POTENTIALS AND THE RANDOM WALK

BY

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1. Introduction

Given an integer $s \geqq 3$, write $e_1, e_2, \cdots, e_s$ for the s coordinate vectors $(1, 0, \cdots, 0), (0, 1, \cdots, 0), \cdots, (0, 0, \cdots, 1)$, spanning the s-dimensional lattice of points with integral coordinates, and let s_n denote the position at time n ($= 0, 1, 2, \cdots$) of a particle performing the standard s-dimensional random walk according to the following rule: fixing the first $n - 1$ steps $s_1, s_2, \cdots, s_{n-1}$, the particle starts afresh at s_{n-1}, jumping next to one of the $2s$ neighbors $s_n = s_{n-1} \pm e_1, s_{n-1} \pm e_2, \cdots, s_{n-1} \pm e_s$ of s_{n-1}, the chance of landing at a particular neighbor being $(2s)^{-1}$.

Given a set B of lattice points, the probability p_B that the random walk hits B at some time $n < +\infty$, as a function of the starting point of the walk, is *excessive* in the sense that $\mathbf{G}p_B \leqq 0$, where $\mathbf{G}$ is Laplace's difference operator:

1.1 $$(\mathbf{G}p)(a) = (2s)^{-1} \sum_{k \leqq s, n=1,2} p(a + (-)^n e_k) - p(a).$$

B. H. Murdoch [1, pp. 13–19] proved that if $p \geqq 0$, and if $\mathbf{G}p = 0$, then p is constant,[2] and, with the help of this result, it follows, as Murdoch himself noted, that p_B is the sum of the *potential* $\mathbf{K}e_B$ and the constant $p_B(\infty)$, where $e_B = -\mathbf{G}p_B$ ($\geqq 0$), $\mathbf{K}e_B$ is the expectation of $\sum_{n \geqq 0} e_B(s_n)$, as a function of the starting point of the walk, and $p_B(\infty)$ is the (constant) probability $P.(\mathbf{B})$ of the event $\mathbf{B}$ that $s_n \in B$ for an infinite number of integers n.

$P.(\mathbf{B})$ is either 0 or 1. When $P.(\mathbf{B}) = 0$, p_B is the greatest *potential* $p \leqq 1$ such that $\mathbf{G}p = 0$ outside B, and, on the strength of the example of the Newtonian potential in 3 dimensions, it is natural to think of e_B as the electrostatic distribution of charge on the conductor B and to introduce the total charge (of e_B) as the capacity $C(B)$ of B.

Given a set B, it is an interesting problem to decide whether $P.(\mathbf{B}) = 0$ or 1; the solution is

1.2 $$P.(\mathbf{B}) = 0 \text{ or } 1 \quad \text{according as} \quad \sum_{n \geqq 0} 2^{-n(s-2)} C(B_n) < \text{ or } = +\infty,$$

where B_n is the intersection of B and the spherical shell $2^n \leqq |a| < 2^{n+1}$. Wiener's test for the singular points of the Newtonian electrostatic potential (see Courant and Hilbert [1, p. 286]) served us as a model, and for this reason we call 1.2 Wiener's test also. B. H. Murdoch [1, pp. 45–47] came close to proving 1.2 and used his method to compute $P.(\mathbf{B})$ for sets B similar to those

Received September 25, 1958.

[1] Fulbright grantee 1957–1958.

[2] J. Capoulade [1] also stated this result and S. Verblunsky [1] and R. Duffin [1, pp. 242–245] proved it. Murdoch's results lie much deeper.

figuring in the last example of Section 6. Similar results hold for the Brownian motion and Newtonian potentials in $s\ (\geqq 2)$ dimensions and for stable processes with exponent $< 2\ (< 1)$ and Riesz potentials in $s \geqq 2\ (\geqq 1)$ dimensions; for the identification of hitting probabilities and electrostatic potentials, see J. L. Doob [2] and G. Hunt [1, 2]; the proof of Wiener's test runs along the lines of Section 5. G. Hunt [1, 2, 3] showed the full scope and power of the connection between Markov processes and potentials, and most of the (nonclassical) results of Sections 3 and 4 are from his papers [1, 2] or from B. H. Murdoch. The present paper is based on lectures given at Kyôto and Fukuoka in April, 1958, in which we tried to present Hunt's things in the simplest setting.

We thank J. L. Doob, N. Ikeda, and D. Ray for spotting several misprints.

2. The random walk

Write W for the space of paths $w: n = 1, 2, \cdots \to s_n(w)$ with values from the s-dimensional lattice, w_m^- for the stopped path $s_n(w_m^-) = s_{n \wedge m}(w)$,[3] w_m^+ for the shifted path $s_n(w_m^+) = s_{n+m}(w)$, $\mathbf{C}$ for the class of Borel subsets of W, and, for $\mathbf{C}$ measurable $m = m(w) \geqq 0$, write $\mathbf{C}_m$ for the Borel algebra of sets

$$A = (w: w_m^- \in C), \qquad C \in \mathbf{C};$$

let $P.(C)$ denote the probability of the event $C \in \mathbf{C}$ for the standard random walk as a function of the starting point of the walk; note that $P.(w_n^+ \in C/\mathbf{C}_n) = P_{s_n}(C)$ for each $n \geqq 0$ and each $C \in \mathbf{C}$; and let $E.(f) = \int f P.(dw)$.

$m = m(w) = 0, 1, 2, \cdots$ is a *Markov time* if $(w: m > n) \in \mathbf{C}_n$ for each $n \geqq 0$; for example, given a set B of lattice points, $m = n_B = \min (n: s_n \in B)$ is Markov.

Given a Markov time m,

2.1 $$P.(P.(w_m^+ \in C/\mathbf{C}_m) = P_{s_m}(C)) = 1, \qquad C \in \mathbf{C};$$

in short, $s_n: n \geqq 0$ starts from scratch at time $n = m$ at the place s_m.

The proof is simple: $(w: m = n) \in \mathbf{C}_n$ for each $n \geqq 0$, and therefore

2.2 $$\begin{aligned} P.(w_m^+ \in C^+, w_m^- \in C^-) &= \textstyle\sum_{n \geqq 0} P.(w_n^+ \in C^+, w_n^- \in C^-, m = n) \\ &= \textstyle\sum_{n \geqq 0} E.(P_{s_n}(C^+), w_n^- \in C^-, m = n) \\ &= E.(P_{s_m}(C^+), w_m^- \in C^-), \qquad C^-, C^+ \in \mathbf{C}. \end{aligned}$$

Given a point $\theta = (\theta_1, \theta_2, \cdots, \theta_s)$ of the s-dimensional torus $[-\pi, \pi)^s$,

2.3 $$\begin{aligned} E_a(e^{is_n \cdot \theta}) &= E_a(E_{s_{n-1}}(e^{is_1 \cdot \theta})) = E_a(e^{is_{n-1} \cdot \theta}) f(\theta) = e^{ia \cdot \theta} f(\theta)^n, \\ f(\theta) &= s^{-1} \textstyle\sum_{k \leqq s} \cos \theta_k, \quad n \geqq 0. \end{aligned}$$ [4]

[3] $m \wedge n$ means the smaller of m and n.

[4] $s_n \cdot \theta$ is the inner product of the vectors s_n and θ.

and inverting 2.3 proves the result of G. Pólya [1, pp. 151–153]:

2.4 $$P_a(s_n = b) = (2\pi)^{-s} \int e^{i(b-a)\cdot\theta} f(\theta)^n \, d\theta, \qquad n \geqq 0.$$

Summing 2.4 for $n = 0, 1, 2, \cdots$ proves

2.5 $$\begin{aligned} K(a, b) &= \sum_{n \geqq 0} P_a(s_n = b) = (2\pi)^{-s} \int e^{i(b-a)\cdot\theta} \frac{d\theta}{g(\theta)} \\ &\leqq (2\pi)^{-s} \int \frac{d\theta}{g(\theta)} < +\infty, \qquad g(\theta) = 1 - f(\theta), \end{aligned}$$

and using the Borel-Cantelli lemma, we infer that

2.6 $$P.(\lim_{n \uparrow +\infty} s_n = \infty) = 1.$$

B. H. Murdoch [1, pp. 23–32] and (for $s = 3$) R. Duffin [1, pp. 238–240, 245–251] estimated $K(a, b)$ for $|a - b| \uparrow +\infty$ up to terms of magnitude $|a - b|^{-s-2}$.

We will want the following simpler result:

2.7 $$\begin{aligned} &\lim_{|b-a| \uparrow +\infty} |b - a|^{s-2} K(a, b) \\ &= \lim_{|b-a| \uparrow +\infty} |b - a|^{s-2} (2\pi)^{-s} \int e^{i(b-a)\cdot\theta} \frac{d\theta}{g(\theta)} = k_1, \\ &k_1 = s \int_0^{+\infty} (2\pi t)^{-s/2} e^{-1/2t} \, dt.^5 \end{aligned}$$

Consider, for the proof, the modified Bessel coefficients

2.8 $$\begin{aligned} I_n(t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\theta} e^{t\cos\theta} \, d\theta \\ &= \frac{e^t}{2\pi} \int_{-\pi}^{\pi} e^{in\theta} e^{-2t\sin^2(\theta/2)} \, d\theta, \qquad n = 0, \pm 1, \pm 2, \cdots, \end{aligned}$$

introduce the (positive) Fourier coefficients

2.9 $$\begin{aligned} &s(2\pi)^{-s} \int e^{ic\cdot\theta} e^{-stg(\theta)} \, d\theta \\ &= s(2\pi)^{-1} \int_{-\pi}^{\pi} e^{il_1\theta_1} e^{-2t\sin^2(\theta_1/2)} \, d\theta_1 \cdots (2\pi)^{-1} \int_{-\pi}^{\pi} e^{il_s\theta_s} e^{-2t\sin^2(\theta_s/2)} \, d\theta_s \\ &= se^{-t} I_{l_1}(t) e^{-t} I_{l_2}(t) \cdots e^{-t} I_{l_s}(t), \qquad t \geqq 0, \quad c = (l_1, l_2, \cdots, l_s), \end{aligned}$$

and note that

2.10 $$\begin{aligned} (2\pi)^{-s} \int e^{ic\cdot\theta} \frac{d\theta}{g(\theta)} &= \int_0^{+\infty} dt \; s(2\pi)^{-s} \int e^{ic\cdot\theta} e^{-stg(\theta)} \, d\theta \\ &= s \int_0^{+\infty} e^{-st} \, dt \; I_{l_1}(t) I_{l_2}(t) \cdots I_{l_s}(t). \end{aligned}$$

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[5] $k_1, k_2, \cdots$ denote positive constants.

Given $\theta \in [-\pi, \pi)^s$, $g(\theta) > k_2 |\theta|^2$, so that

2.11 $$e^{-st|c|^2 g(\theta/|c|)} < e^{-stk_2|\theta|^2}, \qquad t \geqq 0, \quad \theta \in [-\pi|c|, \pi|c|)^s,$$

and

2.12 $$\begin{aligned} &\Big| |c|^{s-2} \int_{k_3|c|^2}^{+\infty} s(2\pi)^{-s} \int_{[-\pi,\pi)^s} e^{ic\cdot\theta} e^{-stg(\theta)}\, d\theta\, dt \\ &\qquad - \int_{k_3}^{+\infty} s(2\pi)^{-s} \int_{|\theta| \leqq \pi|c|} e^{i(c/|c|)\cdot\theta} e^{-(t/2)|\theta|^2}\, d\theta\, dt \Big| \\ &\leqq s(2\pi)^{-s} \int_{k_3}^{+\infty} dt \Big| \int_{[-\pi|c|, \pi|c|)^s} e^{i(c/|c|)\cdot\theta} e^{-st|c|^2 g(\theta/|c|)}\, d\theta \\ &\qquad - \int_{|\theta| \leqq \pi|c|} e^{i(c/|c|)\cdot\theta} e^{-(t/2)|\theta|^2}\, d\theta \Big| \\ &\leqq s(2\pi)^{-s} \int_{k_3}^{+\infty} dt \Big[\int_{|\theta| \leqq \pi|c|} | e^{-st|c|^2 g(\theta/|c|)} - e^{-(t/2)|\theta|^2} |\, d\theta \\ &\qquad + \int_{\pi|c| < |\theta| < s^{1/2}\pi|c|} e^{-stk_2|\theta|^2}\, d\theta \Big] \\ &\to 0, \qquad |c| \uparrow +\infty; \end{aligned}$$

this implies

2.13 $$\begin{aligned} \lim_{|c|\uparrow+\infty} |c|^{s-2} \int_{k_3|c|^2}^{+\infty} s(2\pi)^{-s} \int_{[-\pi,\pi)^s} e^{ic\cdot\theta} e^{-stg(\theta)}\, d\theta\, dt &= \int_{k_3}^{+\infty} dt\ s(2\pi)^{-s} \int_{R^s} e^{i\theta_1} e^{-(t/2)|\theta|^2}\, d\theta \\ &= \int_{k_3}^{+\infty} s(2\pi t)^{-s/2} e^{-1/2t}\, dt \\ &\uparrow s \int_0^{+\infty} (2\pi t)^{-s/2} e^{-1/2t}\, dt = k_1, \qquad k_3 \downarrow 0; \end{aligned}$$

and to complete the proof, it is enough to check that

2.14 $$\lim_{k_3\downarrow 0} \limsup_{|c|\uparrow+\infty} |c|^{s-2} \int_0^{k_3|c|^2} dt\ s(2\pi)^{-s} \int_{[-\pi,\pi)^s} e^{ic\cdot\theta} e^{-stg(\theta)}\, d\theta = 0.$$

But, as is clear from 2.8 and 2.9,

2.15 $$\begin{aligned} &\limsup_{|c|\uparrow+\infty} |c|^{s-2} \int_0^{k_3|c|^2} dt\ s(2\pi)^{-s} \int_{[-\pi,\pi)^s} e^{ic\cdot\theta} e^{-stg(\theta)}\, d\theta \\ &\qquad \leqq s^{s-1} \limsup_{n\uparrow+\infty} n^{s-2} \int_0^{k_3 s^2 n^2} e^{-t} I_n(t) (e^{-t} I_0(t))^{s-1}\, dt, \end{aligned}$$

where n is the greatest of the integers $|l_1|, |l_2|, \cdots, |l_s|$, and since, in view of 2.13,

2.16
$$\lim_{n\uparrow+\infty} n^{s-2}s\int_{k_3s^2n^2}^{+\infty} e^{-t}I_n(t)(e^{-t}I_0(t))^{s-1}\,dt = \lim_{n\uparrow+\infty} n^{s-2}s\int_{k_3s^2n^2}^{+\infty}(2\pi)^{-s}\int_{[-\pi,\pi)^s} e^{in\theta_1}e^{-stg(\theta)}\,d\theta\,dt \uparrow k_1, \qquad k_3\downarrow 0,$$

it is enough to show that

2.17
$$k_1 \geqq \limsup_{n\uparrow+\infty} n^{s-2}s\int_0^{+\infty} e^{-t}I_n(t)(e^{-t}I_0(t))^{s-1}\,dt.$$

With the help of

2.18
$$1 \geqq e^{-t}I_0(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{-2t\sin^2(\theta/2)}\,d\theta \sim (2\pi t)^{-1/2}, \qquad t\uparrow+\infty,$$

and of

2.19
$$\int_0^r e^{-t}I_n(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta}\,\frac{1-e^{-2r\sin^2(\theta/2)}}{2\sin^2(\theta/2)}\,d\theta \leqq n^{-s}, \qquad n\uparrow+\infty, \quad +\infty > r \geqq 0,$$

our task now simplifies to showing that

2.20
$$k_1 \geqq \limsup_{n\uparrow+\infty} n^{s-2}s\int_0^{+\infty} e^{-t}I_n(t)(2\pi t)^{-(s-1)/2}\,dt,$$

and, consulting A. Erdélyi [2, p. 196 (8) and 1, p. 164 (20)], it is seen that

2.21
$$s\int_0^{+\infty} e^{-t}I_n(t)(2\pi t)^{-(s-1)/2}\,dt \sim k_1 n^{2-s}, \qquad n\uparrow+\infty,$$

which completes the proof.

3. Potentials

Write e for nonnegative functions defined for the points of the s-dimensional lattice, and consider the Green operator $\mathbf{K}$ defined in

3.1
$$(\mathbf{K}e)(a) = E_a\left(\sum_{n\geqq 0} e(s_n)\right) = \sum\left[\sum_{n\geqq 0} P_a(s_n = b)\right]e(b) = \sum K(a,b)e(b).$$

$-\mathbf{G}$ is inverse to $\mathbf{K}$; for, if $\mathbf{K}e < +\infty$, then

$$\begin{aligned} -\mathbf{GK}e &= E.\left(\sum_{n\geqq 0} e(s_n)\right) - E.\,E_{s_1}\left(\sum_{n\geqq 0} e(s_n)\right) \\ &= E.\left(\sum_{n\geqq 0} e(s_n)\right) - E.\left(\sum_{n\geqq 0} e(s_{n+1})\right) \\ &= E.(e(s_0)) = e. \end{aligned}$$

A nonnegative function p is *excessive* if $\mathbf{G}p \leqq 0$; it is a *potential* if $p = \mathbf{K}e$ with $e \geqq 0$; if $p = \mathbf{K}e$ ($e \geqq 0$, $p < +\infty$), then, as the reader will check, $\mathbf{G}p = -e \leqq 0$, so that potentials are excessive.

Given an excessive function p,

$$\begin{aligned} p &= E.(\textstyle\sum_{l \leqq n} (p(s_{l-1}) - p(s_l)) + p(s_n)) \\ &= \textstyle\sum_{l \leqq n} E.(p(s_{l-1}) - E_{s_{l-1}}(p(s_1))) + E.(p(s_n)) \\ &= E.(\textstyle\sum_{l \leqq n} -(\mathbf{G}p)(s_{l-1})) + E.(p(s_n)), \end{aligned}$$

and, after putting $-\mathbf{G}p = e$ and $p_\infty = \lim_{n \uparrow +\infty} E.(p(s_n))$, it results that

3.2 $$p = \mathbf{K}e + p_\infty .$$

p_∞ is constant; indeed, it is nonnegative, $\mathbf{G}p_\infty = 0$, and, as B. H. Murdoch proved, *such a nonnegative harmonic function is constant.*

J. L. Kelley's proof[6] that a compact convex set is the convex hull of its extreme points served us as a model for the following simple proof of Murdoch's result.

Write C for the class of nonnegative p with $p(0) = 1$ and $\mathbf{G}p = 0$, label the points $c = (l_1, l_2, \cdots, l_s)$ of the s-dimensional lattice c_1 $(=0)$, $c_2, c_3, \cdots$, and, using the compactness that the estimate

3.3 $$p(l_1, l_2, \cdots, l_s) \leqq (2s)^{|l_1|+|l_2|+\cdots+|l_s|}, \qquad p \in C,$$

provides, put $m_1 = 1$, $C_1 = C$, $m_2 = \max_{p \in C_1} p(c_2)$, $C_2 = C_1 \cap (p: p(c_2) = m_2)$, $m_3 = \max_{p \in C_2} p(c_3)$, $C_3 = C_2 \cap (p: p(c_3) = m_3)$, etc., and select $p_* \in \bigcap_{n \geqq 1} C_n$. $p_*(\pm e_k)$ is then > 0, $p_*(\pm e_k)^{-1} p_*(\cdot \pm e_k) \in C$ for each $k \leqq s$, and

3.4 $$p_* = \sum_{k \leqq s, n=1,2} (2s)^{-1} p_*((-)^n e_k) \frac{p_*(\cdot + (-)^n e_k)}{p_*((-)^n e_k)} .$$

Since $\sum_{k \leqq s, n=1,2} (2s)^{-1} p_*((-)^n e_k) = p_*(0) = 1$, the definition of p_* now implies that $p_*(\pm e_k)^{-1} p_*(\cdot \pm e_k) \in \bigcap_{n \geqq 1} C_n$ for each $k \leqq s$; in short, $p_*(\pm e_k)^{-1} p_*(\cdot \pm e_k) = p_*$ for each $k \leqq s$, and we infer that

3.5 $$p_*(l_1, l_2, \cdots, l_s) = p_*(e_1)^{l_1} p_*(e_2)^{l_2} \cdots p_*(e_s)^{l_s}.$$

But

3.6 $$1 = p_*(0) = E_0(p_*(s_1)) = (1/s) \textstyle\sum_{k \leqq s} \frac{1}{2}(p_*(e_k)^{-1} + p_*(e_k)),$$

proving that $p_*(e_k) = 1$ for each $k \leqq s$; therefore $p_* \equiv 1$; since c_2 was chosen at pleasure, each $p \in C$ is $\leqq 1$; and since, for $p \in C$, $\mathbf{G}p \equiv 0$, $p \equiv 1$ is the sole member of C.

Keeping this result in mind, it is clear from 3.2 that, with the notation $p(\infty) = \liminf_{c \to \infty} p(c)$,

3.7 $$p(\infty) \geqq p_\infty = \lim_{n \uparrow +\infty} E.(p(s_n)) \geqq E.(\liminf_{n \uparrow +\infty} p(s_n)) \geqq p(\infty),$$

and 3.2 goes over into

3.8 $$p = \mathbf{K}e + p(\infty), \qquad p(\infty) = \lim_{n \uparrow +\infty} E.(p(s_n)).$$

[6] See P. T. Bateman [1, pp. 14–15].

Given a set B of lattice points, let n_B denote the hitting time min $(n: s_n \in B)$ and p_B the hitting probability $P.(n_B < +\infty)$.

p_B is excessive; in fact,

$$
\begin{aligned}
e_B &= -\mathbf{G}p_B = p_B - E.(p_B(s_1)) \\
&= P.(n_B < +\infty) - P.(n_B(w_1^+) < +\infty) \qquad 3.9 \\
&= P.(n_B < +\infty, n_B(w_1^+) = +\infty) \geqq 0,
\end{aligned}
$$

and it follows from 2.6 and 3.9 that $e_B = 0$ off the points of[7] ∂B neighboring the (connected) part of the complement of B reaching out to ∞.

Writing $\mathbf{B}$ for the event, $\bigcap_{n \geqq 1} (w: n_B(w_n^+) < +\infty)$ that $s_n \in B$ for an infinite number of times n, it results from

3.10 $$p_B(\infty) = \lim_{n \uparrow +\infty} E.(p_B(s_n)) = \lim_{n \uparrow +\infty} P.(n_B(w_n^+) < +\infty) = P.(\mathbf{B})$$

and from

$$
\begin{aligned}
P.(\mathbf{B}) &= \lim_{n \uparrow +\infty} P.(n_B < +\infty, n_B(w_{n_B+n}^+) < +\infty) \\
&= \lim_{n \uparrow +\infty} E.(n_B < +\infty, P_{s_{n_B}}(n_B(w_n^+) < +\infty)) \qquad 3.11 \\
&= p_B\, P.(\mathbf{B})
\end{aligned}
$$

that $p_B(\infty) = P.(\mathbf{B})$ is 0 or 1.[8]

Using these results, it is not difficult to prove that, for excessive $p = \mathbf{K}e + p(\infty)$,

3.12 $$P.(\lim_{n \uparrow +\infty} p(s_n) = p(\infty)) = 1;$$ [9]

indeed, $p \geqq p(\infty)$, and if $\alpha > p(\infty)$, if A is the set where $p \geqq \alpha$, and if $p_A(\infty) = P.(\mathbf{A}) = 1$, then

3.13 $$(w: l \leqq n \wedge n_A) \in \mathbf{C}_l, \qquad l \geqq 1,$$

and

$$
\begin{aligned}
p &= E.\left[\textstyle\sum_{l \leqq n \wedge n_A} (p(s_{l-1}) - p(s_l)) + p(s_{n \wedge n_A})\right] \\
&= E.\left[\textstyle\sum_{l \leqq n \wedge n_A} e(s_{l-1})\right] + E.(p(s_{n_A}), n_A \leqq n) \qquad 3.14 \\
&\quad + E.(p(s_n), n < n_A) \\
&\geqq \alpha P.(n_A \leqq n) \uparrow \alpha p_A = \alpha, \qquad n \uparrow +\infty,
\end{aligned}
$$

violating $\alpha > p(\infty)$, and we infer that $P.(\mathbf{A}) = 0$ for each $\alpha > p(\infty)$, completing the proof.

We give the proof of the general *maximum principle* of which 3.14 is a special case.

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[7] ∂B is the set of points of B not all of whose neighbors belong to B.

[8] $P.(\mathbf{B}) = 0$ or 1 is a special case of the 0-or-1 law of Hewitt and Savage [1, pp. 493–494].

[9] 3.12 is a special case of the result of J. L. Doob [1, pp. 324–326] that a nonnegative lower semimartingale converges.

Given excessive $p_1 = \mathbf{K}e_1 + p_1(\infty)$, $p_2 = \mathbf{K}e_2 + p_2(\infty)$, if $p_2 \geqq p_1$ on the support B of e_1, and if $p_2(\infty) \geqq p_1(\infty)$ in case $P.(\mathbf{B}) = 0$, then $p_2 \geqq p_1$ on the whole of the s-dimensional lattice; in fact, $e_1(s_n) = 0$ for $n < n_B$, and by using the fact that, for excessive p, $p \geqq p(\infty)$ and $\lim_{n\uparrow+\infty} E.(p(s_n)) = p(\infty)$, it develops that, for $a \notin B$,

3.15
$$\begin{aligned} p_2(a) &= E_a[\textstyle\sum_{l\leqq n\wedge n_B}(p_2(s_{l-1}) - p_2(s_l)) + p_2(s_{n\wedge n_B})] \\ &= E_a[\textstyle\sum_{l\leqq n\wedge n_B} e_2(s_{l-1})] + E_a(p_2(s_{n_B}), n \geqq n_B) \\ &\qquad\qquad + E_a(p_2(s_n), n < n_B) \\ &\to E_a[\textstyle\sum_{l\leqq n_B} e_2(s_{l-1})] + E_a(p_2(s_{n_B}), n_B < +\infty) \\ &\qquad\qquad + p_2(\infty)P_a(n_B = +\infty) \qquad (n\uparrow+\infty) \\ &\geqq E_a[\textstyle\sum_{l\leqq n_B} e_1(s_{l-1})] + E_a(p_1(s_{n_B}), n_B < +\infty) \\ &\qquad\qquad + p_1(\infty)P_a(n_B = +\infty) \\ &= p_1(a). \end{aligned}$$

We learn from 3.15 that p_B is the greatest excessive $p = \mathbf{K}e + p(\infty)$ with $e = 0$ off B, $p \leqq 1$ on B, and $p(\infty) \leqq P.(\mathbf{B})$.

Also (and this will be useful for us in Section 4), $p_B = 1$ on B, so that, if, for two *potentials* p_1 and p_2, $p_2 \geqq p_1$ on B and $e_1, e_2 = 0$ off B, then, writing ep for $\sum_{b\in B} e(b)p(b)$,

3.16 $$e_2(B) = e_2\,p_B = e_B\,p_2 \geqq e_B\,p_1 = e_1\,p_B = e_1(B);$$

in short, $e_2(B) \geqq e_1(B)$, a fact due to Gauss [1, pp. 37–39] for the case of Newtonian potentials.

4. Capacities

Given a set B of lattice points for which $P.(\mathbf{B}) = 0$, its *capacity* $C(B)$ is the total charge

4.1 $$C(B) = e_B(B) = \textstyle\sum_{a\in\partial B} P_a(n_B(w_1^+) = +\infty)$$

of the *electrostatic distribution* e_B.

When $|B|$ (= the number of points of B) $= +\infty$, $C(B) = +\infty$; for, if $C(B) < +\infty$, then (use 2.7) p_B converges to 0 at ∞, and, since $p_B = 1$ on B, $|B| < +\infty$.

We shall therefore confine our attention to the capacities of finite sets B.

The following rules are helpful for computing $C(B)$:

4.2 $$C(B) = C(\partial B),$$

4.3 $$C(B_1) = C(B_2), \qquad B_1 \equiv B_2,$$

4.4 $$C(B) = \max e(B):\quad e \geqq 0,\quad e = 0 \text{ off } B,\quad p = \mathbf{K}e \leqq 1,$$

4.5 $$C(B_1 \cap B_2) + C(B_1 \cup B_2) \leqq C(B_1) + C(B_2).$$

4.2 is clear. $B_1 \equiv B_2$ means that B_1 is congruent to B_2 (with respect to orthogonal transformations with integral entries). 4.3 is then clear. $p = \sum_{b \in B} K(a, b,)e(b) \leqq 1$ implies $p_B - p \geqq 0$, and using 3.16 to compute the (nonnegative) total charge $C(B) - e(B)$ of $e_B - e$ proves 4.4. 4.5 gets a similar proof: the inclusion

$$(w: n_{B_1 \cup B_2} < +\infty, n_{B_2} = +\infty) \subset (w: n_{B_1} < +\infty, n_{B_1 \cap B_2} = +\infty)$$

implies $p_{B_1 \cup B_2} - p_{B_2} \leqq p_{B_1} - p_{B_1 \cap B_2}$; now compute the (nonnegative) total charge of $e_{B_1} + e_{B_2} - e_{B_1 \cup B_2} - e_{B_1 \cap B_2}$.

Given $B, B_1, B_2, \cdots, B_n$, let us write $\mathfrak{m}$ for subsets of $1, 2, \cdots, n$, $|\mathfrak{m}| = l$ for the number of points in $\mathfrak{m}$, and $B_{\mathfrak{m}} = \bigcup_{i \in \mathfrak{m}} B_i$; it is clear from[10]

4.6 $$P.(\bigcap_{l \leqq n} C) = -\sum_{l \leqq n} (-)^l \sum_{|\mathfrak{m}|=l} P.(C_{\mathfrak{m}})$$

that

4.7 $$\begin{aligned} 0 &\leqq P.(n_B = +\infty, n_{B \cup B_l} < +\infty, l \leqq n) \\ &= -P.(n_B < +\infty) + P.(n_{B \cup B_l} < +\infty, l \leqq n) \\ &= -P.(n_B < +\infty) - \sum_{l \leqq n} (-)^l \sum_{|\mathfrak{m}|=l} P.(n_{B \cup B_{\mathfrak{m}}} < +\infty) \\ &= p_B - \sum_{l \leqq n} (-)^l \sum_{|\mathfrak{m}|=l} p_{B \cup B_{\mathfrak{m}}}, \end{aligned}$$

and by using 3.16 to compute the (nonnegative) total charge of $-e_B - \sum_{l \leqq n} (-)^l \sum_{|\mathfrak{m}|=l} e_{B \cup B_{\mathfrak{m}}}$, it results that

4.8 $$C(B) + \sum_{l \leqq n} (-)^l \sum_{|\mathfrak{m}|=l} C(B \cup B_{\mathfrak{m}}) \leqq 0.$$

G. Choquet [1, pp. 147–153] proved the counterpart of 4.8 for Newtonian potentials. 4.7 imitates G. Hunt [1, p. 53]. 4.5 is a special case of 4.8 ($n = 2$, $B = B_1 \cap B_2$).

The following technique for estimating $C(B)$ is useful for Section 6. Given B, if A is the sum of n $(= |B|)$ solid cubes $[0, 1]^s$ centered at the points of B, and if $\hat{C}(A)$ is the Newtonian capacity:

4.9 $$\hat{C}(A) = \max \hat{e}(A): \hat{e} \geqq 0, \hat{e} = 0 \text{ off } A, \hat{p}(\xi) = \int_A |\xi - \eta|^{2-s} \hat{e}(d\eta) \leqq 1,$$

then

4.10 $$k_4 \hat{C}(A) \leqq C(B) \leqq k_5 \hat{C}(A),$$

with k_4, k_5 depending on the dimension number s, but not on B.

To prove the *over*estimate, choose k_5 such that, for ξ in the cube centered at a, the integral $\int |\xi - \eta|^{2-s} d\eta$ extended over the cube centered at b is $\leqq k_5 K(a, b)$, let $\hat{e}(d\eta) = e_B(b)\, d\eta$ on the cube centered at b, and estimate $\hat{p}(\xi) = k_5^{-1} \int_A |\xi - \eta|^{2-s} e(d\eta)$ in terms of p_B; the result is $\hat{p} \leqq 1$, and we conclude that

4.11 $$C(B) = e_B(B) = \hat{e}(A) \leqq k_5 \hat{C}(A).$$

[10] 4.6 is dual to the classical inclusion and exclusion formula.

To prove the *under*estimate, choose k_4 such that, for ξ in the cube centered at a and η in the cube centered at b, $k_4\,K(a, b) \leqq |\,\xi - \eta\,|^{2-s}$, let $e(b)$ be the charge that the Newtonian electrostatic distribution $\hat{e}$ places on the cube centered at b, and estimate $p = k_4 \sum_{b \epsilon B} K(a, b)e(b)$ in terms of

$$\hat{p} = \int |\,\xi - \eta\,|^{2-s}\hat{e}(d\eta);$$

the result is $p \leqq 1$, and we conclude that

4.12 $$C(B) \geqq k_4\, e(B) = k_4\, \hat{e}(A) = k_4\, \hat{C}(A).$$

Given compact $A \subset R^s$,

4.13 $$\hat{C}(\alpha A) = \alpha^{s-2}\hat{C}(A), \qquad \alpha > 0,$$

where αA is the set of points αx with $x \epsilon A$.

We will use 4.13 for getting *under*estimates of $C(B)$; for example, if B_n is the disc

$$(l_1, l_2, \cdots, l_s):\ (l_1^2 + l_2^2)^{1/2} \leqq n, \quad l_3 = l_4 = \cdots = l_s = 0,$$

then $C(B_n) \geqq k_6\, n^{s-2}$ for $n \uparrow +\infty$.

5. Wiener's test

Given a set Q of lattice points, clustering to ∞, let Q_l denote the intersection of Q and the spherical shell $2^l \leqq |\,a\,| < 2^{l+1}$, and let us prove Wiener's test:

5.1 $P.(\mathbf{Q}) = 1$ or 0 according as $\sum_{l \geqq 1} 2^{-l(s-2)}C(Q_l) =$ or $< +\infty$.

When

5.2 $$\sum_{l \geqq 1} 2^{-l(s-2)}C(Q_l) < +\infty,$$

5.3 $$\begin{aligned} p_{Q_l}(a) &= \sum_{b \epsilon Q_l} K(a, b)e_{Q_l}(b) \leqq \sum_{b \epsilon Q_l} k_7|\,a - b\,|^{2-s}e_{Q_l}(b) \\ &\leqq 2k_7\, 2^{-l(s-2)}C(Q_l), \qquad l \uparrow +\infty, \end{aligned}$$

is the general term of a convergent sum, and an application of the first Borel-Cantelli lemma implies

5.4 $$P.(n_{Q_l} = +\infty, l \uparrow +\infty) = 1;$$

2.6 implies $P.(\cup_{l \leqq n} \mathbf{Q}_l) = 0$ for each $n \geqq 1$; and we infer that

5.5 $$P.(\mathbf{Q}) = 0.$$

When

5.6 $$\sum_{l \geqq 1} 2^{-l(s-2)}C(Q_l) = +\infty,$$

$\sum_{l \geqq 1} 2^{-(4l+k)(s-2)}C(Q_{4l+k}) = +\infty$ for $k = 0, 1, 2,$ or 3, and if we suppose, as we can, that $\sum_{l \geqq 1} 2^{-(4l+1)(s-2)}C(Q_{4l+1}) = +\infty$, it is clear that, if m_l is the crossing *time*[11] min $(n:2^l \leqq |s_n| < 2^l + 1)$, and if, for the moment, s_l stands for the crossing *place* s_{m_l}, then for $l \uparrow +\infty$,

5.7
$$\begin{aligned} P_{s_{4l}}(n_{Q_{4l+1}} < m_{4l+4}) &\geqq p_{Q_{4l+1}}(s_{4l}) - E_{s_{4l}}[p_{Q_{4l+1}}(s_{4l+4})] \\ &= \sum_{b\epsilon Q_{4l+1}} (K(s_{4l}, b) - E_{s_{4l}}[K(s_{4l+4}, b)])e_{Q_{4l+1}}(b) \\ &\geqq k_8[\tfrac{2}{3}(2^{4l+2} - 2^{4l})^{2-s} - \tfrac{3}{2}(2^{4l+4} - 2^{4l+2})^{2-s}]C(Q_{4l+1}) \\ &\geqq k_9\, 2^{-(4l+1)(s-2)}C(Q_{4l+1}) = t_l, \end{aligned}$$

which implies

5.8
$$\begin{aligned} &P_{s_{4l}}(n_Q = +\infty) \leqq E_{s_{4l}}[n_{Q_{4l+1}} \geqq m_{4l+4}, P_{s_{4l+4}}(n_Q = +\infty)] \\ &\leqq E_{s_{4l}}[n_{Q_{4l+1}} \geqq m_{4l+4}, E_{s_{4l+4}}(n_{Q_{4l+5}} \geqq m_{4l+8}, P_{s_{4l+8}}(n_Q = +\infty))], \quad \text{etc.} \\ &\leqq (1 - t_l)(1 - t_{l+1})(1 - t_{l+2}) \cdots \\ &= 0, \end{aligned}$$

and we conclude that

5.9
$$P.(n_Q(w_{m_{4l}}^+) < +\infty) = E.(P_{s_{4l}}(n_Q < +\infty)) = 1, \qquad l \uparrow +\infty.$$

which completes the proof of 5.1.

6. Thorns

Given nonnegative $i(1) \leqq i(2) \leqq \cdots$, let Q denote the *thorn*

$$(l_1, l_2, \cdots, l_s): \quad (l_1^2 + l_2^2 + \cdots + l_{s-1}^2)^{1/2} \leqq i(l_s), \quad l_s \geqq 1.$$

We use Wiener's test to prove that for $s \geqq 4$ dimensions

6.1 $P.(\mathbf{Q}) = 1$ or 0 according as $\sum_{n\geqq 1}(2^{-n}i(2^n))^{s-3} =$ or $< +\infty$;[12]

as an example, if $l \geqq 1$, and if $i(n) = n(\lg n \lg_2 n \cdots (\lg_k n)^{\alpha})^{-1/s-3}$,[13] then $P.(\mathbf{Q}) = 1$ (0) for $\alpha \leqq 1$ (>1).

When $\lim\sup_{n\uparrow+\infty} n^{-1}i(n) > \alpha > 0$, $2^{-n}i(2^n) > \alpha/2$ for an infinite number of integers n; for such n, Q_n contains the set Q_n^- of lattice points of a sphere of diameter $\geqq \min(1, \alpha)2^n$; $C(Q_n) \geqq C(Q_n^-)$; $C(Q_n)$ is then $\geqq k_{10}\, 2^{n(s-2)}$; and the upshot is

6.2
$$+\infty = \sum_{n\geqq 1} (2^{-n}i(2^n))^{s-3} = \sum_{n\geqq 1} 2^{-n(s-2)}C(Q_n),$$

which checks with 6.1.

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[11] $m_l < +\infty$ for paths crossing from $|a| \leqq 2^l$ to $|a| > 2^l + 1$; for if

$$(l_1^2 + l_2^2 + \cdots + l_s^2)^{1/2} \leqq 2^l,$$

then $(l_1 \pm 1)^2 + l_2^2 + \cdots + l_s^2 = l_1^2 + l_2^2 + \cdots + l_s^2 \pm 2l_1 + 1 \leqq 2^{2l} + 2\cdot 2^l + 1 = (2^l + 1)^2$.

[12] 6.1 is to be compared with Lebesgue's thorn: see Courant and Hilbert [1, pp. 272–274].

[13] $\lg_1 = \lg$ and $\lg_n = \lg(\lg_{n-1})$ for $n \geqq 2$.

When

6.3 $$\lim_{n\uparrow+\infty} n^{-1}i(n) = 0,$$

Q_n is long and thin, a sphere is not a good approximation, and we consider instead the ellipsoids $Q_n^- \subset Q$ and $Q_n^+ \supset Q$:

$$Q_n^- = (l_1, l_2, \cdots, l_s): \quad \frac{l_1^2 + l_2^2 + \cdots + l_{s-1}^2}{i(2^n)^2} + \frac{(l_s - 3\cdot 2^{-n-1})^2}{2^{2(n-1)}} \leqq 1,$$

$$Q_n^+ = (l_1, l_2, \cdots, l_s): \quad \frac{l_1^2 + l_2^2 + \cdots + l_{s-1}^2}{2\cdot i(2^{n+1})^2} + \frac{(l_s - 3\cdot 2^{n-1})^2}{2\cdot 2^{2(n-1)}} \leqq 1$$

and compare the capacities $C(Q_n^-)$, $C(Q_n^+)$ to the Newtonian capacity of the solid ellipsoid

$$E = (x_1, x_2, \cdots, x_s): \quad \frac{x_1^2}{e_1^2} + \frac{x_2^2}{e_2^2} + \cdots + \frac{x_s^2}{e_s^2} \leqq 1, \qquad e_1, e_2, \cdots, e_s > 0.$$

The Newtonian capacity of E is known; up to a factor depending on the dimension number, it is the reciprocal of the elliptic integral

6.4 $$\mathbf{e} = \int_0^{+\infty} \frac{dt}{\sqrt{(e_1^2 + t)(e_2^2 + t)\cdots(e_s^2 + t)}}.$$

The reader will find a neat proof of 6.4 for $s = 3$ in G. Chrystal [1, p. 30].

When $\alpha = e_1/e_s = e_2/e_s = \cdots = e_{s-1}/e_s$,

6.5
$$\begin{aligned}\mathbf{e} &= \int_0^{+\infty} (\alpha^2 e_s^2 + t)^{-(s-1)/2}(e_s^2 + t)^{-1/2}\,dt \\ &= e_s^{2-s}\int_0^{+\infty} (\alpha^2 + t)^{-(s-1)/2}(1 + t)^{-1/2}\,dt \\ &\sim e_s^{2-s}\int_{\alpha^2}^{1} t^{-(s-1)/2}\,dt \\ &\sim e_s^{2-s}\tfrac{1}{2}(s - 3)\alpha^{-(s-3)}, \qquad \alpha \downarrow 0;\end{aligned}$$

by using 4.10, it is plain that

6.6
$$\begin{aligned}&k_{11}\, 2^{+n(s-2)}(2^{-n}i(2^n))^{s-3} < C(Q_n^-) \\ &\leqq C(Q) \leqq C(Q_n^+) < k_{12}\, 2^{+n(s-2)}(2^{-(n+1)}i(2^{n+1}))^{s-3}, \qquad n\uparrow+\infty;\end{aligned}$$

and an application of Wiener's test completes the proof of 6.1.

When $s = 3$, $P.(\mathbf{Q}) = 1$ even for the thinnest thorn $Q = \cup_{n\geqq 1}(0, 0, n)$; $C(Q_n)$ is then $> k_{13}\, 2^n$, and Wiener's sum is $+\infty$.

We consider, instead, the set $Q = \cup_{n\geqq 1}(0, 0, i(n))$ with integral $i(1) < i(2) < \cdots$ and prove that if

6.7 $$i(n) - i(n-1) \geqq \lg i(n-1), \qquad n\uparrow+\infty,$$

then

6.8 $P.(\mathbf{Q}) = 1$ or 0 according as $\sum_{n \geqq 1} i(n)^{-1} = $ or $< +\infty$;

as an example, if $i(n) = [n \lg n \lg_2 n \cdots (\lg_k n)^{\alpha}]$,[14] then $P.(\mathbf{Q}) = 1$ (0) for $\alpha \leqq 1$ (>1).

Granting

6.9 $$k_{14} \mid Q_n \mid \leqq C(Q_n) \leqq k_{15} \mid Q_n \mid, \qquad n \uparrow +\infty,$$

it is clear that

6.10 $$k_{14} \sum_{2^n \leqq i(l) < 2^{n+1}} i(l)^{-1} \leqq k_{14} 2^{-n} \mid Q_n \mid \leqq 2^{-n} C(Q_n) \leqq k_{15} 2^{-n} \mid Q_n \mid \leqq 2k_{15} \sum_{2^n \leqq i(l) < 2^{n+1}} i(l)^{-1},$$

and an application of Wiener's test proves 6.8. $C(Q_n) \leqq k_{15} \mid Q_n \mid$ with $k_{15} = K(0, 0)^{-1}$ is immediate from 4.5, and to complete the proof, it is enough to use 6.7 and 2.7 to check the estimate

6.11 $$\sum_{b \in Q_n} K(a, b) \leqq k_{16} \sum_{1 \leqq l \leqq |Q_n|} (ln \lg 2)^{-1} < k_{17} n^{-1} \lg \mid Q_n \mid \leqq k_{17} \lg 2, \qquad a \in Q_n, \quad n \uparrow +\infty,$$

and to infer, from 4.4, that

6.12 $$C(Q_n) \geqq k_{14} \mid Q_n \mid, \qquad k_{14} = (k_{17} \lg 2)^{-1}, \qquad n \uparrow +\infty.$$

Problem. When $i(n): n \geqq 1$ is the set of prime numbers, is $P.(\mathbf{Q}) = 1$? Were Gauss's law $n/\lg n$ for the number of primes $\leqq n$ *exact*, we could assert that $+\infty > k_{18} \geqq \sum_{b \in Q_n} K(a, b)$ for $a \in Q_n$ and conclude, as in 6.12, that $C(Q_n) \geqq k_{18}^{-1} \mid Q_n \mid \geqq k_{19} n^{-1} 2^n$ and that $\sum_{n \geqq 1} 2^{-n} C(Q_n) = +\infty$.

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[14] $[\gamma]$ is the greatest integer $\leqq \gamma$.

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