BOOLEAN RINGS AND BANACH LATTICES[1]

BY

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1. Introduction

Let X be a Banach lattice of measurable functions. If $\chi_e \in X$ is the characteristic function of a set e, $\Phi(e) = \| \chi_e \|$ is a function defined on a certain Boolean ring of sets. In this paper we consider the following problem. If a function $\Phi(e)$ is given on a Boolean ring B, what are the conditions under which B can be imbedded into a vector lattice X and Φ extended into a norm on X? Under what conditions on Φ is it possible to postulate some additional properties of X? Answers to such questions are given in Sections 2, 3, 5. This leads in Section 6 to a natural generalization of certain spaces introduced by one of the authors [4] under the name of spaces Λ. We consider abstract Boolean rings B and correspondingly functions in the sense of Carathéodory [3]. The reader may substitute for this, if he so wishes, Boolean rings of sets and point-functions. This substitution would not lead to any simplification of the proofs.

2. Extension of a multiply subadditive function into a norm

Let B be a Boolean ring, i.e., a distributive, relatively complemented lattice with zero element (a Boolean ring is a Boolean algebra if and only if it contains a unit). Let $\Phi(e)$ be a real valued function defined on B. We will discuss extensions of B into a vector lattice S such that unions of disjoint elements of B become sums, intersections become products, and the order is preserved, and at the same time extensions of Φ into a seminorm on S.

The smallest extension of B of this kind is the vector lattice S of step-functions. The elements of S are formal sums $x = \sum_{k=1}^{n} a_k e_k$ (where e_k is also the characteristic function of the set e_k) with an obvious identification rule (see [5], [3]).

A seminorm $P(x)$ on a vector space satisfies the following relations:

$$\text{(a)} \quad P(x) \geqq 0, \qquad \text{(b)} \quad P(ax) = |a| P(x),$$

$$\text{(c)} \quad P(x_1 + x_2) \leqq P(x_1) + P(x_2).$$

Other natural conditions for $P(x)$ are

$$\text{(d)} \quad P(x) \leqq P(y) \quad \text{for } 0 \leqq x \leqq y, \qquad \text{(e)} \quad P(|x|) = P(x).$$

THEOREM 1. (α). *A real valued function Φ on B has an extension P onto S which is a seminorm (we call such Φ norm-generating) if and only if Φ satisfies*

(i) $$\Phi(e) \leqq \sum_{k=1}^{n} |a_k| \Phi(e_k) \qquad \textit{for } e = \sum a_k e_k, \quad e, e_k \in B;$$

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there is a maximal extension of this type. (β). *The function* Φ *has an extension* P *which satisfies* (a)–(c) *and* (d) *or* (e) *if and only if* Φ *is norm-generating and increasing*:

$$\text{(ii)} \qquad \Phi(e_1) \leqq \Phi(e_2) \qquad \text{for } e_1 \subset e_2,$$

or, alternatively, increasing and multiply subadditive:

$$\text{(iii)} \qquad p\Phi(e) \leqq \sum_{k=1}^{n} \Phi(e_k) \quad \text{whenever} \quad pe = \sum_{k=1}^{n} e_k .$$

Proof. (α) The necessity of the condition is obvious. If it is satisfied, we first conclude (applying it to the representation $e = e - e + e$) that $\Phi(e) \geqq 0$. Put

$$\text{(1)} \qquad P_0(x) = \inf \sum_{k=1}^{n} | a_k | \Phi(e_k),$$

where the infimum is taken for all representations $x = \sum a_k e_k$ of x with $e_k \in B$. Clearly P_0 satisfies (a)–(c), and (i) implies that P_0 is an extension of Φ. If P is any seminorm which satisfies $P(e) \leqq \Phi(e)$ for $e \in B$, then for $x = \sum a_k e_k$, $P(x) \leqq \sum | a_k | \Phi(e_k)$; hence $P(x) \leqq P_0(x)$.

(β) Condition (ii) is necessary for (d). It is also necessary for (e), since if $e_1 \cap e_2 = 0$, (e) and (c) imply

$$2\Phi(e_1 + e_2) = P(e_1 + e_2) + P(e_1 - e_2) \geqq P(2e_1) = 2\Phi(e_1).$$

Assume now that (i) and (ii) hold. We define for $x \geqq 0$

$$\text{(2)} \qquad P_1(x) = \inf \sum_{k=1}^{n} a_k \Phi(e_k) \qquad \text{for } x = \sum a_k e_k, \quad a_k \geqq 0,$$

$$\text{(3)} \qquad P_1(x) = P_1(| x |), \qquad x \in S.$$

From (i) we see that $P_1(e) = \Phi(e)$, $e \in B$, and it is clear that P_1 satisfies (a)–(c) (with $a \geqq 0$ in (b)) on the cone C of positive elements of S. If $0 \leqq x \leqq y$ and $y = \sum a_k e_k$, $a_k \geqq 0$, there exists a representation [2, p. 19]

$$x = \sum_{k,l} a_{kl} e_{kl}, \qquad a_{kl} \geqq 0, \quad \sum_l a_{kl} \leqq a_k, \quad e_{kl} \subset e_k .$$

Hence by (ii)

$$\sum_{k,l} a_{kl} \Phi(e_{kl}) \leqq \sum_k \Phi(e_k) \sum_l a_{kl} \leqq \sum_k a_k \Phi(e_k)$$

and $P_1(x) \leqq \sum a_k \Phi(e_k)$, so that $P_1(x) \leqq P_1(y)$. Thus P_1 satisfies (d) on C, and hence (a)–(e) on S. Also P_1 is the *largest* extension of this kind. In this proof we used the condition (i) only with $a_k \geqq 0$. It follows that this special case of (i) is equivalent to the general case, if (ii) is assumed. But this is also equivalent [5, p. 457] to (iii). This completes the proof.

We conclude by remarking that the extension P is a norm if and only if

$$\text{(iv)} \qquad \Phi(e) > 0 \qquad \text{for } e \neq 0.$$

3. Norms additive for covariant elements

We shall now discuss functions Φ which possess an extension P additive for certain positive elements of S. Let $x = \sum a_k e_k$, $a_k \geqq 0$. There exists an extension P with the properties (a)–(e) and

$$\text{(4)} \qquad P(x) = \sum_{k=1}^{n} a_k \Phi(e_k)$$

if and only if (4) is true for the seminorm P_1 instead of P, in other words, if and only if

$$\sum_{k=1}^{n} a_k \Phi(e_k) \leqq \sum_{l=1}^{m} a_l' \Phi(e_l') \quad \text{whenever } \sum a_k e_k = \sum a_l' e_l', \quad a_l' \geqq 0. \tag{5}$$

For if there is an extension P satisfying (4), then

$$P_1(x) \leqq \sum a_k \Phi(e_k) = P(x) \leqq P_1(x).$$

Each positive element x of S has a representation

$$x = \sum_{k=1}^{n} a_k e_k, \qquad a_k > 0, \quad e_1 \supset \cdots \supset e_n, \tag{6}$$

which is unique if the e_k are assumed different. Two positive functions $x(t)$, $y(t)$ are *covariant* if the differences $x(t) - x(t')$ and $y(t) - y(t')$ have always the same sign. Two positive elements $x, y \in S$ are covariant if there exist representations $x = \sum a_k e_k$, $y = \sum b_l f_l$ of type (6) in which all e_k, f_l are comparable in B.

We now ask whether an increasing, multiply subadditive $\Phi(e)$ has an extension which is additive for covariant elements. As a necessary condition we have

$$\sum_{k=1}^{n} a_k \Phi(e_k) \leqq \sum_{l=1}^{m} a_l' \Phi(e_l') \quad \text{for } \sum a_k e_k = \sum a_l' e_l', \quad a_k \geqq 0, \quad a_l' \geqq 0, \quad e_1 \supset \cdots \supset e_n. \tag{7}$$

This condition is also sufficient. We prove more, replacing (7) by the simpler condition (v), which is implied by (7).

THEOREM 2. (α). *An increasing function Φ on B has an extension P on S which satisfies* (a)–(e) *and is additive for covariant elements if and only if Φ is concave, i.e., satisfies*

$$\Phi(e_1 \cup e_2) + \Phi(e_1 \cap e_2) \leqq \Phi(e_1) + \Phi(e_2). \tag{v}$$

(β). *A concave increasing function Φ is multiply subadditive.* (γ). *If the desired extension exists, it is given by $P_1(x)$ of Section* 2.

Proof. If Φ satisfies (v), we first prove that

$$\sum_{k=1}^{n} \Phi(e_k) \leqq \sum_{l=1}^{m} \Phi(e_l') \qquad \text{if } \sum e_k \leqq \sum e_l', \quad e_1 \supset \cdots \supset e_n. \tag{8}$$

From (v) we derive $\Phi(e_1' \cup e_2') \leqq \Phi(e_1') + \Phi(e_2')$ and the subadditivity property

$$\Phi(e) \leqq \sum_1^m \Phi(e_l') \qquad \text{if } e \subset \bigcup e_l'.$$

This is (8) with $n = 1$; we prove (8) by induction on n. Let $\sum_1^n e_k \leqq \sum_1^m e_l'$, $e_1 \supset \cdots \supset e_n$. We put

$$g_l = e_l' \cap (e_1' \cup \cdots \cup e_{l-1}'); \qquad f_l = e_l' - g_l, \qquad l = 1, 2, \cdots, m.$$

Then the f_l are disjoint, and $\bigcup f_l = \bigcup e_l' \supset \bigcup e_k = e_1$. Multiplying $\sum_1^n e_k \leqq \sum_1^m f_l + \sum_2^m g_l$ with e_1 we obtain

$$\sum_1^n e_k \leqq e_1 + \sum_2^m g_l,$$

so that $\sum_2^n e_k \leqq \sum_2^m g_l$, and by the induction hypothesis,

$$\sum_{k=2}^n \Phi(e_k) \leqq \sum_{l=2}^m \Phi(g_l). \tag{9}$$

Next we have by (v)

$$\Phi(e_1' \cup \cdots \cup e_l') \leqq \Phi(e_1' \cup \cdots \cup e_{l-1}') + \Phi(e_l') - \Phi(g_l), \qquad l = 2, \cdots, m. \tag{10}$$

Adding (9) and all relations (10) we obtain

$$\sum_2^n \Phi(e_k) + \Phi(e_1' \cup \cdots \cup e_m') \leqq \sum_{l=1}^m \Phi(e_l'),$$

which implies (8). If a_k, a_l' are all rational in (7), this inequality reduces to (8). Finally, in the general case, we have only to approximate the a_k by rationals from below, the a_l' by rationals from above, and pass to the limit.

Since (7) implies (i) and (iii), we see that Φ is norm-generating and multiply subadditive.

It remains to show that if $P_1(x)$ is defined by (2), then $P_1(x + y) = P_1(x) + P_1(y)$ for covariant positive x, $y \in S$. Writing $x = \sum a_k e_k$, $y = \sum b_l f_l$ in form (6), we arrange the e_k, f_l to obtain a single decreasing sequence $g_1 \subset \cdots \subset g_{n+m}$. If c_i denote the corresponding a_k or b_l, we have

$$\sum c_i \Phi(g_i) = \sum a_k \Phi(e_k) + \sum b_l \Phi(f_l) = P_1(x) + P_1(y).$$

This completes the proof.

4. Extension of Φ onto a complete Boolean ring

We now assume that the Boolean ring B is contained in a σ-ring $\mathfrak{B}$ where countable unions and intersections are defined. A subring B^* of $\mathfrak{B}$ is relatively σ-complete if each sequence $e_n \in B^*$ with $e_n \subset e \in B^*$ has $\bigcup_1^\infty e_n \in B^*$. In what follows, Φ always will be a strictly positive increasing multiply subadditive function on B. Our purpose is to imbed B into a relatively σ-complete B^* and to extend Φ into a function Φ^* on B^* which has some continuity properties. We say that Φ is *σ-subadditive* on B if

(vi) $\qquad \Phi(e) \leqq \sum_1^\infty \Phi(e_n) \qquad$ whenever $e, e_n \in B, \quad e \subset \bigcup_1^\infty e_n$.

Also, Φ is *multiply σ-subadditive* if

(vii) $p\Phi(e) \leqq \sum_1^\infty \Phi(e_n)$ whenever e, $e_n \in B$ and e is covered p times by the e_n.

The last statement means that if g_n is the maximal element covered p times by $e_1, \cdots, e_n$ (g_n is obtained from the $e_1, \cdots, e_n$ by finite intersections and unions), then $e \subset \bigcup_1^\infty g_n$.

THEOREM 3. *There exists a multiply σ-subadditive extension Φ^* of Φ onto a relatively σ-complete subring B^* of $\mathfrak{B}$, $B \subset B^*$, if (and only if) Φ satisfies* (vi) *on B.*

Proof. Let B_0 be the set of $f \in \mathfrak{B}$ which admit a covering $f \subset \bigcup_1^\infty e_n$, $e_n \in B$, and put

$$\Phi^*(f) = \inf \sum \Phi(e_n), \qquad f \in B_0, \tag{11}$$

where the infimum is taken for all possible coverings. Let B^* be the subset of $f \in B_0$ with $\Phi^*(f) < \infty$. Only the fact that Φ^* is multiply σ-subadditive on B^* requires a proof. From the definition of Φ^* it follows that we have only to prove (vii) in case $e \in B^*$, $e_n \in B$. If $g_n \in B$ is the largest element covered p times by $e_1, \cdots, e_n$, then by the multiple subadditivity of Φ, $p\Phi(g_n) \leqq \sum_1^\infty \Phi(e_k)$. Also by (vi), $\Phi(g_m - g_n) \leqq \sum_{n+1}^m \Phi(e_k)$. Combining this with $\Phi(e) \leqq \Phi(g_{n_1}) + \sum_1^\infty \Phi(g_{n_{k+1}} - g_{n_k})$, and taking n_k sufficiently rapidly increasing, we obtain (vii).

In the same way we can prove:

(12) If $f_n \in B^*$ increase, and $\Phi^*(f_m - f_n) \to 0$ for $n, m \to \infty$, then $f = \bigcup_1^\infty f_n \in B^*$ and $\Phi^*(f - f_n) \to 0$.

The order relation $\subset$ defines *order convergence* [1, p. 50] on B. We shall call a function Φ *order continuous* if order convergence $e_n \to e$ implies $\Phi(e_n) \to \Phi(e)$.

THEOREM 4. *Let Φ be an increasing multiply subadditive function on B. There exist a relatively σ-complete ring B_0^*, $B \subset B_0^* \subset \mathfrak{B}$, and an order continuous extension Φ^* of Φ onto B_0^* if and only if Φ satisfies* (vi) *and*

$$\Phi(e_{n+p} - e_n) \to 0 \quad \textit{if } e_n, e \in B, \quad e_n \textit{ increase, and } e_n \to e. \tag{viii}$$

Proof. The necessity of the condition is obvious. To prove the sufficiency, we take B_0^* to be the subring of B^* of all $f \in B^*$ with the property that for each $\varepsilon > 0$ there is a representation

$$f = e + \varphi - \psi \qquad \text{with } e \in B, \quad \varphi, \psi \in B^*, \quad \Phi^*(\varphi) < \varepsilon, \quad \Phi^*(\psi) < \varepsilon.$$

The main difficulty is to prove that if $f_n \in B_0^*$ are increasing, and $f = \bigcup_1^\infty f_n \in B^*$, then $\Phi^*(f - f_n) \to 0$. We can write $f_n = e_n + \varphi_n - \psi_n$. The elements $\bar{e}_n = \bigcup_{k=1}^n e_k$ increase and are contained in $e_0 + \psi$, with $e_0 \in B$, $\Phi^*(\psi) < \varepsilon$, if the φ_n, ψ_n are chosen properly. For the $\bar{e}_n' = \bar{e}_n \cap e_0$ we have (viii). Then the relations between the $\bar{e}_n'$, $\bar{e}_n$, f_n imply $\Phi^*(f_{n+p} - f_n) < 2\varepsilon$ for all large n, and the desired conclusion follows from (12).

Remark. One sees easily that if B is relatively σ-complete, then (viii) is equivalent to

$$\Phi(e_n) \to 0 \quad \text{if } e_n \in B \text{ decrease and converge to } 0. \tag{ix}$$

We mention some examples. The outer Lebesgue measure μe of subsets

e of [0, 1] satisfies (i)–(vii), but not (viii). Another example is the function $\Phi(e) = \sup_n \mu(e \cap E_n)$, where $E_n = (2^{-n}, 2^{-n+1})$, $n = 1, 2, \cdots$, which satisfies all conditions (i)–(ix) except (v).

5. Extension of S into a Banach lattice

Let B, $\mathfrak{B}$, Φ have the same meaning as in Section 4; B in addition is relatively σ-complete, and Φ and B satisfy the first half of (12). P is a norm which extends Φ onto S and satisfies (d) and (e). We wish to imbed S into a Banach lattice X and to extend P into a norm of X. The possibility of this extension is discussed in Theorem 5 below; it is interesting to note that this possibility depends only on the properties of Φ and not on the mode of extension of Φ into P.

A Banach lattice is a Banach space and a vector lattice in which the norm has the property that $| x | \leqq | y |$ implies $\| x \| \leqq \| y \|$ (from this also the continuity of the lattice operations follows).

THEOREM 5. (α). *There exists an extension of S into a Banach lattice X such that B is a closed sublattice of X and of the norm P of S into the norm of X if and only if Φ satisfies* (vi). *In this case X can be taken relatively σ-complete.* (β). *There is an extension of S into a Banach lattice such that order convergence implies norm convergence if and only if in addition Φ satisfies* (ix).

Proof. We show that the condition (vi) is necessary. Let $e \subset \bigcup e_n$ and $\sum \Phi(e_n) < \infty$. We put $\bar{e}_n = e \cap \bigcup_1^n e_k$; then

$$\| \bar{e}_{n+p} - \bar{e}_n \| \leqq \sum_{n+1}^{n+p} \Phi(e_k) \to 0;$$

hence e_n converges in norm to an element $x \in X$. From

$$0 \leqq \| (\bar{e}_n - x) \cup 0 \| \leqq \| (\bar{e}_{n+p} - x) \cup 0 \| \leqq \| \bar{e}_{n+p} - x \| \to 0$$

we see that $(\bar{e}_n - x) \cup 0 = 0$, $\bar{e}_n \leqq x$. Since $\bigcup \bar{e}_n = e$, we have $\bar{e}_n \leqq e \leqq x$; hence $\bar{e}_n \to e$. Therefore $\Phi(e) = \lim \Phi(\bar{e}_n) \leqq \sum \Phi(e_k)$.

Sufficiency is proved by a direct construction of X. A positive real function x (see [3]) is a map $x(\alpha)$ from the positive reals into B such that $\bigcup_{a>\alpha} x(a) = x(\alpha)$. Functions x_1, x_2 are disjoint if $x_1(\alpha)$, $x_2(\alpha)$ are disjoint for each $\alpha > 0$. Arbitrary real functions are differences of disjoint positive functions. We now define for $x \geqq 0$

$$\| x \| = \sup_{0 \leqq \bar{x} \leqq x} P(\bar{x}), \qquad \bar{x} \in S, \tag{13}$$

and put $\| x \| = \| \, | x | \, \|$ for an arbitrary function x. Then X is the set of all x with $\| x \| < \infty$. Only the proofs of the subadditivity of the norm and the norm completeness require some care. In the first proof, the following device is used. For each $x \geqq 0$, $\| x \|$ may be approximated by the norm of a special element $0 \leqq x' \leqq x$ which does not take arbitrarily large or arbitrarily small positive values, and $\| x' \|$ in its turn by the norm of an element $\bar{x} \in S$, $\bar{x} \geqq x'$.

The subspace S is not always dense in X. This can be true even for all extensions P of a norm-generating function Φ. For example, if $x(t) = 2^n$ on E_n, $n = 1, 2, \cdots$ in the last example of Section 4, then $x \in X$, but $\| x - \bar{x} \| = 1$ for each $0 \leqq \bar{x} \leqq x$, $\bar{x} \in S$.

6. Spaces Λ_Φ

The phenomenon just described cannot happen for a certain class of spaces which we shall discuss now. Let X be a Banach lattice of functions on a relatively σ-complete ring B; this implies by definition that $x \in X$ if $0 \leqq |x| \leqq y$, $y \in X$. Let the norm on X be additive for covariant functions. In particular, for this norm the function $\Phi(e) = \| e \|$ will satisfy (v). We shall show that for $x \geqq 0$, $x \in X$

$$\|x\| = \int_0^\infty \Phi(x(\alpha))\, d\alpha. \tag{14}$$

Let x_n be defined by $x_n(\alpha) = x(n + n^{-1})$ for $\alpha > n$, $x_n(\alpha) = x(\alpha + n^{-1})$ for $n^{-1} \leqq \alpha \leqq n$. Then $x - x_n$ and x_n are covariant; hence $\| x - x_n \| = \| x \| - \| x_n \|$. The x_n increase and $x_n \to x$; therefore $\| x_n \| \to \| x \|$, $\| x - x_n \| \to 0$. Each x_n can be approximated uniformly by elements of S. It follows that S is dense in X. Moreover we have $\| x \| = \sup \| \bar{x} \|$ for the elements $\bar{x} \in S$ of the form

$$\bar{x} = \textstyle\sum_{k=1}^n a_k\, x(a_1 + \cdots + a_k), \qquad a_k > 0.$$

This gives

$$\|x\| = \sup \sum a_k\, \Phi(x(a_1 + \cdots + a_k)) = \int_0^\infty \Phi(x(\alpha))\, d\alpha.$$

For a given concave Φ, the Banach lattice of all functions with the norm (14) is called a space Λ_Φ. We obtain in this way

THEOREM 6. *Let X be a Banach lattice of functions on B with order continuous norm. Then X is a space Λ_Φ if and only if the norm in X is additive for covariant functions.*

Spaces Λ considered in [4] were of the following type. Let $x(t)$ be measurable functions on a measure space S with a countably additive measure μ. We select a nonnegative measurable function $\varphi(t)$ and put

$$\|x\| = \sup \int_S \varphi x'\, d\mu, \tag{15}$$

where the supremum is taken for all functions $x'(t)$ equimeasurable with $x(t)$. It is fairly easy to see that the space $\Lambda_{\varphi,\mu}$ with the norm (15) is a space of type Λ_Φ. The converse is not in general true. However we can find necessary and sufficient conditions for Φ in order that Λ_Φ have a concrete representation as a space $\Lambda_{\varphi,\mu}$. The tools of this proof are: a theory of equimeasurable functions for arbitrary measure spaces, and theorems similar to those of

D. Maharam [6] about the representation of a function $\Phi(e)$ in form $\Phi(e) = F(\mu e)$, where μ is a measure. We hope to come back to these results in a separate publication.

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