# INVERSION OF TOEPLITZ MATRICES II[1]

BY
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## 1. Introduction

With a function $\varphi(\theta) \in L_1(0, 2\pi)$, $\varphi(\theta) \sim \sum_{-\infty}^{\infty} c_k e^{ik\theta}$, is associated the *semi-infinite Toeplitz matrix* $T_\varphi = (c_{j-k})_{0 \leqq j,k < \infty}$. In case $\sum | c_k | < \infty$, $T_\varphi$ represents a bounded operator on the space $l_\infty^+$ of bounded sequences

$$X = \{x_0, x_1, \cdots\},$$

and in [1] a necessary and sufficient condition was found for the invertibility of $T_\varphi$ (i.e., the existence of a bounded inverse for $T_\varphi$), namely that $\varphi(\theta) \neq 0$ and $\Delta_{-\pi \leqq \theta \leqq \pi} \arg \varphi(\theta) = 0$. If $\varphi(\theta) \in L_\infty$, $T_\varphi$ represents a bounded operator on the space $l_2^+$ of square-summable sequences, and in §3 of [1] sufficient conditions were obtained for invertibility in this situation.

The purpose of the present paper is to obtain conditions which are necessary as well as sufficient for invertibility of $T_\varphi$ as an operator on $l_2^+$. That the situation is quite different in the $l_\infty^+$ and $l_2^+$ cases can be seen, for instance, from the fact that in the former, the set of $\varphi$ for which $T_\varphi$ is invertible forms a group, while in the latter we may have $T_\varphi$ invertible but $T_{\varphi^2}$ not (Corollary 2 of Theorem IV).

As in all problems of Wiener-Hopf type, and this is one, the basic idea is a certain type of factorization. In our case, the idea is that of writing $T_\varphi$ as the product of triangular Toeplitz matrices (which amounts to a factorization of $\varphi$), the question of invertibility for these being simpler since any two triangular Toeplitz matrices of the same type commute. Thus, roughly speaking, if $\varphi$ is sufficiently nice, we can factor $T_\varphi$ and then invert each factor, thus obtaining the inverse of $T_\varphi$. This gives rise to sufficient conditions for invertibility, as in [1, §3]. Now in the $l_\infty^+$ theory it turned out that the $\varphi$'s for which this could be carried out were *exactly* those giving rise to invertible Toeplitz matrices; thus the invertibility of $T_\varphi$ implies the existence of a suitable factorization of $\varphi$. It is the content of Theorem I of the present paper that this situation prevails also in the $l_2^+$ case. From this result we easily settle the invertibility question for triangular and self-adjoint Toeplitz matrices.

For general Toeplitz matrices we have been unable to find a simple criterion for invertibility; there is one however (Theorem IV) in case $\arg \varphi(\theta)$ is reasonably well-behaved.

Before proceeding, we introduce some notation. For $f(\theta) \in L_p(0, 2\pi)$,

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$1 \leqq p \leqq \infty$, $f(\theta) \sim \sum_{-\infty}^{\infty} a_k e^{ik\theta}$, we shall say that $f \in L_p^+$ (resp. $L_p^-$) if $a_k = 0$ for $k < 0$ (resp. $k > 0$). Thus $f \in L_p^+$ means there exists an $F(z)$ belonging to $H_p$ of the unit circle [3, Chapter 7] such that $F(e^{i\theta}) = f(\theta)$ pp., and $f(\theta) \in L_p^-$ means $\overline{f(\theta)} \in L_p^+$.

For $f \in L_1$, $Cf$ will denote the conjugate function of $f$,

$$Cf(\omega) = \frac{1}{2\pi} \operatorname{PV} \int_0^{2\pi} f(\theta) \cot \frac{1}{2} (\omega - \theta)\, d\theta \qquad \text{pp};$$

$Mf$ will be the mean of $f$,

$$Mf = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta;$$

and the operator $P$ is defined by

$$Pf = \tfrac{1}{2}(f + Mf + i\, Cf). \tag{1}$$

If $f \in L_p$ with $1 < p < \infty$, then also $Cf \in L_p$, and the Fourier series of $Cf$ is the conjugate series of the Fourier series of $f$ [3, §7.21]. It follows that if $f(\theta) \sim \sum_{-\infty}^{\infty} a_k e^{ik\theta}$, then $Pf(\theta) \sim \sum_0^{\infty} a_k e^{ik\theta}$; thus for $1 < p < \infty$, $P$ projects $L_p$ onto $L_p^+$.

Throughout this paper $\varphi(\theta)$ will be bounded, and $T_\varphi$ will be considered an operator on $l_2^+$. Now $l_2^+$ is imbedded in a natural way in the space $l_2$ of square-summable doubly infinite sequences $X = \{\cdots, x_{-1}, x_0, x_1, \cdots\}$. If we define the isomorphism $\mathcal{U}\colon l_2 \to L_2$ in the obvious way, then $\mathcal{U} l_2^+ = L_2^+$ and $\mathcal{U} T_\varphi \mathcal{U}^{-1} = P\varphi$. (Here $P\varphi$ means, not $P$ applied to $\varphi$, but the operator consisting of multiplication by $\varphi$ followed by $P$; ambiguities of this sort will appear occasionally but should cause no difficulty.) The Toeplitz matrix $T_\varphi$ and the operator $P\varphi$ may therefore be discussed interchangeably.

## 2. A general theorem

THEOREM I. *A necessary and sufficient condition for the invertibility of $T_\varphi$ is the existence of functions $\varphi_+(\theta)$ and $\varphi_-(\theta)$, in $L_2^+$ and $L_2^-$ respectively, such that*

(a) $\varphi(\theta) = \varphi_+(\theta)\varphi_-(\theta)$;

(b) $1/\varphi_+ \in L_2^+$ *and* $1/\varphi_- \in L_2^-$;

(c) *for* $f \in L_2$, $Sf = \varphi_+^{-1} P \varphi_-^{-1} f \in L_2$, *and* $f \to Sf$ *is a bounded operator on* $L_2$.

We first prove the conditions sufficient for invertibility of $T_\varphi$, or equivalently that of $P\varphi$; in fact we shall show that $S$, when restricted to $L_2^+$ is just $(P\varphi)^{-1}$. Let $f \in L_\infty^+$. Then

$$P\varphi S f = P\varphi_- P \varphi_-^{-1} f = Pf - P\varphi_-(I - P)\varphi_-^{-1} f, \tag{2}$$

where $I$ represents the identity operator. Now $g = \varphi_-(I - P)\varphi_-^{-1} f \in L_1^-$, and $Mg = 0$. It follows from this that $Pg = 0$. For let $\sigma_n(\theta)$ be the Fejér means of $g(\theta)$. Then clearly $P\sigma_n = 0$ for all $n$. Since $\sigma_n \to g\ (L_1)$, we have $P\sigma_n \to Pg\ (L_p)$ for any $p$ in $0 < p < 1$ [3, §7.3 (ii)]. Thus $Pg = 0$, and (2) gives $P\varphi Sf = Pf = f$ since $f \in L_\infty^+$. Since $P\varphi S$ is a bounded operator, we have $P\varphi Sf = f$ for all $f \in L_2^+$, i.e., $S$ is a right inverse for $P\varphi$. To show that $S$ is also a left inverse, again let $f \in L_\infty^+$. We have

$$SP\varphi f = \varphi_+^{-1} P\varphi_+ f - \varphi_+^{-1} P\lambda_-^{-1}(I - P)\varphi f.$$

By an argument similar to the one above, we see the second term on the right is zero; moreover since $\varphi_+ f \in L_2^+$, we have $P\varphi_+ f = \varphi_+ f$, and the first term on the right is $f$. Consequently $SP\varphi f = f$ for $f \in L_\infty^+$, and so for $f \in L_2^+$. Thus $S$ is a left inverse for $P\varphi$, and the sufficiency is proved.

To prove the conditions necessary, assume $T\varphi$ is invertible, and denote the inverse matrix by $(s_{jk})_{0 \leqq j,k < \infty}$. Define

$$\sigma_{jk} = \sum_{l \leqq \min(j,k)} s_{j-l,0}\, s_{0,k-l}\ ;$$

we shall prove

$$\sum_{k=0}^{\infty} c_{h-k}\, \sigma_{kj} = s_{00}\, \delta_{hj}\ , \qquad h, j \geqq 0. \tag{3}$$

Note that since $\sum_{j=0}^{\infty} |\, s_{jk}\,|^2 < \infty$ for each $k$, and $\sum_{k=0}^{\infty} |\, s_{jk}\,|^2 < \infty$ for each $j$, similar statements hold for $\sigma_{jk}$, so the left side of (3) converges absolutely. We have

$$\begin{aligned}
\sum_{k=0}^{\infty} c_{h-k}\, \sigma_{kj} &= \sum_{k=0}^{\infty} c_{h-k} \sum_{l \leqq \min(k,j)} s_{k-l,0}\, s_{0,j-l} \\
&= \sum_{k=0}^{\infty} c_{h-k} \sum_{l<j;\, l \leqq k} s_{k-l,0}\, s_{0,j-l} + \sum_{k=j}^{\infty} c_{h-k}\, s_{k-j,0}\, s_{00} \\
&= \sum_{l=0}^{j-1} s_{0,j-l} \sum_{k=l}^{\infty} c_{h-k}\, s_{k-l,0} + \sum_{k=j}^{\infty} c_{h-k}\, s_{k-j,0}\, s_{00} \\
&= \sum_{l=0}^{j-1} s_{0,j-l} \sum_{k=0}^{\infty} c_{h-k-l}\, s_{k0} + \sum_{k=0}^{\infty} c_{h-j-k}\, s_{k0}\, s_{00}\ . \qquad (4)
\end{aligned}$$

Now since $(s_{jk})$ is the inverse of $T_\varphi = (c_{j-k})$, we have

$$\sum_{k=0}^{\infty} c_{h-k}\, s_{kl} = \sum_{k=0}^{\infty} s_{hk}\, c_{k-l} = \delta_{hl}\ , \qquad h, l \geqq 0. \tag{5}$$

Thus if $j \leqq h$, the inner sum of the first term of (4) is always zero for $0 \leqq l \leqq j - 1$, so the entire first term is zero. Moreover the second term is $\delta_{hj}\, s_{00}$. This proves (3) in case $j \leqq h$.

To obtain the result for $j > h$, we note that by (5)

$$0 = \sum_{l=0}^{\infty} s_{0l}\, c_{l-j-k+h} = c_{h-j-k}\, s_{00} + \sum_{l=1}^{\infty} c_{l+h-j-k}\, s_{0l}\ ,$$

so

$$\begin{aligned}
\sum_{k=0}^{\infty} c_{h-j-k}\, s_{k0}\, s_{00} &= -\sum_{k=0}^{\infty} s_{k0} \sum_{l=1}^{\infty} c_{l+h-j-k}\, s_{0l} \\
&= -\sum_{l=1}^{\infty} s_{0l} \sum_{k=0}^{\infty} c_{l+h-j-k}\, s_{k0} \qquad (6) \\
&= -\sum_{l=1}^{j-k} s_{0l} \sum_{k=0}^{\infty} c_{l+h-j-k}\, s_{k0} \\
&= -\sum_{l=h}^{j-1} s_{0,j-l} \sum_{k=0}^{\infty} c_{h-l-k}\, s_{k0}\ .
\end{aligned}$$

Now if $j > h$, we see from (5) that the outer summation in the first term of (4) may begin with $l = h$, so we have just shown that the sum of the two terms of (4) is zero, which verifies (3) in the case $j > h$. We must still, however, justify the step leading to (6), this being not completely trivial. Let $\Psi(z) = \sum_{k=0}^{\infty} s_{k0} z^k$ for $|z| < 1$. Then

$$\sum_{k=0}^{\infty} s_{k0}\, r^k c_{l+h-j-k} = \frac{1}{2\pi}\int_0^{2\pi} \Psi(re^{-i\theta})\varphi(\theta)\, e^{i(j-h)\theta}\, e^{-il\theta}\, d\theta.$$

Since

$$\underset{r\to 1-}{\text{l.i.m.}^{(2)}}\ \Psi(re^{-i\theta})\varphi(\theta)e^{i(j-h)\theta} = \Psi(e^{-i\theta})\varphi(\theta)e^{i(j-h)\theta}$$

(note that $\Psi(z) \in H_2$ and $\varphi \in L_\infty$), we have

$$\lim_{r\to 1-} \sum_{l=-\infty}^{\infty} \Big| \sum_{k=0}^{\infty} s_{k0}\, c_{l+h-j-k}(r^k - 1) \Big|^2 = 0.$$

Consequently,

$$\begin{aligned}\sum_{l=1}^{\infty} s_{0l} \sum_{k=0}^{\infty} s_{k0}\, c_{l+h-j-k} &= \lim_{r\to 1-} \sum_{l=1}^{\infty} s_{0l} \sum_{k=0}^{\infty} s_{k0}\, c_{l+h-j-k}\, r^k \\ &= \lim_{r\to 1-} \sum_{k=0}^{\infty} s_{k0}\, r^k \sum_{l=1}^{\infty} c_{l+h-j-k}\, s_{0l} = \sum_{k=0}^{\infty} s_{k0} \sum_{l=1}^{\infty} c_{l+h-j-k}\, s_{0l}\end{aligned}$$

since the last series converges. This completes the justification of (6) and therefore the proof of (3).

It follows from (3) and the invertibility of $T_\varphi$ that

$$\sigma_{kj} = s_{00}\, s_{kj}\,. \tag{7}$$

Next we show that $s_{00} \neq 0$. Assume $s_{00} = 0$; then by (7), $\sigma_{kj} = 0$ for all $k$, $j$. Assume $s_{01} = \cdots = s_{0,n-1} = s_{10} = \cdots = s_{n-1,0} = 0$. We shall show $s_{0n} = s_{n0} = 0$. For $i \geqq n$,

$$0 = \sigma_{in} = \sum_{k\leqq n} s_{i-k,0}\, s_{0,n-k} = s_{i0}\, s_{0n}\,.$$

If $s_{0n} \neq 0$, we would have $s_{i0} = 0$ for $i \geqq n$. Thus we would have $s_{i0} = 0$ for all $i$, i.e., the first column of the invertible matrix $T_\varphi^{-1}$ consists entirely of zeros. Since this cannot be, we must have $s_{0n} = 0$. A similar argument shows $s_{n0} = 0$. But now we have proved by induction that $s_{0n} = s_{n0} = 0$ for all $n$, which again cannot be. Thus our assumption $s_{00} = 0$ was incorrect.

Introduce the functions

$$\psi_+(\theta) \sim \sum_{k=0}^{\infty} s_{k0}\, e^{ik\theta}, \qquad \psi_-(\theta) \sim \sum_{k=0}^{\infty} s_{0k}\, e^{-ik\theta}$$

belonging to $L_2^+$ and $L_2^-$, respectively. We have, for $j \geqq 0$,

$$\begin{aligned}\psi_+(\theta)P\psi_-(\theta)e^{ij\theta} &= \psi_+(\theta)\sum_{k=0}^{j} s_{0k}\, e^{i(j-k)\theta} \\ &= \sum_{l=0}^{\infty} s_{l0}\, e^{il\theta} \sum_{k=0}^{j} s_{0,j-k}\, e^{ik\theta} \\ &= \sum_{n=0}^{\infty} e^{in\theta} \sum_{k\leqq j; k\leqq n} s_{0,j-k}\, s_{n-k,0} \\ &= \sum_{n=0}^{\infty} \sigma_{nj}\, e^{in\theta} = s_{00} \sum_{n=0}^{\infty} s_{nj}\, e^{in\theta}\end{aligned} \tag{8}$$

by (7). But if $S$ denotes the inverse of $P\varphi$ as an operator on $L_2^+$, we have

$$s_{nj} = (Se^{ij\theta}, e^{in\theta}),$$

so $Se^{ij\theta} = \sum_{n=0}^{\infty} s_{nj}\, e^{in\theta}$. Therefore by (8)

$$\psi_+(\theta)P\psi_-(\theta)e^{ij\theta} = s_{00}\, Se^{ij\theta}, \qquad j \geqq 0,$$

from which we conclude $\psi_+ P\psi_- f = s_{00}\, Sf$ for any trigonometric polynomial $f \in L_2^+$. To prove this for an arbitrary $f \in L_2^+$, let $\{s_N\}$ denote its sequence of partial sums. Then since $S$ is a bounded operator

$$\text{(9)} \qquad s_{00}\, Sf \overset{(2)}{=} \text{l.i.m.}_{N\to\infty}\, s_{00}\, Ss_N \overset{(2)}{=} \text{l.i.m.}_{N\to\infty}\, \psi_+ P\psi_-\, s_N .$$

Now since $\psi_- \in L_2$, we have

$$\overset{(1)}{\text{l.i.m.}}_{N\to\infty}\, \psi_-\, s_N = \psi_- f,$$

so that

$$\overset{(p)}{\text{l.i.m.}}_{N\to\infty}\, P\psi_-\, s_N = P\psi_- f$$

for any $p < 1$. (This follows easily from [3, Theorem 7.24 (i)].) Therefore, for a suitable subsequence $N'$,

$$P\psi_- f = \lim_{N'\to\infty} P\psi_-\, s_{N'} .$$

We obtain from (9) therefore that

$$\text{(10)} \qquad s_{00}\, Sf = \psi_+ P\psi_- f, \qquad f \in L_2^+.$$

Setting $f(\theta) \equiv 1$ and applying $P\varphi$ to both sides of (10), we obtain $s_{00} = P\varphi\psi_+ P\psi_-$. Since $P\psi_-$ is a constant (nonzero since $s_{00} \neq 0$), so is $P\varphi\psi_+$. Thus

$$\text{(11)} \qquad \varphi\psi_+ \in L_2^-.$$

Now the adjoint of $P\varphi$ is $P\bar{\varphi}$ (since that of $T_\varphi$ is $T_{\bar{\varphi}}$), and that of $\psi_+ P\psi_-$ (which we know to be bounded by (10)) is $\bar{\psi}_- P\bar{\psi}_+$. Therefore

$$(P\bar{\varphi})(\bar{\psi}_- P\bar{\psi}_+)f = s_{00} f, \qquad f \in L_2^+.$$

Setting $f(\theta) \equiv 1$ we see as above that $P\bar{\varphi}\bar{\psi}_-$ is a constant, so $\bar{\varphi}\bar{\psi}_- \in L_2^-$; hence

$$\text{(12)} \qquad \varphi\psi_- \in L_2^+.$$

Since $\psi_- \in L_2^-$, (11) gives $\varphi\psi_+ \psi_- \in L_1^-$, and since $\psi_+ \in L_2^+$, (12) gives $\varphi\psi_+ \psi_- \in L_1^+$. Hence $\varphi\psi_+ \psi_- = \alpha$, a constant. Since $S \neq 0$, we have $\psi_+ \not\equiv 0$ and $\psi_- \not\equiv 0$, from which it follows that neither $\psi_+$ nor $\psi_-$ is zero on a set of positive measure. (In fact $\psi \in L_2^+$ implies $\log |\psi| \in L_1$ [2].) Since, moreover, $\varphi \not\equiv 0$, we deduce $\alpha \neq 0$. Applying (10) to

$$f = P\psi_-^{-1} = P\varphi(\varphi\psi_-)^{-1} = \alpha^{-1}P\varphi\psi_+ ,$$

we obtain

$$s_{00}\,\alpha^{-1}\psi_+ = \psi_+ P\psi_- P\psi_-^{-1} = \psi_+ .$$

Therefore $\alpha = s_{00}$, and so

$$\varphi\psi_+\psi_- = s_{00}\,. \tag{13}$$

Finally, set $\varphi_+(\theta) = \psi_+(\theta)^{-1}$ and $\varphi_-(\theta) = s_{00}\,\psi_-(\theta)^{-1}$. (11)–(13) show that $\varphi_+(\theta)$ and $\varphi_+(\theta)^{-1}$ are in $L_2^+$, that $\varphi_-(\theta)$ and $\varphi_-(\theta)^{-1}$ are in $L_2^-$, and that $\varphi = \varphi_+\varphi_-$. Thus conditions (a) and (b) of the theorem are satisfied. As for (c), we know from (10) that for some constant $A$ we have

$$\|\,\varphi_+^{-1}P\varphi_-^{-1}f\,\|_2 \leqq A\,\|\,f\,\|_2\,, \qquad f\in L_2^+.$$

For general $f \in L_2$,

$$\varphi_+^{-1}P\varphi_-^{-1}f = \varphi_+^{-1}P\varphi_-^{-1}Pf + \varphi_+^{-1}P\varphi_-^{-1}(I-P)f = \varphi_+^{-1}P\varphi_-^{-1}Pf$$

by the argument used in the proof of sufficiency. Thus

$$\|\,\varphi_+^{-1}P\varphi_-^{-1}f\,\|_2 = \|\,\varphi_+^{-1}P\varphi_-^{-1}Pf\,\|_2 \leqq A\,\|\,Pf\,\|_2 \leqq A\,\|\,f\,\|_2\,,$$

and this completes the proof.

Corollary. *If $T_\varphi$ is invertible, then $1/\varphi \in L_\infty$.*

*Proof.* It suffices, in view of Theorem I, to show the following: If $\psi_1$, $\psi_2 \in L_2$ are such that $\psi_1 P\psi_2$ represents a bounded operator on $L_2$, then $\psi_1, \psi_2 \in L_\infty$. Let $f \in L_\infty$, $\psi_2(\theta)f(\theta) \sim \sum_{-\infty}^{\infty} a_k\,e^{ik\theta}$. Then for $n > 0$

$$e^{-in\theta}P\psi_2(\theta)f(\theta)e^{in\theta} \sim \textstyle\sum_{k=-n}^{\infty} a_k\,e^{ik\theta},$$

so $e^{-in\theta}P\psi_2(\theta)f(\theta)e^{in\theta} \to \psi_2(\theta)f(\theta)$ in $L_2$ as $n \to \infty$. By choosing a subsequence we have convergence pp. Then

$$|\,\psi_1(\theta)P\psi_2(\theta)f(\theta)e^{in\theta}\,| \to |\,\psi_1(\theta)\psi_2(\theta)f(\theta)\,| \qquad \text{pp.}$$

But

$$\|\,\psi_1(\theta)P\psi_2(\theta)f(\theta)e^{in\theta}\,\|_2 \leqq A\,\|\,f(\theta)e^{in\theta}\,\|_2 = A\,\|\,f\,\|_2$$

for an appropriate $A$. It follows from Fatou's lemma that $\psi_1\psi_2 f \in L_2$ and $\|\,\psi_1\psi_2 f\,\|_2 \leqq A\,\|\,f\,\|_2$. This holds for all $f \in L_\infty$, so $\psi_1, \psi_2 \in L_\infty$.

## 3. Special theorems

Lemma 1. *If either $\varphi_1 \in L_\infty^-$ or $\varphi_2 \in L_\infty^+$, we have $T_{\varphi_1}T_{\varphi_2} = T_{\varphi_1\varphi_2}$.*

*Proof.* Let $\varphi_1(\theta) \sim \sum a_k\,e^{ik\theta}$, $\varphi_2(\theta) \sim \sum b_k\,e^{ik\theta}$. Then $T_{\varphi_1}T_{\varphi_2}$ has $j$, $k$ entry

$$\textstyle\sum_{l=0}^{\infty} a_{j-l}\,b_{l-k}\,.$$

If either $a_k = 0$ for $k > 0$ or $b_k = 0$ for $k < 0$, the summation may begin with $l = -\infty$. Thus the $j$, $k$ entry of $T_{\varphi_1}T_{\varphi_2}$ is

$$\textstyle\sum_{l=-\infty}^{\infty} a_{j-l}\,b_{l-k} = \sum_{l=-\infty}^{\infty} a_{j-k-l}\,b_l\,,$$

which is the $(j-k)^{\text{th}}$ Fourier coefficient of $\varphi_1\varphi_2$.

THEOREM II. *Let* $\varphi \in L_\infty^+$ *(resp.* $L_\infty^-$*). Then* $T_\varphi$ *is invertible if and only if* $1/\varphi \in L_\infty^+$ *(resp.* $L_\infty^-$*), in which case* $T_\varphi^{-1} = T_{1/\varphi}$.

If $\varphi$, $1/\varphi \in L_\infty^+$ (resp. $L_\infty^-$), then by Lemma 1 we have $T_\varphi T_{1/\varphi} = T_{1/\varphi} T_\varphi = I$, so the sufficiency is proved. To prove necessity, we shall assume $\varphi \in L_\infty^+$, the result for $L_\infty^-$ following by taking adjoints. With $\varphi_+(\theta)$ and $\varphi_-(\theta)$ as in Theorem I, we have $\varphi\varphi_+^{-1} = \varphi_-$. Since $\varphi \in L_\infty^+$ and $\varphi_+^{-1} \in L_2^+$, we have $\varphi\varphi_+^{-1} \in L_2^+$. Moreover $\varphi_- \in L_2^-$. Thus $\varphi\varphi_+^{-1} = \varphi_- = \alpha$, a nonzero constant. Then $\varphi^{-1} = \alpha^{-1}\varphi_+^{-1} \in L_2^+$. Since, by the corollary to Theorem I, $\varphi^{-1} \in L_\infty$, we have $\varphi^{-1} \in L_\infty^+$.

THEOREM III. *Assume* $\varphi$ *is real, i.e.,* $T_\varphi$ *is self-adjoint. Then* $T_\varphi$ *is invertible if and only if either* ess sup $\varphi < 0$ *or* ess inf $\varphi > 0$.

If, for example, ess inf $\varphi = m > 0$, we have for $f \in L_2^+$,

$$(P\varphi f, f) = (\varphi f, f) \geqq m \| f \|_2^2 ,$$

so that $P\varphi$ is positive definite and therefore invertible.

Suppose now that $T\varphi$ is invertible, and let $\varphi_+$, $\varphi_-$ be as given by Theorem I. Then since $\varphi$ is real, $\varphi_+\varphi_- = \bar{\varphi}_+\bar{\varphi}_-$, or $\bar{\varphi}_- \varphi_+^{-1} = \varphi_- \bar{\varphi}_+^{-1}$. The function on the left belongs to $L_1^+$, and that on the right to $L_1^-$. Thus each is a constant $\alpha$. Then $\varphi_- = \alpha\bar{\varphi}_+$, so $\varphi = \varphi_- \varphi_+ = \alpha \mid \varphi_+ \mid^2$. Therefore either ess inf $\varphi \geqq 0$, or ess sup $\varphi \leqq 0$. But since $1/\varphi \in L_\infty$, equality cannot occur.

The following series of lemmas leads to invertibility criteria for $T_\varphi$ in case $\varphi$ possesses a sufficiently well-behaved argument.

LEMMA 2. *If* $\psi \in L_2^+$ *and* $\Re\psi \in L_\infty$, *then* $e^\psi$, $e^{-\psi} \in L_\infty^+$.

*Proof.* Let $\Psi(z)$ in $H_2$ of the unit circle be such that $\Psi(e^{i\theta}) = \psi(\theta)$. The Poisson integral representation shows that $\Re\Psi(z)$ is bounded in $|z| < 1$, so $e^{\pm\Psi(z)}$ belongs to $H_\infty$, which yields the conclusion of the lemma.

LEMMA 3. *Assume* $\varphi = \varphi_1\varphi_2$, *where* $\varphi_1$, $\varphi_1^{-1}$, $\varphi_2 \in L_\infty$, *and there may be defined an* arg $\varphi_1(\theta)$ *which belongs to* $L_2$ *and whose conjugate function belongs to* $L_\infty$. *Then* $T_\varphi$ *and* $T_{\varphi_2}$ *are equivalent, i.e.,* $T_\varphi = UT_{\varphi_2}V$ *for invertible* $U$, $V$.

*Proof.* Set $\log \varphi_1 = \log |\varphi_1| + i \arg \varphi_1$;

$$\log \varphi_1(\theta) \sim \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}.$$

A simple computation shows

$$2\Re P \log \varphi_1 = \log |\varphi_1| - C \arg \varphi_1 + \Re a_0 ,$$

so $\Re P \log \varphi_1$ is bounded. Since

$$\Re(I - P) \log \varphi_1 = \log |\varphi_1| - \Re P \log \varphi_1 ,$$

this is also bounded. Set

$$\psi_+ = \exp\,(P \log \varphi_1), \qquad \psi_- = \exp\,\big((I - P) \log \varphi_1\big).$$

It follows from Lemma 2 that $\psi_+$, $\psi_+^{-1} \in L_\infty^+$ and $\psi_-$, $\psi_-^{-1} \in L_\infty^-$. Since $\varphi = \psi_- \varphi_2 \psi_+$, Lemma 1 gives $T_\varphi = T_{\psi_-} T_{\varphi_2} T_{\psi_+}$, and by Theorem II, $T_{\psi_-}$ and $T_{\psi_+}$ are invertible.

LEMMA 4. *If $1/\varphi \in L_\infty$, $T_\varphi$ and $T_{\operatorname{sgn} \varphi}$ are equivalent.*

*Proof.* We write $\varphi = |\varphi| \operatorname{sgn} \varphi$, which is a factorization satisfying the conditions of Lemma 3 since we may take $\arg |\varphi| \equiv 0$.

It follows from the lemma that we may restrict our attention to $\varphi$ of absolute value 1. We shall assume that $\arg \varphi(\theta)$ is smooth except for a finite number of jumps. Next to a constant, the simplest such function is

$$J(\theta) = \theta - 2\pi[\theta/2\pi].$$

Thus $J(\theta) = \theta$ for $0 \leqq \theta < 2\pi$ and has period $2\pi$; it is continuous except for a jump of $-2\pi$ at $\theta = 0 \pmod{2\pi}$.

LEMMA 5. *Let $\theta_1, \cdots, \theta_n$ be distinct* (mod $2\pi$), *$\alpha_1, \cdots, \alpha_n$ real with $|\alpha_k| < \frac{1}{2}$ $(k = 1, \cdots, n)$. Then if*

$$\varphi(\theta) = \exp\left(i \textstyle\sum_{k=1}^n \alpha_k J(\theta - \theta_k)\right),$$

*$T_\varphi$ is invertible.*

*Proof.* Set

$$\varphi_+(\theta) = \prod_{k=1}^n (1 - e^{i(\theta - \theta_k)})^{\alpha_k}, \qquad \varphi_-(\theta) = e^{-i\pi \Sigma \alpha_k} \prod_{k=1}^n (1 - e^{-i(\theta - \theta_k)})^{-\alpha_k},$$

where the convention $-\pi/2 < \arg(1 - e^{i\theta}) \leqq \pi/2$ makes the powers unambiguous. That $\varphi(\theta) = \varphi_+(\theta)\varphi_-(\theta)$ (except possibly for $\theta = \theta_1, \cdots, \theta_n$) is easily verified. Now $\varphi_+(\theta)$ is the boundary function of

$$\Phi_+(z) = \prod_{k=1}^n (1 - ze^{-i\theta_k})^{\alpha_k}, \qquad |z| < 1,$$

and both $\Phi_+(z)$ and $\Phi_+(z)^{-1}$ belong to $H_2$. Therefore $\varphi_+(\theta)$, $\varphi_+(\theta)^{-1} \in L_2^+$. Similarly $\varphi_-(\theta)$, $\varphi_-(\theta)^{-1} \in L_2^-$, so we have verified conditions (a) and (b) of Theorem I; (c) remains. Since $\varphi_+^{-1}\varphi_-^{-1} = \varphi^{-1} \in L_\infty$, it suffices to show $\varphi_+^{-1} P \varphi_+$ is a bounded operator, or, by (1), that $\varphi_+^{-1} C \varphi_+$ is a bounded operator. For almost all $\omega$

$$\begin{aligned} \varphi_+^{-1} C \varphi_+ f(\omega) &= \varphi_+(\omega)^{-1} \frac{1}{2\pi} \operatorname{PV} \int_0^{2\pi} \varphi_+(\theta) f(\theta) \cot \frac{1}{2}(\omega - \theta)\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{\varphi_+(\theta)}{\varphi_+(\omega)} - 1 \right) f(\theta) \cot \frac{1}{2}(\omega - \theta)\, d\theta + Cf(\omega), \end{aligned}$$

where in the last integral the PV has been dropped since the integrand is in $L_1$. We know $\| Cf \| \leqq \| f \|$. (A norm without a subscript will mean $L_2$-norm.) Moreover

$$\left\| \int_0^{2\pi} \left| \frac{\varphi_+(\theta)}{\varphi_+(\omega)} - 1 \right| |f(\theta)|\, d\theta \right\| \leqq \left\{ \int_0^{2\pi} \int_0^{2\pi} \left| \frac{\varphi_+(\theta)}{\varphi_+(\omega)} - 1 \right|^2 d\theta\, d\omega \right\}^{1/2} \| f \|.$$

Therefore, since

$$\cot \frac{1}{2}(\omega - \theta) - \left(\frac{2}{\omega - \theta} + \frac{2}{\omega - \theta - 2\pi} + \frac{2}{\omega - \theta + 2\pi}\right)$$

is bounded for $0 < \omega, \theta < 2\pi$, it suffices to prove

$$\left\| \int_0^{2\pi} \left(\frac{\varphi_+(\theta)}{\varphi_+(\omega)} - 1\right) f(\theta) \left(\frac{1}{\omega - \theta} + \frac{1}{\omega - \theta - 2\pi} + \frac{1}{\omega - \theta + 2\pi}\right) d\theta \right\| \leqq A \|f\|,$$

which reduces to inequalities for three integrals. We consider the first, the others being entirely analogous. The relevant inequality is implied by one of the form

$$\int_0^{2\pi} g(\omega)\, d\omega \int_0^{2\pi} \left| \frac{\varphi_+(\theta)}{\varphi_+(\omega)} - 1 \right| \frac{f(\theta)}{|\omega - \theta|} d\theta \leqq A \|f\| \; \|g\|$$

holding for all nonnegative $f, g \in L_2$.

For any finite index set $L$ we have

$$\prod_{k \in L} (\xi_k + 1) = 1 + \sum_{K \subset L; K \neq 0} \prod_{k \in K} \xi_k,$$

or, replacing $\xi_k$ by $\xi_k - 1$,

$$\prod_{k \in L} \xi_k - 1 = \sum_{K \subset L; K \neq 0} \prod_{k \in K} (\xi_k - 1).$$

In our situation $L = 1, \cdots, n$, and

$$\xi_k = \left(\frac{1 - e^{i(\theta - \theta_k)}}{1 - e^{i(\omega - \theta_k)}}\right)^{\alpha_k},$$

so it suffices to prove, for each nonempty $K \subset L$, an inequality of the form

$$\int_0^{2\pi} g(\omega)\, d\omega \int_0^{2\pi} \prod_{k \in K} \left| \left(\frac{1 - e^{i(\theta - \theta_k)}}{1 - e^{i(\omega - \theta_k)}}\right)^{\alpha_k} - 1 \right| \frac{f(\theta)}{|\omega - \theta|} d\theta \leqq A \|f\| \; \|g\|.$$

Split the interval $(0, 2\pi)$ into subintervals $I_k$ with $\theta_k$ in the interior of $I_k$; then split $I_k$ into $I_k^{-1}$, $I_k^0$, $I_k^1$ with $\theta_k$ in the interior of $I_k^0$. Then it suffices to show that for each $m, m' \in L$, $-1 \leqq \varepsilon, \varepsilon' \leqq 1$,

$$Q = \int_{I_{m'}^{\varepsilon'}} g(\omega)\, d\omega \int_{I_m^{\varepsilon}} \prod_{k \in K} \left| \left(\frac{1 - e^{i(\theta - \theta_k)}}{1 - e^{i(\omega - \theta_k)}}\right)^{\alpha_k} - 1 \right| \frac{f(\theta)}{|\omega - \theta|} d\theta \leqq A \|f\| \; \|g\|,$$

*Case* 1. The intervals $I_m^{\varepsilon}$, $I_{m'}^{\varepsilon'}$ are not adjacent. Then $1/|\omega - \theta|$ is bounded, and

$$\prod_{k \in K} \left| \left(\frac{1 - e^{i(\theta - \theta_k)}}{1 - e^{i(\omega - \theta_k)}}\right)^{\alpha_k} - 1 \right| \leqq A \,|1 - e^{i(\theta - \theta_m)}|^{-|\alpha_m|}\, |1 - e^{i(\omega - \theta_{m'})}|^{-|\alpha_{m'}|},$$

so

$$Q \leqq A \int_0^{2\pi} \frac{g(\omega)}{|1 - e^{i(\omega - \theta_{m'})}|^{|\alpha_{m'}|}} d\omega \int_0^{2\pi} \frac{f(\theta)}{|1 - e^{i(\theta - \theta_m)}|^{|\alpha_m|}} d\theta \leqq A \|g\| \; \|f\|.$$

*Case* 2. The intervals are adjacent but $m' \neq m$. In this case, no $\theta_k$ touches $I_m^\varepsilon \cup I_{m'}^{\varepsilon'}$. It follows that

$$\frac{1 - e^{i(\theta-\theta_k)}}{1 - e^{i(\omega-\theta_k)}}$$

is continuous on $I_m^\varepsilon \times I_{m'}^{\varepsilon'}$ and is in fact $1 + O(|\omega - \theta|)$ there. Consequently

$$\prod_{k \in K} \left| \left( \frac{1 - e^{i(\theta-\theta_k)}}{1 - e^{i(\omega-\theta_k)}} \right)^{\alpha_k} - 1 \right| = O(|\omega - \theta|) \tag{14}$$

on $I_m^\varepsilon \times I_{m'}^{\varepsilon'}$, so $Q \leqq A \|f\| \|g\|$.

*Case* 3. The intervals are adjacent and $m' = m$. If $m \notin K$, there is a bound of the type of (14) on $I_m^\varepsilon \times I_{m'}^{\varepsilon'}$, and $Q \leqq A \|f\| \|g\|$. We assume therefore that $m \in K$. Then

$$\begin{aligned} Q &\leqq A \int_{I_m} g(\omega)\, d\omega \int_{I_m} \left| \left( \frac{1 - e^{i(\theta-\theta_m)}}{1 - e^{i(\omega-\theta_m)}} \right)^{\alpha_m} - 1 \right| \frac{f(\theta)}{|\omega - \theta|}\, d\theta \\ &= A \int_0^{2\pi} g_1(\omega)\, d\omega \int_0^{2\pi} \left| \left( \frac{1 - e^{i\theta}}{1 - e^{i\omega}} \right)^{\alpha} - 1 \right| \frac{f_1(\theta)}{|\omega - \theta|}\, d\theta, \end{aligned}$$

where we have set $\alpha = \alpha_m$, and

$$f_1(\theta) = \begin{cases} f(\theta + \theta_m), & \theta + \theta_m \in I_m, \\ 0, & \text{otherwise}, \end{cases} \qquad g_1(\theta) = \begin{cases} g(\theta + \theta_m), & \theta + \theta_m \in I_m, \\ 0, & \text{otherwise}. \end{cases}$$

By symmetry it is clear we may assume $\alpha > 0$. We next use a device suggested by H. Pollard. We change variables:

$$Q \leqq A \int_0^\infty \frac{du}{|u - 1|} \int_{\substack{0<\omega<2\pi \\ 0<u\omega<2\pi}} \left| \left( \frac{1 - e^{iu\omega}}{1 - e^{i\omega}} \right)^{\alpha} - 1 \right| f_1(u\omega) g_1(\omega)\, d\omega.$$

Now

$$\left| \frac{1 - e^{iu\omega}}{1 - e^{i\omega}} - 1 \right| = \left| \frac{1 - e^{i(u-1)\omega}}{1 - e^{i\omega}} \right| \leqq A\, |u - 1|$$

for $0 < \omega < 2\pi$, $0 < u\omega < 2\pi$. Therefore

$$\left| \left( \frac{1 - e^{iu\omega}}{1 - e^{i\omega}} \right)^{\alpha} - 1 \right| \leqq \begin{cases} A\, |u - 1| & \text{for all } u, \\ A(u - 1)^{\alpha} & \text{for } u \geqq 2. \end{cases}$$

Thus

$$\begin{aligned} &\int_0^2 \frac{du}{|u - 1|} \int_{\substack{0<\omega<2\pi \\ 0<u\omega<2\pi}} \left| \left( \frac{1 - e^{iu\omega}}{1 - e^{i\omega}} \right)^{\alpha} - 1 \right| f_1(u\omega) g_1(\omega)\, d\omega \\ &\quad \leqq A \int_0^2 du \int_{\substack{0<\omega<2\pi \\ 0<u\omega<2\pi}} f_1(u\omega) g_1(\omega)\, d\omega \\ &\quad\quad \leqq A \int_0^2 du \left\{ \int_{0<u\omega<2\pi} f_1(u\omega)^2\, d\omega \right\}^{1/2} \left\{ \int_{0<\omega<2\pi} g_1(\omega)^2\, d\omega \right\}^{1/2} \\ &\quad\quad = A \|f\| \|g\| \int_0^2 u^{-1/2}\, du = A \|f\| \|g\|, \end{aligned}$$

and similarly

$$\int_2^\infty \frac{2}{u-1} \int_{\substack{0<\omega<2\pi \\ 0<u\omega<2\pi}} \left| \left( \frac{1-e^{iu\omega}}{1-e^{i\omega}} \right)^\alpha - 1 \right| f_1(u\omega) g_1(\omega)\, d\omega$$

$$\leqq A \,\| f \| \;\| g \| \int_2^\infty \frac{du}{u^{1/2}(u-1)^{1-\alpha}} = A \,\| f \| \;\| g \| .$$

This completes the proof of Lemma 5.

Call a periodic function $f(\theta)$ *nice* if it is continuous and either

(a) $f(\theta)$ has an absolutely convergent Fourier series, or

(b) the modulus of continuity $\omega(\delta)$ of $f(\theta)$ is such that $\omega(\delta)/\delta$ is integrable near $\delta = 0$.

THEOREM IV. *Assume* $1/\varphi \in L_\infty$ *and that there may be defined an* $\arg \varphi(\theta)$ *which is continuous except for jumps at* $\theta_1, \cdots, \theta_m \pmod{2\pi}$. *Defining*

$$\alpha_k = (1/2\pi)\, \{\arg \varphi(\theta_k+) - \arg \varphi(\theta_k-)\},$$

*assume that the continuous function*

$$H(\theta) = \arg \varphi(\theta) + \textstyle\sum_{k=1}^m \alpha_k\, J(\theta - \theta_k)$$

*is nice. Write* $\alpha_k = \beta_k + \gamma_k$, *where* $\beta_k$ *is an integer and* $-\frac{1}{2} < \gamma_k \leqq \frac{1}{2}$.

*A necessary condition that* $T_\varphi$ *be invertible is that each* $\gamma_k < \frac{1}{2}$. *If this holds, then*

(i) $\sum \beta_k = 0$ *implies* $T_\varphi$ *invertible*;

(ii) $\sum \beta_k < 0$ *implies* $T_\varphi$ *is one-one with range a subspace of deficiency* $-\sum \beta_k$ ;

(iii) $\sum \beta_k > 0$ *implies* $T_\varphi$ *is onto and has null space of dimension* $\sum \beta_k$ .

By Lemma 4 we may assume $|\varphi| \equiv 1$. Consider first the case when each $\gamma_k < \frac{1}{2}$. We have $\varphi = \varphi_1 \varphi_2 \varphi_3$, where

$$\varphi_1(\theta) = e^{iH(\theta)}, \qquad \varphi_2(\theta) = e^{-i\Sigma\beta_k J(\theta-\theta_k)}, \qquad \varphi_3(\theta) = e^{-\Sigma\gamma_k J(\theta-\theta_k)}.$$

By Lemma 3 (using the fact that $H$ nice implies $CH$ bounded) $T_\varphi$ is equivalent to $T_{\varphi_2 \varphi_3}$. Since each $\beta_k$ is an integer, $\beta_k(J(\theta) - \theta)$ is an integral multiple of $2\pi$ for all $\theta$, so

$$\varphi_2(\theta) = e^{-i\Sigma\beta_k\theta} e^{i\Sigma\beta_k\theta_k} = \text{constant}\cdot e^{in\theta},$$

where we have set

$$n = -\sum \beta_k .$$

Thus if we denote by $e_n$ the function whose value at $\theta$ is $e^{in\theta}$, $T_\varphi$ is equivalent to $T_{e_n \varphi_3}$. Lemma 5 tells us that $T_{\varphi_3}$ is invertible. Thus we have (i). If $n > 0$, we have by Lemma 1

$$T_{e_n \varphi_3} = T_{\varphi_3} T_{e_n} ,$$

the operator $T_{e_n}$ being one-one with range of deficiency $n$. Similarly $n < 0$ gives

$$T_{e_n \varphi_n} = T_{e_n} T_{\varphi_3},$$

and $T_{e_n}$ is onto and has null space of dimension $-n$. Therefore (ii) and (iii) are proved.

To show $T_\varphi$ is not invertible if some $\gamma_k = \frac{1}{2}$, we approximate by noninvertible matrices. Assume first that $\sum \beta_k \geqq 0$, and for small positive $\varepsilon$ set

$$\varphi_\varepsilon(\theta) = \exp\left(i[\arg \varphi(\theta) - \varepsilon \textstyle\sum_{k=1}^m J(\theta - \theta_k)]\right),$$

so that

$$\arg \varphi_\varepsilon(\theta) = \arg \varphi(\theta) - \varepsilon \textstyle\sum_{k=1}^m J(\theta - \theta_k).$$

Denote by $\alpha_k^\varepsilon$ the jumps of $\arg \varphi_\varepsilon(\theta)$ with corresponding $\beta_k^\varepsilon$, $\gamma_k^\varepsilon$. Since $\alpha_k^\varepsilon > \alpha_k$ we have $\beta_k^\varepsilon \geqq \beta_k$ for all $k$, and $\beta_k^\varepsilon > \beta_k$ if $\gamma_k = \frac{1}{2}$. Therefore $\sum \beta_k^\varepsilon > 0$. Since, for small enough $\varepsilon$, no $\gamma_k^\varepsilon = \frac{1}{2}$, we may apply (iii) to conclude that $T_{\varphi_\varepsilon}$ is not invertible. Since $\varphi_\varepsilon \to \varphi$ uniformly as $\varepsilon \to 0$, $T_{\varphi_\varepsilon} \to T_\varphi$ in norm, so $T_\varphi$ is not invertible. A similar argument takes care of the case $\sum \beta_k \leqq 0$.

COROLLARY 1. *Assume $\varphi(\theta)$ is nice and $\varphi(\theta) \neq 0$. Set*

$$n = (1/2\pi)\Delta_{0 \leqq \theta \leqq 2\pi} \arg \varphi(\theta).$$

*Then*

(i) *$n = 0$ implies $T_\varphi$ invertible;*

(ii) *$n > 0$ implies $T_\varphi$ is one-one with range a subspace of deficiency $n$;*

(iii) *$n < 0$ implies $T_\varphi$ is onto and has null space of dimension $-n$.*

*Proof.* If $\arg \varphi(\theta)$ is continuous for $0 < \theta < 2\pi$, it has a jump of $-2\pi n$ at $\theta = 0$. Therefore $\gamma = 0$, $\beta = -n$, and $H(\theta) = \arg \psi(\theta)$, where we have set $\psi(\theta) = \varphi(\theta)e^{-in\theta}$. The result will follow from Theorem IV if $\arg \psi(\theta)$ is nice, and so certainly if $\log \psi(\theta)$ is nice. In case $\varphi$ has an absolutely convergent Fourier series, so does $\psi$, and since $\Delta \arg \psi = 0$, $\log \psi$ has an absolutely convergent Fourier series (see [1], Lemma of §2) and so is nice. If the modulus of continuity of $\varphi$ is $\omega(\delta)$, then that of $\log \psi$ is at most $A\omega(\delta)$. Thus in either case $\varphi$ nice implies $\log \psi$ nice.

COROLLARY 2. *There is a $\varphi$ such that $T_\varphi$ is invertible while $T_{\varphi^2}$ is not.*

*Proof.* We need only take $\varphi(\theta) = e^{i\alpha\theta}$ $(0 \leqq \theta < 2\pi)$ where $\frac{1}{4} < \alpha < \frac{1}{2}$.

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