

$$h = \int_{R \cup B_e} K(\cdot, \xi)\nu(d\xi). \tag{3.10}$$

Indeed, ν is the measure μ^h, concentrated on $R^h \cup B_e^h$. Moreover, h is concordant if and only if μ^h is concentrated on the boundary B_e.

Clearly formula (3.10) defines an excessive function h whenever ν is one of the measures described, and by (2.11) the function h is concordant if and only if ν is concentrated on B_e. If h is excessive and integrable, on the other hand, equation (3.1) and Theorem 2.1 ensure the existence of at least one representation (3.10), with ν the measure μ^h. Only the unique determination of ν by h remains to be proved. All proofs rest on the observation preceding the theorem, that $K(\cdot, \xi)$ ranges over a set of distinct excessive functions as ξ ranges over $R \cup B_e$. We shall sketch one proof in the spirit of the paper.

Suppose first that h is $K(\cdot, \xi)$, with ξ in $R \cup B_e$. Then ν must be the unit mass at ξ, because h is extreme. Hence μ^h is the unit mass at ξ, by (3.1), and consequently every h-chain almost certainly has ξ for final state.

Now consider a general h, with representation (3.10). We shall construct an h-chain which exhibits ν as μ^h. The underlying space Ω is the set of all sequences $\omega \equiv (r_0, r_1, \cdots, r_d)$, with d a positive integer or $+\infty$, and r_n in R for n finite or in $R \cup B_e$ for n infinite; the Borel field $\mathcal{B}$ is the obvious one, generated by sets defined by finitely many conditions on the length d and the individual terms r_n. Let Λ be the set of sequences for which d is not less than the integer n, the terms $r_0, \cdots, r_n$ have prescribed values, and the final term r_d lies in C. We define $\mathcal{P}\{\Lambda\}$ by equation (3.7), with μ^h replaced by ν. One easily verifies that this prescription determines a measure $\mathcal{P}$ on $\mathcal{B}$. Now take $(x, 0, \beta)$ to be the identity function on Ω; that is to say, $x(n, \omega)$ is r_n and $\beta(\omega)$ is d. Clearly, $(x, 0, \beta)$ is an h-chain having γ^h for initial measure. Moreover, the chain has a conditional distribution relative to $x(\beta)$, and the conditional distribution can be taken without exception to be that of a $K(\cdot, x(\beta))$-chain; these assertions follow immediately from the construction. In view of the paragraph above, the definition of $x(\beta)$ as r_d is consistent with the definition (2.16) as the limit of $x(n)$. Matters being so, $x(\beta)$ has ν for distribution measure by construction and μ^h for distribution measure by definition; so the two measures coincide.

Let us consider more particularly the representation of a bounded concordant function h. The measure μ^h is concentrated on B_e^h, and by (3.4) it is dominated by some multiple of the measure μ associated with the excessive function 1. So we may write

$$h = \int_{B_e} K(\cdot, \xi)f(\xi)\mu_B(d\xi), \tag{3.11}$$

with f a version of the Radon-Nikodym derivative of μ_B^h relative to μ_B.

4. Boundary values

The ultimate behavior of an approximate h-chain is completely summarized by its final state, in a sense which we proceed to explain.

The sample space for random chains on R^h is the set of all triples (y, c, d), with c either $-\infty$ or an integer, d either $+\infty$ or an integer not less than c, and y a function $k \to y(k)$ having values in R^h and defined for integers k satisfying $c \leqq k \leqq d$. The sample space is provided with a field of measurable sets, generated by the sets defined by a pair of conditions $c \leqq n \leqq d$ and $y(n) = r$, with n a given integer and r a point of R^h. Of course, it is often advantageous to complete the field under some prescribed measure.

Final and initial sets in the sample space are of especial interest. A final set C is one which is measurable and which satisfies the following two conditions:

(4.1) *The triple (y, c, d) belongs to C if and only if $(y^*, c+1, d+1)$ does so; here $y^*(k) = y(k-1)$.*

(4.2) *The triple (y, c, d) belongs to C if and only if (y', c', d) does so; here c' is any integer satisfying $c \leqq c' \leqq d$, and y' is the restriction of y.*

The first condition expresses the invariance of C under the shift transformation of the sample space; the second, the irrelevance of all but the ultimate behavior of an element of the sample space. Initial sets are defined similarly.

The final sets form a Borel field closely related to h-excessive functions. Given a positive function f on the sample space, measurable over the final sets, define a function g on R^h by setting

$$g(r) \equiv \mathcal{E}\{f(x, 0, \beta)\}, \tag{4.3}$$

with $(x, 0, \beta)$ a true h-chain starting at r and defined over a space Ω of unit mass. (Here we interpret $(x, 0, \beta)$ as a function from Ω to the sample space, and $f(x, 0, \beta)$ as the composition of functions; later we shall sometimes write $(x, 0, \beta)$ for the element $(x(\omega), 0, \beta(\omega))$ of the sample space in order to simplify notation.) The function g is easily seen to be h-excessive, even h-concordant if f is bounded and vanishes at all elements (y, c, d) for which d is finite.

Suppose h to be excessive and integrable for the reference measure γ. Given an approximate h-chain (x, α, β), defined over $(\Omega, \mathcal{B}, \mathcal{P})$, we denote by $\mathcal{B}_\beta$ the least subfield of $\mathcal{B}$ which is complete under $\mathcal{P}$ and which contains every set defined by a condition $(x, \alpha, \beta) \in C$, with C a final set, and we denote by $\mathcal{B}_{x(\beta)}$ the completed subfield of $\mathcal{B}$ generated by the final state $x(\beta)$ of the chain. The opening sentence of the section may now be expressed more precisely:

PROPOSITION 4.1. *If (x, α, β) is an approximate h-chain, with h excessive and integrable for γ, then the fields $\mathcal{B}_\beta$ and $\mathcal{B}_{x(\beta)}$ coincide.*

Clearly $\mathcal{B}_\beta$ includes $\mathcal{B}_{x(\beta)}$. On the other hand, a set in $\mathcal{B}_\beta$ on which β is finite must belong to $\mathcal{B}_{x(\beta)}$, since it evidently differs only by a null set from one defined by conditions on the final state. We are left with proving that a function is measurable over $\mathcal{B}_{x(\beta)}$ if it is measurable over $\mathcal{B}_\beta$ and vanishes where β is finite.

We shall first complete the proof supposing h to be the constant 1 and $(x, 0, \beta)$ to be a true 1-chain having γ for initial distribution measure. We also make the following conventions, which involve no further loss of generality and permit the free use of the language of probability: The measure γ has unit mass, so also the space Ω; the expression $x(n, \omega)$ stands for b if n exceeds $\beta(\omega)$, with b an object not in R; and an excessive function has the value zero at b.

Consider a bounded positive function ϕ which is measurable over $\mathcal{B}_\beta$ and vanishes where β is finite. There is a concordant function g satisfying

$$g(x(n)) = \mathcal{E}\{\phi \mid x(n)\}, \tag{4.4}$$

as one sees from the fourth paragraph of the section. This relation determines g, since $x(0)$ takes on every possible value with strictly positive probability. Conversely, g determines ϕ almost everywhere, since

$$\begin{aligned} \lim_{n\to\infty} g(x(n)) &\equiv \lim_{n\to\infty} \mathcal{E}\{\phi \mid x(n)\} \\ &\equiv \lim \mathcal{E}\{\phi \mid x(0), \cdots, x(n)\} \\ &= \mathcal{E}\{\phi \mid x(0), x(1), \cdots\} = \phi, \end{aligned} \tag{4.5}$$

the first step being justified by the Markoff property, the second by martingale theory, the third by the measurability of ϕ over the field generated by all the $x(n)$.

If g is bounded concordant, on the other hand, then the $g(x(n))$ form a bounded martingale and $g(x(n))$ tends almost certainly to a bounded limit function ϕ satisfying (4.4). Moreover, ϕ is measurable over $\mathcal{B}_\beta$ and null where β is finite, because g is concordant.

Let g and ϕ be such a pair. According to (2.19) with 1 for h and to (3.11) with g for h, there is a bounded Borel measurable function f on $^*R$, vanishing outside B, such that

$$g(x(n)) = \mathcal{E}\{f(x(\beta)) \mid x(n)\}. \tag{4.6}$$

By martingale theory and the Markoff property again,

$$\begin{aligned} f(x(\beta)) &= \mathcal{E}\{f(x(\beta)) \mid x(0), x(1), \cdots\} \\ &= \lim_{n\to\infty} \mathcal{E}\{f(x(\beta)) \mid x(0), \cdots, x(n)\} \\ &= \lim_{n\to\infty} \mathcal{E}\{f(x(\beta)) \mid x(n)\}. \end{aligned} \tag{4.7}$$

On comparing the last four equations we find that ϕ coincides with $f(x(\beta))$. So ϕ is certainly measurable over $\mathcal{B}_{x(\beta)}$, and the proposition holds for the chain $(x, 0, \beta)$.

One next establishes the proposition for a true 1-chain starting at a given point of R by considering the chain $(x, 0, \beta)$ with the initial point held fast; then for every true 1-chain; then for every approximate 1-chain by approximating with true 1-chains. Finally, one obtains the full assertion of the proposition on replacing P by P^h and R by R^h.

During the proof it was shown that $g(x(n))$ almost certainly approaches $f(x(\beta))$. In this statement, too, $(x, 0, \beta)$ may obviously be replaced by any approximate 1-chain. The next theorem extends this result.

THEOREM 4.2. *Let h be concordant and integrable for γ, let f be the Radon-Nikodym derivative of μ^h relative to μ_B, and let (x, α, β) be an approximate 1-chain. Then, as n increases, $h(x(n))$ almost certainly approaches $f(x(\beta))$ wherever β is infinite.*

It suffices to prove the theorem for the 1-chain $(x, 0, \beta)$ having γ for initial distribution measure; the measure μ_B, one will recall, is the restriction to B of the distribution measure μ of the final state of this chain.

Let k denote the excessive function $1 + h$. The function h/k is bounded k-concordant and integrable for γ^k; so it has a representation

$$\frac{h}{k} = \int_{B_e} K^k(\cdot, \xi)g(\xi)\mu^k(d\xi), \tag{4.8}$$

which is the relative version of (3.11) with k replacing 1. Moreover, by the relative version of the remark preceding the theorem, $h(x'(n))/k(x'(n))$ almost certainly approaches $g(x'(\beta'))$ wherever β' is infinite, provided $(x', 0, \beta')$ is a k-chain. We now make several observations.

One may take $(x', 0, \beta')$ to be the set-theoretic sum of $(x, 0, \beta)$ and some other chain, as in the proof of Proposition 3.1.

The measure μ^k is the sum of μ and μ^h. Multiplying both members of (4.8) by k yields a representation of h that must be the same as (3.11), according to Theorem 3.2. Since $g\,d\mu_B + g\,d\mu_B^h$ therefore coincides with $f\,d\mu_B$, the function g must coincide with $f/(1 + f)$ except on a set null for μ_B, and with 1 except on a set null for the singular part of μ_B^h relative to μ_B.

The $h(x(n))$ form a positive martingale with finite expectations $\int h\,d\gamma$, provided one interprets $h(x(n))$ as zero for n greater than β. So $h(x(n))$ almost certainly remains bounded as n increases.

These remarks taken together evidently prove the theorem. Incidentally, the theorem remains valid for h only excessive and integrable for γ. In verification, one notes that if k is a finite function of the form

$$k(r) \equiv \int_R K(r, s)\nu(ds), \tag{4.9}$$

with ν a positive measure, then $k(x(n))$ approaches zero as n becomes infinite, because k has an interpretation similar to the one given for G in (1.19).

We study next the resolutivity of boundary functions, using the arguments of Brelot with the simplifications made possible by the results we have already obtained. In the remainder of the section functions on $^*R$ may be of variable sign, but of course excessive or concordant functions are understood to be positive. The reference measure γ is fixed, and μ is the distribution measure of the final state of a 1-chain $(x, 0, \beta)$ having γ for initial distribution.

Given a function u on R, denote by u^* the function on B defined as

$$u^*(\xi) \equiv \inf_U \sup_r u(r), \tag{4.10}$$

where U ranges over the neighborhoods of ξ in the topology of $^*R$, and r ranges over the points of $U \cap R$. The function u_* is defined similarly as the inferior limit on approach to the boundary.

PROPOSITION 4.3. *If a function u on R satisfies the conditions*

$$u_* \geqq 0, \qquad u \geqq Pu > -\infty, \tag{4.11}$$

then it is positive, and therefore excessive.

Implicit in (4.11) is the hypothesis that Pu is well-defined, being either $+\infty$ or finite. In order to argue by contradiction, suppose u were somewhere strictly negative. Then u would attain its minimum at some point r_0, by the first part of (4.11) and the compactness of $^*R$ and B. Since $u(r_0)$ dominates $P_n\, u.(r_0)$, as one sees from the second part of (4.11), the measure $P_n(r_0, ds)$ necessarily would have unit mass and would be concentrated on the set where u takes on its minimum. This set would therefore extend to the boundary of R, because as n increases $P_n(r_0, s)$ approaches zero for each s. So, finally, u_* would take on the strictly negative value $u(r_0)$, contradicting the first part of (4.11).

PROPOSITION 4.4. *If k is excessive, then $\int_R k\, d\gamma$ dominates $\int_B k_*\, d\mu_B$.*

We assume $\int k\, d\gamma$ to be finite, the alternative being trivial. Consider the 1-chain $(x, 0, \beta)$ having γ for initial distribution measure. The random variables $k(x(n))$ form a positive supermartingale with finite expectations, provided one understands $k(x(n))$ to be zero for n exceeding β. Thus, after extending k_* to $^*R$ so as to vanish on R, we have

$$\begin{aligned} \int_R k\, d\gamma &= \mathcal{E}\{k(x(0))\} \geqq \mathcal{E}\,\{\lim k(x(n))\}. \\ &\geqq \mathcal{E}\{k_*(x(\beta))\} = \int_B k_*\, d\mu_B. \end{aligned} \tag{4.12}$$

The last two steps are valid because $x(n)$ almost certainly approaches $x(\beta)$ wherever β is infinite and because k_* is Borel measurable, being lower semicontinuous on B.

Given a subset E of $^*R$ measurable for μ, denote by $p_E(r)$ the probability that a 1-chain starting at r has at least one of its states in E, perhaps the final one. The function p_E so defined on R is clearly excessive, bounded by 1 and constantly 1 on $E \cap R$ Just as clearly, $p_E + p_F$ dominates $p_{E \cup F}$, and

p_F dominates p_E if F includes E. If ε is strictly positive, there is a neighborhood V of E such that

$$\int_R (p_V - p_E)\, d\gamma < \varepsilon. \tag{4.13}$$

In verifying this assertion, we consider again the 1-chain $(x, 0, \beta)$ having γ for initial distribution measure. Suppose V runs through such a sequence that $V \cap R$ decreases to $E \cap R$ and $\mu_B(V \cap B)$ decreases to $\mu_B(E \cap B)$. The integral in (4.13) is the probability that some term in $x(0), \cdots, x(\beta)$ belongs to V minus the probability that some term belongs to E; it therefore decreases to zero, by the choice of the sequence of neighborhoods and by the definition of μ_B.

The function p_E is concordant if E is a subset of B. It is countably additive in E if E is restricted to the subsets of B that are measurable for μ_B. The function p_B plays a special role in subsequent arguments. It is the concordant approximation of the constant 1, which itself is not usually concordant; in the present treatment the multiples of p_B take the place of the constant functions in the standard treatments of resolutivity.

Proposition 4.5. *Let E be a subset of B measurable for μ_B, let χ be the characteristic function of E, and let ε be a strictly positive number. Then there exist open neighborhoods V of E and W of $B - E$ in $*R$ satisfying*

$$(p_V)_* \geqq \chi, \qquad (p_B - p_W)^* \leqq \chi, \qquad \int (p_V + p_W - p_B)\, d\gamma < \varepsilon. \tag{4.14}$$

The first two conditions are fulfilled for every choice of V and W since p_V, for example, has the constant value 1 on V. The relations $p_B = p_E + p_{B-E}$ and (4.13) together show that the third condition can also be fulfilled.

We proceed to define resolutivity. Given a function f on B, perhaps not measurable for μ_B, we say that a function u on R is an upper function for f if u_* dominates f and if u has a form $cp_B + g$, with c a finite constant and g excessive; similarly, v is a lower function if f dominates v^* and if v has the form $dp_B - h$, with d a finite constant and h excessive. The difference $u - v$ is excessive, by Proposition 4.3; so every upper function dominates every lower function. The function f is said to be resolutive if the integral $\int (u - v)\, d\gamma$ can be made arbitrarily small by choosing u and v suitably. Note that finiteness of the integral implies finiteness of $\int | u |\, d\gamma$ and $\int | v |\, d\gamma$, because u is bounded below and v is bounded above.

If f is resolutive, the infimum (or lower envelope) of the upper functions coincides with the supremum of the lower functions; this finite intermediate function will be denoted by $L(f)$.

If f is resolutive and if λ is a constant, then λf is resolutive, and $L(\lambda f)$ is just $\lambda L(f)$. If f is the sum of f_1 and f_2, both resolutive, then f also is resolutive, and $L(f)$ coincides with $L(f_1) + L(f_2)$.

THEOREM 4.6. *An arbitrary function on B is resolutive if and only if it is measurable integrable for μ_B. If f is resolutive, then*

$$L(f) = \int_B K(\cdot, \xi) f(\xi) \mu_B(d\xi), \tag{4.15}$$

and $L(f)$ coincides with $PL(f)$.

First suppose f resolutive, with upper and lower functions u and v satisfying

$$u = cp_B + g, \qquad v = dp_B - h, \qquad \int (u - v)\, d\gamma < \varepsilon.$$

The integral being finite, g is integrable for γ; hence g_* is integrable for μ_B, by Proposition 4.4; so u_* is integrable for μ_B. Similarly, v^* also is integrable for μ_B. Finally

$$\int_B (u_* - v^*)\, d\mu_B \leqq \int_B (u - v)_*\, d\mu_B \leqq \int_R (u - v)\, d\gamma < \varepsilon,$$

by Proposition 4.4 applied to the excessive function $u - v$. The measurability and integrability of f follow on letting ε decrease, since f lies between u_* and v^*.

Next suppose f measurable integrable for μ_B. If f is the characteristic function of a set in B, then it is resolutive, by Proposition 4.5, and the assertions concerning $L(f)$ follow from (4.13) and (2.19). By linearity, the theorem holds if f is a finite linear combination of characteristic functions, and moreover $\int f\, d\mu_B$ then lies between $\int u\, d\gamma$ and $\int v\, d\gamma$ for every choice of upper function u and lower function v. If f is a countable sum $\sum f_k$, each f_k being positive and assuming only two values (both finite), we choose positive upper and lower functions u_k and v_k for f_k so that $\int (u_k - v_k)\, d\gamma$ is less than ε_k, the ε_k having a finite sum ε; on taking u to be $\sum u_k$ we obtain an upper function for f, by the remark in the preceding sentence, and on taking v to be $v_1 + \cdots + v_n$ we obtain a lower function; clearly, $\int (u - v)\, d\gamma$ can be made less than 2ε by choosing n suitably, so that f is resolutive; and (4.15) holds because the right member obviously lies between u and v. Now, every positive integrable function is such a countable sum. Finally, all assertions concerning an integrable function are proved by writing it as the difference of two positive integrable functions.

5. The entrance boundary

Our treatment of the entrance boundary rests on the correspondence between approximate h-chains and h-excessive measures used in proving Theorem 1.1. It is necessary first to recall some results of potential theory.

Fix a sequence of finite sets D_n that increase to R, and denote $D_n \cap R^h$ by D_n^h. An h-excessive measure η, finite on finite sets, determines a sequence of finite measures ν_n with the following properties: The measure ν_n is concentrated on D_n^h. The measure η dominates $\nu_n G^h$, and it coincides with the latter on D_n^h. The measures $\nu_n G^h$ increase to η as n increases. Finally, the sequence (ν_n) is coherent in the following sense: Construct an h-chain $(x, 0, \beta)$, defined over a measure space Ω and having ν_n for initial distribution measure, and let τ be the time the chain first meets D_m, for a given m less than n; that is to say, τ is $+\infty$ if $x(k)$ never lies in D_m, and otherwise it is the least k for which $x(k)$ belongs to D_m. Then ν_m is the distribution measure of the random point $x(\tau)$, which is defined over the part of Ω where τ is finite.

These statements, which lie a little outside the scope of the paper, are proved in [7] for a more complicated situation.

An approximate h-chain (x, α, β) determines an h-excessive measure by the formula

$$\eta(C) \equiv \int_\Omega \sum_{\alpha \leqq k \leqq \beta} \chi(x(k))\mathcal{P}(d\omega), \qquad C \subset R^h, \tag{5.1}$$

which is only (1.12) repeated for clarity; here χ is the characteristic function of C, and Ω is the underlying measure space of the chain. Suppose η finite on finite sets. Then, as noted in the proof of Theorem 1.1, the random times giving the chain its structure of approximate h-chain may be taken to be the times the chain meets the sets D_n^h. When α_n is chosen so, the corresponding true h-chain $(x_n, 0, \beta_n)$ has precisely ν_n for intial distribution measure, because the measure assigned to this chain by formula (5.1) agrees with η on D_n^h and has the form νG^h, the measure ν being the distribution measure of $x_n(0)$ and hence concentrated on D_n^h also. Consequently, η determines the chain (x, α, β) in the sense that it determines the structure of an approximating sequence of true h-chains.

It is more important, in the present discussion, that conversely formula (5.1) yields every h-excessive measure which is finite on finite sets. We shall therefore sketch a means of assigning canonically to each such measure η an approximate h-chain (x, α, β) related to η by (5.1). The underlying measure space is the sample space of §4, with the measure determined by the following construction. Consider the set C of elements (y, c, d) of the sample space satisfying the conditions

$$c \leqq i, \qquad d \geqq j, \qquad y(k) = r_k \quad \text{for} \quad i \leqq k \leqq j, \tag{5.2}$$

with i and j given integers and the r_k points of R^h. Choose n so great that all r_k belong to D_n^h, and construct an h-chain $(z, 0, \varepsilon)$ defined over a measure space Ω and having for initial distribution the measure ν_n on D_n^h mentioned earlier. Let τ_m be the time this chain meets D_m^h, for m not exceeding n; thus, τ_n certainly vanishes, though the other τ_m may be infinite. Next, define another random chain $(z', \delta', \varepsilon')$ over Ω by setting

$$\delta' = -\tau_m, \qquad \varepsilon' = \varepsilon - \tau_m, \qquad z'(k) = z(\tau_m + k), \tag{5.3}$$

on the set in Ω where $\tau_1, \cdots, \tau_{m-1}$ are infinite but τ_m is finite. We now assign to the set C in the sample space the measure of the set in Ω where the relations

$$(5.4)\qquad \delta' \leqq i, \quad \varepsilon' \geqq j, \quad z'(k) = r_k \text{ for } i \leqq k \leqq j,$$

hold. This prescription does not depend on the choice of n, as one easily sees, and it determines a finitely additive measure on the sample space. One next extends the measure to a countably additive measure on the sample space by a straightforward adaptation of Kolmogoroff's argument; we shall not present the details, for the proof should be conducted in a more general setting. Finally, take (x, α, β) to be the identity function on the sample space provided with the measure described. By construction this chain is evidently an approximate h-chain satisfying (5.1). We shall speak of it as the chain canonically associated with η and P^h. Usually it is not a Markoff chain, but it has the property that conditional probability distributions exist.

Suppose two approximate h-chains determine h-excessive measures η_1 and η_2. Then, clearly, the set-theoretic union of the two chains, as defined in the proof of Proposition 4.1, is an approximate h-chain determining $\eta_1 + \eta_2$. Similarly, if η is determined by an approximate h-chain, then a positive multiple $\lambda\eta$ is determined by the same chain, the measure on the underlying space being the original one multiplied by λ. So relation (5.1) is linear in a definite sense.

We shall now complete R^h in order to obtain an entrance boundary. Fix a strictly positive reference function g, subject to the requirement that Gg be finite everywhere, and denote by g^h the function g/h on R^h. The function $G^h g^h$, denoted by E^h from now on, is finite and strictly positive on R^h. Introduce the measures

$$(5.5)\qquad K_*^h(r, ds) \equiv \frac{G^h(r, ds)}{E^h(r)} = K_*(r, ds)h(s),$$

which satisfy

$$(5.6)\qquad \int_{R^h} K_*^h(r, ds) g^h(s) = 1.$$

In (5.5) the measure $K_*(r, ds)$ is the one obtained on taking h to be a strictly positive constant. Choose a strictly positive function δ on R so that $\sum \delta(s)$ is finite, and set

$$(5.7)\qquad d_3^h(r, s) \equiv \sum_{t \in R^h} | K_*^h(r, t) - K_*^h(s, t) | g^h(t)\delta(t),$$

thus defining a metric d_3^h on R^h. A sequence of points r_n is a Cauchy sequence in this metric if and only if the numbers $K_*^h(r_n, s)$ form a Cauchy sequence for each s in R^h. Choose also a metric d_1 on R under which the completion of R is the Alexandroff compactification of R, and denote by d_1^h the restriction of d_1 to R^h. Finally, complete R^h under the metric $d_1^h + d_3^h$ to obtain a compact metric space R_*^h in which R^h is imbedded as a dense open subspace with the discrete topology. The Martin entrance boundary (relative to g) is the compact set $R_*^h - R^h$, denoted by A^h. Just as in §2, the spaces R_*^h and A^h

may be considered subspaces of R_* and A, which are obtained on taking h to be a strictly positive constant.

Let ρ be a point of A^h. The formula

$$K_*^h(\rho, C) \equiv \lim_{r\to\rho} K_*^h(r, C), \qquad r \in R^h, \tag{5.8}$$

defines an h-excessive measure $K_*^h(\rho, ds)$ which is finite on finite sets and satisfies relations similar to (2.9) and (2.10). One now defines extreme h-excessive measures and the subset A_e^h of A^h, following the treatment in §2 of the dual situation. The dual of Theorem 2.1 then takes the following form.

THEOREM 5.1. *Let (x, α, β) be an approximate h-chain, and η the h-excessive measure assigned to this chain by* (5.1). *Suppose g^h to be integrable for η. Then, almost certainly, either α is finite and $x(\alpha)$ a point of R^h, or else α is infinite and $x(n)$ converges in the topology of R_*^h to a point of A_e^h as $n \to -\infty$.*

We shall prove the theorem by reducing it to Theorem 2.1.

Clearly, the values of $x(n)$ almost certainly belong to the set S comprising the points s for which $\eta(s)$ is strictly positive. By Theorem 1.1, reversing the sense of time turns (x, α, β) into a Q-chain, with

$$Q(s, dr) \equiv \frac{P^h(r, s)}{\eta(s)}\, \eta(dr), \qquad s \in S. \tag{5.9}$$

Now make the definitions of §2, replacing R^h, P^h, γ^h by S, Q, $g^h\, d\eta$. The kernel

$$H(s, dr) \equiv \frac{G^h(r, s)}{\eta(s)}\, \eta(dr) \tag{5.10}$$

replaces G^h, and

$$L(s, r) \equiv \frac{H(s, r)}{\int_S H(s, r) g^h(s) \eta(ds)} = \frac{K_*^h(r, s)}{\eta(s)} \tag{5.11}$$

replaces $K^h(r, s)$. Consequently, the space $*S$ replacing $*R^h$ may be taken to be a compact subspace of R_*^h, the boundary set corresponding to B_e^h then becoming a subspace of A_e^h, as one verifies directly from the definitions. Matters being so, Theorem 2.1 applied to the reversed chain yields Theorem 5.1 at once.

The argument enables one to state and prove the dual of every result obtained in the preceding sections. In particular, every approximate h-chain satisfying the hypotheses of Theorem 5.1 has a reasonably defined initial state $x(\alpha)$, and every h-excessive measure for which g^h is integrable has a canonical representation as a linear composite of extreme h-excessive measures; the second assertion follows from our having proved that every such measure is related by (5.1) to some approximate h-chain.

The entrance and exit boundaries can be used to decompose an approximately stationary chain into simpler chains. Let h be an excessive function, and η an excessive measure; we suppose $h(r)$ and $\eta(r)$ to be finite for every point r, deleting some points of R, if necessary, and replacing P, h, η by their restrictions. Construct the approximate h-chain (x, α, β) canonically associated with P^h and the h-excessive measure $h\, d\eta$. The initial behavior of this chain is determined by $h\, d\eta$, the final behavior by h; any other approximate h-chain with the same initial and final behavior can be approximated by true h-chains structurally similar to the ones approximating (x, α, β). The mass of the chain is the limit of the integral $\int h\, d\nu_n$, where the ν_n are chosen as in the beginning of the section so that $\nu_n G$ increases to η. The mass is therefore the capacity of R relative to h and η, as defined in [7]; and it vanishes unless there is some point at which both h and η are strictly positive. In constructing the boundaries, choose the reference measure γ and the reference function g to make the integrals $\int h\, d\gamma$ and $\int g\, d\eta$ finite. The initial and final states of the chain are then defined almost certainly and have a joint distribution measure $\sigma\,(d\rho, d\xi)$ on the product of the spaces $R \cup A_e$ and $R \cup B_e$.

We say the chain is simple if h is $K(\cdot, \xi)$ for some ξ in $R \cup B_e$ and η is $K_*(\rho, \cdot)$ for some ρ in $R \cup A_e$, and we denote by $\mathcal{P}_{\rho,\xi}$ the corresponding measure on the sample space of chains on R. Such a chain has ρ for initial state, ξ for final state.

Evidently the measure on the sample space corresponding to the more general chain (x, α, β) can be expressed in terms of $\sigma\,(d\rho, d\xi)$ and the $\mathcal{P}_{\rho,\xi}$, provided the measure σ is concentrated on the set of pairs (ρ, ξ) for which $\mathcal{P}_{\rho,\xi}$ has finite mass. So (x, α, β) may be regarded as a linear composite of simple chains. The measure σ, moreover, may be written explicitly in terms of the measures occurring in the Martin representations of h and η.

6. Complements

Several remarks are collected here that would have interrupted the argument had they been inserted at the relevant points.

(*a*) *Language.* I have followed [7] in terminology, rather than [5], because the discussion has been limited to positive functions and measures except for an awkward passage in §4. Doob's more systematic language is preferable in treating resolutivity or in stating results most generally.

(*b*) *Approximately stationary chains.* The notion of approximately stationary chain and the technique of the relative theory illuminate the discussion of potential theory in [7].

The following examples show the need of some form of approximate h-chains in studying excessive measures and the entrance boundary. Take R to be the integers and $P(r, s)$ to be $p(r - s)$, with p a probability distribution on the integers. The measure η which attributes unit mass to each integer certainly is excessive, but it is related by (5.1) to a true 1-chain only if p is concentrated at a single point different from zero. For simplicity, we shall prove this assertion only under the additional hypotheses that $\sum rp(r)$ converges absolutely and has the value 1, so that any approximate 1-chain related to η by (5.1) has underlying space of unit mass and initial and terminal times almost certainly infinite. Suppose p not concentrated at a single point and x a true 1-chain giving rise to η. Clearly, $P_k(r, s)$ approaches zero uniformly in r and s as k increases. Now, denoting by q_n the distribution measure of $x(n)$, we have

$$q_n(s) = \sum_r q_{n-k}(r)P_k(r, s), \tag{6.1}$$

and the right member approaches zero as k increases, because q_{n-k} has unit mass. So $q_n(s)$ vanishes for every n and s, an obvious contradiction.

On the other hand, η is related by (5.1) to strongly approximate 1-chains that can be constructed from the chain canonically associated with η and P by suitably shifting the origin of time on the sample paths.

Dropping the additional hypotheses on p, take $p(r)$ to be $c/(1 + r^2)$ with c chosen to make p a probability distribution. It is likely that then no strongly approximate 1-chain stands in the relation (5.1) to η.

(c) *Completing the state space.* Our completion of R in defining the exit boundary differs from the one in [5], first in defining K and next in strengthening the metric. Our definition of K is easily reduced to Doob's. Simply adjoin a new point c to R, take $P(r, c)$ to vanish for all r, and take $P(c, s)$ to be $\gamma(s)$ for s in R; here we assume the reference measure to have unit mass. The K defined by Doob, with c replacing 1 and $R \cup \{c\}$ replacing S, then gives our K upon restriction to R.

The following example justifies strengthening the metric. Take R to be the set comprising the symbol 1 and all finite strings $1k_1 \cdots k_n$ with the k_i integers; from $1k_1 \cdots k_n$ one passes to the point $1k_1 \cdots k_n k_{n+1}$ with strictly positive probability $p(k_{n+1})$, where p is a probability distribution on the integers. In the metric of [5], as one verifies easily, the point $1k_1 \cdots k_n k_{n+1}$ approaches $1k_1 \cdots k_n$ as k_{n+1} tends to infinity in any manner; furthermore, in the completion of R under this metric, every point of R is a limit point of the complement of R. The arguments of Brelot and Martin clearly require some modification to fit such a situation, whereas they apply as they stand to the space $^*R$ defined in §2.

As a matter of record, Doob had in mind a metric like ours but neglected part of it in preparing his paper for publication. Of course, only the set $R \cup B_e$ counts in the end, and on this set the metric d_2 alone may be preferable to $d_1 + d_2$; at any rate, the two determine the same mode of approach to the boundary.

Except for slightly strengthening the metric, I have repeated Doob's method of completing R so as not to bewilder the reader. It is now time to remark that none of our proofs used the topology essentially and that another method of completing R would serve as well. Change the metric d_2 to a stronger metric d_2' by replacing the function δ by the constant 1 in (2.7). A sequence (s_n) is a Cauchy sequence for d_2' if and only if the functions $K(\cdot, s_n)$ form a Cauchy sequence in the space $\mathfrak{L}^1(\gamma)$ of functions integrable for γ. Denote by R' the completion of R under the metric $d_1 + d_2'$, by B' the complement of R in R', by B_e' the analogue of B_e. As sets, B_e' and B_e may be identified, for they are in natural one-one correspondence with the same set of extreme excessive functions. Although R' is not compact, all our results hold with $*R$ replaced by R'. To see this, consider an arbitrary 1-chain (x, α, β). As we have proved, almost certainly the function $K(\cdot, x(n))$ converges pointwise to $K(\cdot, x(\beta))$ as n increases to β, almost certainly $x(\beta)$ belongs to $R \cup B_e$, and almost certainly

$$\int_R \gamma(dr)K(r, x(\beta)) = 1. \tag{6.2}$$

These statements, together with the positivity of the functions, ensure that $K(\cdot, x(n))$ almost certainly converges to $K(\cdot, x(\beta))$ in the norm of $\mathfrak{L}^1(\gamma)$, that is to say, $x(n)$ converges to $x(\beta)$ in the topology of R'.

(*d*) *The reference measure.* The reference measure γ may be chosen so that countably many prescribed finite excessive functions are integrable for γ; and an arbitrary excessive function is usually treated by disregarding the set where it is infinite. In many circumstances, therefore, one obtains a satisfactory exit boundary by choosing γ appropriately. One would sometimes prefer to speak of a total exit boundary, however, independent of the reference measures and forming with R a topological space. It does not seem possible to define such a space by patching together the completions $*R$ for all choices of reference measures, although it is easy to patch together the partial completions $R \cup B_e$.

Often the reference measure can be chosen to make every finite excessive function integrable. If, for example, every point of R can be reached from one point c in finitely many steps, then $\sum 2^{-n}P_n(c, ds)$ is such a measure. When matters are so, one obtains a satisfactory universal completion by choosing a single reference measure.

The reference measure serves only to define the excessive measure γG. So, instead of singling out γ, we could single out some excessive measure ζ that is finite on finite sets. Definitions and proofs would remain about the same, in view of the beginning paragraphs of §5.

(*e*) *Resolutivity.* We have defined what may be called strong resolutivity. Weak resolutivity is defined, in contrast, by requiring that for each point r of R the difference $u(r) - v(r)$ can be made small by suitable choice of the upper function u and lower function v. The two notions agree if every finite excessive measure is integrable for the reference measure. Weak resolutivity can be treated in the same manner as strong resolutivity, the family of measures $K(r, \xi)\mu_B(d\xi)$ replacing the single measure μ_B.

(*f*) *Duality.* The argument reducing Theorem 5.1 to Theorem 2.1 differs slightly from the one found in [5] and [6]. Let (x, α, β) be an approximate h-chain defined over Ω and related to η by (5.1), and define (x_n, α_n, β_n) over Ω_n by momentarily identifying Ω_n with Ω and setting

$$(6.3) \qquad \alpha_n = \alpha + n, \qquad \beta_n = \beta + n, \qquad x_n(k) = x(k - n).$$

The set-theoretic sum of all these chains, as n ranges over the integers, is then a true h-chain having η for stationary distribution measure. One customarily uses such chains in deriving initial properties from final properties; they are not suitable, however, for establishing the duals of some of the results following Theorem 2.1.

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