HOMOTOPY GROUPS, COMMUTATORS, AND Γ-GROUPS

BY
Daniel M. Kan

1. Introduction

In [7] J. H. C. Whitehead introduced for a simply connected complex K an exact sequence

$$\to H_{n+1}(K) \xrightarrow{\nu_{n+1}} \Gamma_n(K) \xrightarrow{\lambda_n} \pi_n(K) \xrightarrow{\mu_n} H_n(K) \to$$

(called Γ-sequence) involving the homotopy groups $\pi_n(K)$, the homology groups $H_n(K)$, and a new kind of groups $\Gamma_n(K)$, called the Γ-groups of K.

As was shown in [3], *homology groups are*, in a certain precise sense, *"obtained from the homotopy groups by abelianization."* The above exact sequence suggests that between Γ-groups and homotopy groups a dual relationship might exist. It is the purpose of this note to show that this is indeed the case, and that the Γ-groups are, in a similar sense, *"obtained from the homotopy groups by taking commutator subgroups."*

The result will be stated in terms of c.s.s. complexes and c.s.s. groups. We shall freely use the notation and results of [3] and [4].

The main step in the argument is a rather curious lemma on connected c.s.s. groups. It states that for a connected c.s.s. group F and any integer $n \geqq 2$, every n-simplex in the commutator subgroup of F is homotopic with an n-simplex in the commutator subgroup of the $(n - 1)$-skeleton of F.

2. The main lemma

We shall state a lemma which describes a rather surprising property of connected c.s.s. groups. The lemma shows how connectedness, although its definition involves only 0-simplices and 1-simplices, influences quite strongly the behaviour of a c.s.s. group in all higher dimensions. This explains somewhat why connectedness is such a strong condition to impose on a c.s.s. group or, equivalently, (cf. [4], §§9 and 11) why simple connectedness is such a strong condition to impose on a CW-complex or a c.s.s. complex.

For another application of this lemma see [6].

Let F be a c.s.s. group; denote by $[F, F] \subset F$ the *commutator subgroup*, i.e., the (c.s.s.) subgroup such that $[F, F]_n = [F_n, F_n]$ for all n; and for every integer $s \geqq 0$ let $F^s \subset F$ be the *s-skeleton*, i.e., the smallest (c.s.s.) subgroup containing F_s. Then we have

Lemma 2.1. *Let F be a connected c.s.s. group, and let $\sigma \in [F, F]_n$, where $n \geqq 2$. Then there exist elements*

$$\phi \in [F^{n-1}, F^{n-1}]_n \quad \textit{and} \quad \rho \in [F^n, F^n]_{n+1}$$

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such that

$$\rho\varepsilon^{n+1} = \sigma\cdot\phi^{-1}, \qquad \rho\varepsilon^{i} = e_n \quad \textit{for} \quad 0 \leqq i \leqq n,$$

or, equivalently, such that

$$(\rho^{-1}\cdot\sigma\eta^{n}) : \sigma \sim \phi \qquad \text{([3], \textit{Definition} 2.2).}$$

Proof. The proof of Lemma 2.1 is almost the same as that of [6], Lemma (3.5) ([6], §8). The only difference is that, instead of [6], Lemma $(8.1)_k$, the following lemma has to be used.

LEMMA 2.2_k. *Let F be a connected c.s.s. group, and let $\sigma \in F_n$ and $\tau \in F_k$, where $n \geqq 2$ and $k \leqq n$. Then there exist elements*

$$\phi \in [F^{n-1}, F^{n-1}]_n \quad \textit{and} \quad \rho \in [F^{n}, F^{n}]_{n+1}$$

such that

$$\rho\varepsilon^{n+1} = [\sigma, \tau\eta^{k} \cdots \eta^{n-1}]\cdot\phi^{-1},$$

$$\rho\varepsilon^{i} = e_n, \qquad 0 \leqq i < n+1.$$

Proof. We first prove the case $k = 0$. The connectedness of F implies the existence of a $\psi \in F_1$ such that $\psi\varepsilon^0 = e_0$ and $\psi\varepsilon^1 = \tau$. Let

$$\gamma = \textstyle\prod_{i=0}^{n} [\sigma\eta^{i}, \psi\eta^{0} \cdots \eta^{i-1}\eta^{i+1} \cdots \eta^{n}]^{(-1)^{n+i+1}}.$$

Then it is readily verified that $\gamma\varepsilon^{i} \in [F^{n-1}, F^{n-1}]_n$ for $0 \leqq i < n+1$. Let $\nu = m(\gamma\varepsilon^{0}, \cdots, \gamma\varepsilon^{n})$, where the function m is as in [6], §5, and let

$$\rho = \gamma^{-1}\cdot\nu$$

$$\phi = \nu^{-1}\varepsilon^{n+1}\cdot\textstyle\prod_{i=0}^{n-1} [\sigma\varepsilon^{n}\eta^{i}, \psi\eta^{0} \cdots \eta^{i-1}\eta^{i+1} \cdots \eta^{n-1}]^{(-1)^{n+i+1}};$$

then a simple computation yields that ϕ and ρ have the desired properties.

The proof for $k > 0$ is the same as that of [6], Lemma $(8.1)_k$.

3. Application of the main lemma

Let F be a c.s.s. group. Denote by $j^n : F^{n-1} \to F^n$ and $j : F^n \to F$ the inclusion maps, and for any c.s.s. group G let $\pi_n(G) = \pi_n(G; e_0)$. Then Lemma 2.1 together with J. C. Moore's definition of the homotopy groups of a c.s.s. group ([3], §5) implies

COROLLARY 3.1. *Let F be a connected c.s.s. group. Then for every integer $n \geqq 2$ the map*

$$\pi_n([j^n, j^n]) : \pi_n([F^{n-1}, F^{n-1}]) \to \pi_n([F^n, F^n])$$

is an epimorphism, and the map

$$\pi_n([j, j]) : \pi_n([F^n, F^n]) \to \pi_n([F, F])$$

is an isomorphism.

We will need the following application of this corollary.

Let F be a c.s.s. group, let its (-1)-skeleton F^{-1} be the subgroup containing only the identity in every dimension, and for every integer $s \geqq -1$ let $B^s = F^s/[F^s, F^s]$ be "F^s made abelian." Then the sequence

$$[F^s, F^s] \xrightarrow{q^s} F^s \xrightarrow{p^s} B^s,$$

where q^s denotes the inclusion map and p^s the projection, is a fibre sequence ([3], §3) with which is associated an exact homotopy sequence. Hence in the following diagram both horizontal sequences are exact; and clearly the rectangle is commutative.

$$\begin{array}{ccccccc}
\pi_n(B^{n-2}) & \xrightarrow{\partial} & \pi_{n-1}([F^{n-2}, F^{n-2}]) & \xrightarrow{\pi_{n-1}(q^{n-2})} & \pi_{n-1}(F^{n-2}) & \xrightarrow{\pi_{n-1}(p^{n-2})} & \pi_{n-1}(B^{n-2}) \\
 & & \downarrow{\scriptstyle \pi_{n-1}([j^{n-1}, j^{n-1}])} & & \downarrow{\scriptstyle \pi_{n-1}(j^{n-1})} & & \\
\pi_n(B^{n-1}) & \xrightarrow{\partial} & \pi_{n-1}([F^{n-1}, F^{n-1}]) & \xrightarrow{\pi_{n-1}(q^{n-1})} & \pi_{n-1}(F^{n-1}). & &
\end{array}$$

Clearly B^s coincides with its own s-skeleton, and as the homotopy groups of the s-skeleton of an abelian c.s.s. group vanish in dimension $> s$ (this is readily verified by using the equivalence of the notions of abelian c.s.s. group and chain complex of [1]), it follows that $\pi_{n-1}(q^{n-2})$ is an isomorphism and $\pi_{n-1}(q^{n-1})$ a monomorphism. Hence $\pi_{n-1}(q^{n-1})$ induces an isomorphism

$$\phi_{n-1} : \textit{image } \pi_{n-1}([j^{n-1}, j^{n-1}]) \approx \textit{image } \pi_{n-1}(j^{n-1}).$$

Denote by $\chi_{n-1}(F)$ the composition of ϕ_{n-1}^{-1} with

$$\pi_{n-1}([j, j]) : \pi_{n-1}([F^{n-1}, F^{n-1}]) \to \pi_{n-1}([F, F]).$$

Then we have

PROPOSITION 3.2. *Let F be a connected c.s.s. group. Then the map*

$$\chi_{n-1}(F) : \textit{image } \pi_{n-1}(j^{n-1}) \to \pi_{n-1}([F, F])$$

is an isomorphism for $n > 2$ and $n = 1$. If F is free, ([4], *Definition* 5.1), *then this is also the case for $n = 2$.*

Proof. For $n > 2$ the proposition is an immediate consequence of Corollary 3.1, while for $n = 1$ it follows from the fact that $\pi_0(F^{-1}) = 1$ and $\pi_0([F, F]) = 1$.

If F is free, then application of [3], Theorem 17.6 and the exactness of the homotopy sequence of the fibre sequence ([3], §3)

$$[F, F] \to F \to F/[F, F]$$

yields that $\pi_1([F, F]) = 1$. The proposition then follows from the fact that $\pi_1(F^0) = 1$.

Remark 3.3. One might ask if the freeness condition in the second half of Proposition 3.2 could be dropped, or more generally (see the proof of Proposition 3.2) if the freeness condition could be omitted in [3], Theorem 17.6, the analogue for c.s.s. groups of the Hurewicz theorem. The answer is negative; a counterexample will be given in §6.

4. Γ-groups and γ-groups

This section deals with the c.s.s. analogue of J. H. C. Whitehead's definition of the Γ-groups ([2], p. 105) and with new groups, called γ-groups. The latter are in some sense "obtained from the homotopy groups by taking commutator subgroups." To be more exact: if in the definition of homotopy groups of [3], §8 we insert at a certain stage the operation of taking commutator subgroups, then we obtain a definition of the γ-groups. It will be shown (Theorem 4.3) that for simply connected complexes the Γ-groups and γ-groups are isomorphic.

Only reduced complexes will be considered, i.e., c.s.s. complexes with only one 0-simplex. This restriction is not essential; its main advantage is that there is no need to indicate the base point.

DEFINITION 4.1. Let K be a reduced complex, and for every integer $n \geqq 0$ let K^n be its n-skeleton (i.e., the smallest subcomplex containing K_n) and $i^n \colon K^{n-1} \to K^n$ the inclusion map. Then $\Gamma_n(K)$, the n^{th} *Γ-group of K*, is defined by

$$\Gamma_n(K) = \mathit{image}\ \big(\pi_n(i^n) \colon \pi_n(K^{n-1}) \to \pi_n(K^n)\big).$$

DEFINITION 4.2. Let K be a reduced complex, and let GK be as in [3], §7. (GK is a free c.s.s. group which has the homotopy type of the loops on K.) Then for every integer $n > 0$ we define $\gamma_n(K)$, the n^{th} *γ-group of K*, by

$$\gamma_n(K) = \pi_{n-1}([GK, GK]).$$

In order to be able to compare the groups $\Gamma_n(K)$ and $\gamma_n(K)$, we will define a homomorphism $\psi_n \colon \Gamma_n(K) \to \gamma_n(K)$ as follows. Let $i \colon K^n \to K$ be the inclusion map. Then it follows immediately from the definition of the functor G ([4], §10) that $G(i) \colon GK^n \to GK$ maps GK^n isomorphically onto the $(n-1)$-skeleton $G^{n-1}K$ of GK. We therefore may identify GK^n with $G^{n-1}K$ under this isomorphism. By [3], §8 there exist natural isomorphisms $\partial \colon \pi_n(K) \approx \pi_{n-1}(GK)$. Hence the diagram

$$\begin{array}{ccc} \pi_n(K^{n-1}) & \xrightarrow{\ \pi_n(i^n)\ } & \pi_n(K^n) \\ \partial \big\downarrow \approx & & \partial \big\downarrow \approx \\ \pi_{n-1}(G^{n-2}K) & \xrightarrow{\ \pi_{n-1}(j^{n-1})\ } & \pi_{n-1}(G^{n-1}K) \end{array}$$

is commutative, and it follows that ∂ induces isomorphisms

$$\partial \colon \Gamma_n(K) \approx \mathit{image}\ \pi_{n-1}(j^{n-1}).$$

Define $\psi_n : \Gamma_n(K) \to \gamma_n(K)$ as the composition

$$\Gamma_n(K) \xrightarrow[\approx]{\partial} \textit{image } \pi_{n-1}(j^{n-1}) \xrightarrow{\chi_{n-1}(GK)} \gamma_n(K).$$

Then our main result is

THEOREM 4.3. *Let K be a reduced complex, such that $\pi_1(K) = 1$. Then*

$$\psi_n : \Gamma_n(K) \to \gamma_n(K)$$

is an isomorphism for all $n > 0$.

Proof. This is an immediate consequence of Proposition 3.2 and [4], Proposition 10.2.

In order to compare the homotopy groups of the commutator subgroup of a c.s.s. group F with the γ-groups of its classifying complex $\overline{W}F$, consider the map $\alpha'(i): G\overline{W}F \to F$ of [4], §11. This map induces homomorphisms

$$\pi_{n-1}([\alpha'(i), \alpha'(i)]): \gamma_n(\overline{W}F) \to \pi_{n-1}([F, F]).$$

That these homomorphisms need not be isomorphisms may be seen by taking for F an abelian c.s.s. group. However

THEOREM 4.4. *Let F be a free c.s.s. group* ([4], §5). *Then*

$$\pi_{n-1}([\alpha'(i), \alpha'(i)]): \gamma_n(\overline{W}F) \to \pi_{n-1}(| F, F |)$$

is an isomorphism for all $n > 0$.

Proof. By [4], Theorem 11.3, $\alpha'(i)$ is a loop homotopy equivalence. Clearly the functor "taking the commutator subgroup" is a c.s.s. functor in the sense of [5], Definition 5.2, and hence the theorem follows from [5], Theorem 5.3.

5. The Γ-sequence and γ-sequence

DEFINITION 5.1. Let K be a reduced complex, let GK be as in [3], §7, and let $AK = GK/[GK, GK]$, i.e., AK is "GK made abelian." Then we define the *γ-sequence of K* as the homotopy sequence of the fibre sequence ([3], §3)

$$[GK, GK] \xrightarrow{q} GK \xrightarrow{p} AK,$$

where q denotes the inclusion map and p the projection.

An immediate consequence of Definition 4.1 and [3], Proposition 3.5 is

PROPOSITION 5.2. *The γ-sequence is exact.*

For simply connected complexes the γ-sequence is isomorphic with the Γ-sequence of J. H. C. Whitehead. In fact

THEOREM 5.3. *Let K be a reduced complex, such that $\pi_1(K) = 1$. Then we have a commutative diagram*

5.4

$$\begin{array}{ccccccccc}
\to H_{n+1}(K) & \xrightarrow{\nu_{n+1}} & \Gamma_n(K) & \xrightarrow{\lambda_n} & \pi_n(K) & \xrightarrow{\mu_n} & H_n(K) \to \\
\approx \downarrow \alpha_{n+1} & & \approx \downarrow \psi_n & & \approx \downarrow \partial & & \approx \downarrow \alpha_n \\
\to \pi_n(AK) & \xrightarrow{\partial_n(q,p)} & \pi_{n-1}([GK, GK]) & \xrightarrow{\pi_{n-1}(q)} & \pi_{n-1}(GK) & \xrightarrow{\pi_{n-1}(p)} & \pi_{n-1}(AK) \to
\end{array}$$

where the upper row is the Γ*-sequence of* K ([2], *p.* 105), *and where the isomorphism* $\alpha_n : H_n(K) \approx \pi_{n-1}(AK)$ *is as in* [3], §15.

Proof. The map $\mu_n : \pi_n(K) \to H_n(K)$ is the Hurewicz homomorphism, and hence, by [3], Theorem 16.1, the rectangle on the right of Diagram 5.4 is commutative.

The map $\lambda_n : \Gamma_n(K) \to \pi_n(K)$ is the one induced by the inclusion map $i: K^n \to K$. Commutativity in the rectangle in the middle therefore follows from the commutativity of the diagram

$$\begin{array}{ccccc}
\pi_{n-1}([G^{n-1}K, G^{n-1}K]) & \xrightarrow{\pi_{n-1}(q^{n-1})} & \pi_{n-1}(G^{n-1}K) & \underset{\approx}{\xrightarrow{\partial}} & \pi_n(K^n) \\
\downarrow \pi_{n-1}([j, j]) & & \downarrow \pi_{n-1}(j) & & \downarrow \pi_n(i) \\
\pi_{n-1}([GK, GK]) & \xrightarrow{\pi_{n-1}(q)} & \pi_{n-1}(GK) & \underset{\approx}{\xrightarrow{\partial}} & \pi_n(K).
\end{array}$$

The proof of the fact that commutativity also holds in the rectangle on the left is similar, although more complicated, as at this point the simple connectedness of K has to be used. The details will be left to the reader.

6. A counterexample

Let F be a c.s.s. group. Because of the exactness of the homotopy sequence of the fibre sequence

$$[F, F] \to F \to F/[F, F],$$

the analogue for c.s.s. groups of the Hurewicz theorem ([3], Theorem 17.6) is equivalent to the statement that for a *free* c.s.s. group which is $(n-1)$-connected, its commutator subgroup is n-connected. The following example shows that this statement becomes false if the word free is omitted.

Let $n > 0$, and let K be a c.s.s. complex of which the only nondegenerate simplices are

(i) one 0-simplex ϕ,
(ii) two n-simplices π and ρ,
(iii) two $(n+1)$-simplices σ and τ with faces

$$\sigma\varepsilon^0 = \pi, \qquad \tau\varepsilon^0 = \rho,$$
$$\sigma\varepsilon^i = \phi\eta^0 \cdots \eta^{n-1}, \qquad \tau\varepsilon^i = \phi\eta^0 \cdots \eta^{n-1}, \qquad i > 0.$$

Let GK be as in [3], §7, and let R be the c.s.s. group obtained from GK by addition of the relation

$$[\bar{\pi}, \bar{\rho}] = e_{n-1} .$$

It is readily verified that R is $(n-1)$-connected. However

PROPOSITION 6.1. $\pi_n([R, R]) \neq 1$.

Proof. GK has the homotopy type of the loops on K ([3], §7) and hence is contractible. Let $p: GK \rightarrow R$ denote the projection, and let $Q = kernel\ p$. Then the contractibility of $[GK, GK]$ together with the exactness of the homotopy sequence of the fibre sequence

$$kernel\ [p, p] \rightarrow [GK, GK] \rightarrow [R, R]$$

implies that it suffices to show that $\pi_{n-1}(kernel\ [p, p]) \neq 1$. We shall do this using the notation and results of [3], §18.

Let $q: R_n \rightarrow G_n K$ be the function such that for every element $\alpha \in R_n$

(i) $pq\alpha = \alpha$,

(ii) $length\ q\alpha = length\ \alpha$,

(iii) $q\alpha$ is such that $\bar{\rho}\eta^i$ or $\bar{\rho}^{-1}\eta^i$ is never followed by $\bar{\pi}\eta^i$ or $\bar{\pi}^{-1}\eta^i$ $(0 \leqq i \leqq n-1)$. Then clearly the elements $q\alpha$, where $\alpha \in R_n$, form a Schreier system of representatives for the cosets of Q_n in $G_n K$, and it follows from the Kurosch-Schreier theorem that Q_n is freely generated by elements of the form

$$\beta \cdot [\bar{\rho}^t, \bar{\pi}]\eta^i \cdot \beta^{-1},$$

where the elements $\beta \in G_n K$ are suitably chosen.

For every integer i with $0 \leqq i \leqq n-1$, the elements $\gamma\eta^i$, where $\gamma \in {}^{(n-1-i)}Q_i$, form a Schreier system of representatives for the cosets of ${}^{(n-i)}Q_i$ in ${}^{(n-1-i)}Q_{i+1}$. Hence by the Kurosch-Schreier theorem ${}^{(1)}Q_{n-1}$ is freely generated by elements of the form

$$\beta_{n-1} \cdot [\bar{\rho}^t, \bar{\pi}]\eta^{n-1} \cdot \beta_{n-1}^{-1} \cdot \beta \cdot [\bar{\pi}, \bar{\rho}^t]\eta^{n-1} \cdot \beta^{-1}, \qquad \beta_i \cdot [\bar{\rho}^t, \bar{\pi}]\eta^i \cdot \beta_i^{-1},$$

where $i < n-1$, and the elements $\beta, \beta_0, \cdots, \beta_{n-1} \in Q_n$ are suitably chosen. As

$$\beta_{n-1} \cdot [\bar{\rho}^t, \bar{\pi}]\eta^{n-1} \cdot \beta_{n-1}^{-1} \cdot \beta \cdot [\bar{\pi}, \bar{\rho}^t]\eta^{n-1} \cdot \beta^{-1} \in [G_n K, [G_n K, G_n K]],$$

iterated application of the Kurosch-Schreier theorem yields that ${}^{(n-i)}Q_i$ is freely generated by certain elements of $[G_n K, [G_n K, G_n K]]$ and elements of the form

$$\beta_j \cdot [\bar{\rho}^t, \bar{\pi}]\eta^j \cdot \beta_j^{-1},$$

where $j < i$, and the elements $\beta_j \in {}^{(n-1-i)}Q_{i+1}$ are suitably chosen. Hence

$${}^{(n)}Q_0 = \tilde{Q}_n \subset [G_n K, [G_n K, G_n K]],$$

and it follows ([3], §5) that

$$\tilde{Q}_{n-1} \cap image\ \tilde{\partial}_n \subset [G_{n-1} K, [G_{n-1} K, G_{n-1} K]].$$

However we have

$$[\bar{\pi}, \bar{\rho}]\varepsilon^i = e_{n-2}, \qquad 0 \leqq i \leqq n-1,$$

$$[\bar{\pi}, \bar{\rho}] \not\subset [G_{n-1} K, [G_{n-1} K, G_{n-1} K]],$$

and consequently $[\bar{\pi}, \bar{\rho}]$ represents a nontrivial element of $\pi_{n-1}(Q)$ and hence of $\pi_{n-1}(\textit{kernel}\ [p, p])$.

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Massachusetts Institute of Technology
Cambridge, Massachusetts