MODULES OF EXTENSIONS OVER DEDEKIND RINGS[1]

BY

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This paper is devoted to the study of the structure of the R-module $\operatorname{Ext}_R^1(A, C)$ where R is a Dedekind ring and A, C are R-modules. In the first part of the paper the particular objects of study are the submodules $I \operatorname{Ext}_R^1(A, C)$ where I is a nonzero ideal of R. For any R-module M we show that IM is the kernel of the connecting homomorphism

$$\delta : M \to \operatorname{Ext}_R^1(I^{-1}/R, M)$$

associated with the exact sequence $0 \to R \to I^{-1} \to I^{-1}/R \to 0$. Using the associativity law

$$\operatorname{Ext}_R^1(I^{-1}/R, \operatorname{Ext}_R^1(A, C)) \approx \operatorname{Ext}_R^1(\operatorname{Tor}_1^R(I^{-1}/R, A), C),$$

we show that $I \operatorname{Ext}(A, C)$ is the kernel of the homomorphism

$$\operatorname{Ext}(f_I, C) : \operatorname{Ext}_R^1(A, C) \to \operatorname{Ext}_R^1(\operatorname{Tor}_1^R(I^{-1}/R, A), C)$$

where $f_I : \operatorname{Tor}_1^R(I^{-1}/R, A) \to A$ is the appropriate connecting homomorphism. By investigating the structure of $\operatorname{Tor}_1^R(I^{-1}/R, A)$ we obtain various properties of $I \operatorname{Ext}_R^1(A, C)$.

The elements of $\operatorname{Ext}_R^1(A, C)$ are in 1-1 correspondence with the equivalence classes of extensions of C by A. Suppose $(\mathfrak{e}) : 0 \to C \to E \to A \to 0$ is such an extension. We show that its characteristic class $\chi(\mathfrak{e})$ lies in $I \operatorname{Ext}_R^1(A, C)$ if and only if $JC = C \cap JE$ for every ideal J containing I.

In the second part of the paper we begin to determine the kinds of modules $\operatorname{Ext}_R^1(A, C)$ can be. The main result is that, mod its submodule of divisible elements, $\operatorname{Ext}_R^1(A, C)$ is a direct product of its P-adic completions, P ranging over the prime ideals of R. It follows that a module M which is either torsion-free or torsion is isomorphic to $\operatorname{Ext}_R^1(A, C)$ for some modules A and C if and only if it is the direct product of its P-adic completions. In particular R itself has this property if and only if it is a complete discrete valuation ring.

The final section records some properties of A and C which follow from the relation $\operatorname{Ext}_R^1(A, C) = 0$. One curious result is that, if R is not a complete discrete valuation ring, then $\operatorname{Hom}_R(A, R) = 0 = \operatorname{Ext}_R^1(A, R)$ implies that $A = 0$. We also show that if A is a module such that $\operatorname{Ext}_R^1(A, C) = 0$ for every torsion module C, then every submodule of A with countable rank is projective. This extends a result of Baer [1] to modules over Dedekind rings.

The properties of Dedekind rings that are needed here are developed in [2, VII] and in [4]. In the first two sections of this paper the base rings can

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be more general. In the first section any ring with unity will do; in the second any hereditary ring.

1. An *extension* of C by A is an exact sequence

$$(\mathfrak{e}) \qquad 0 \to C \xrightarrow{\phi} E \xrightarrow{\psi} A \to 0.$$

The *characteristic class* $\chi(\mathfrak{e})$ of $(\mathfrak{e})$ is Θi where Θ: Hom $(C, C) \to \mathrm{Ext}^1(A, C)$ is the connecting homomorphism defined by $(\mathfrak{e})$ and i is the identity endomorphism of C. It is shown in [2, XIV, 1] that the correspondence $(\mathfrak{e}) \to \chi(\mathfrak{e})$ is 1–1 between equivalence classes of extensions of C by A and the elements of $\mathrm{Ext}^1(A, C)$. The extension $(\mathfrak{e})$ is *equivalent* to an extension

$$(\mathfrak{e}') \qquad 0 \to C \xrightarrow{\phi'} E' \xrightarrow{\psi'} A \to 0$$

if there is a homomorphism $k: E \to E'$ such that the diagram

$$\begin{array}{cccccccccc} (\mathfrak{e}) & 0 \to & C & \xrightarrow{\phi} & E & \xrightarrow{\psi} & A & \to 0 \\ & & \| & & \downarrow k & & \| & \\ (\mathfrak{e}') & 0 \to & C & \longrightarrow & E' & \xrightarrow{\psi'} & A & \to 0 \end{array}$$

commutes.

Let

$$0 \to M \xrightarrow{\beta} P \xrightarrow{\alpha} A \to 0$$

be an exact sequence with P projective. Then a commutative diagram

$$(1) \qquad \begin{array}{cccccccccc} & 0 \to & M & \xrightarrow{\beta} & P & \xrightarrow{\alpha} & A & \to 0 \\ & & \downarrow \gamma & & \downarrow \tau & & \| & \\ (\mathfrak{e}) & 0 \to & C & \xrightarrow{\phi} & E & \xrightarrow{\psi} & A & \to 0 \end{array}$$

exists, and $\chi(\mathfrak{e}) = \vartheta\gamma$ where ϑ: Hom $(M, C) \to \mathrm{Ext}^1(A, C)$ is the connecting homomorphism induced by the top row of (1). We also have an exact sequence

$$(2) \qquad 0 \to M \xrightarrow{\mu} C + P \xrightarrow{\eta} E \to 0,$$

where

$$(3) \qquad \mu m = (-\gamma m, \beta m) \quad \text{and} \quad \eta(c, p) = \phi c + \tau p.$$

The various homomorphisms in (1) and (2) are related by

$$(4) \qquad \psi c = \eta(c, o), \qquad \tau p = \eta(o, p), \qquad \phi\eta(c, p) = \alpha p.$$

It is shown in [2, XIV, 1] that $\chi(\mathfrak{e}) = \vartheta\gamma$ if and only if a sequence (2) exists such that $\alpha, \beta, \gamma, \mu, \eta, \phi, \psi$ are related as in (3) and (4). Then the remaining homomorphism τ is defined by (4) and a diagram (1) results.

THEOREM 1.1. *The diagram*

$$\begin{array}{llllllllll} (\mathfrak{e}) & 0 \rightarrow & C & \rightarrow & E & \rightarrow & A & \rightarrow 0 \\ & & \| & & & & \downarrow f & \\ (\mathfrak{e}') & 0 \rightarrow & C & \rightarrow & E' & \rightarrow & A & \rightarrow 0 \end{array}$$

can be made commutative by a homomorphism $h:E \rightarrow E'$ *if and only if* $\chi(\mathfrak{e}) = \text{Ext}\,(f, C)\chi(\mathfrak{e}')$.

The implication in the forward direction is an immediate consequence of the definition of χ and the elementary properties of Ext. To show the opposite implication we start from (1), (2), and the commutative diagram

$$\begin{array}{ccccccccc} 0 \rightarrow & M & \xrightarrow{\beta} & P & \xrightarrow{\alpha} & A & \rightarrow 0 \\ & \downarrow \gamma' & & \downarrow \tau' & & \downarrow f & \\ 0 \rightarrow & C & \xrightarrow{\phi'} & E' & \xrightarrow{\psi'} & A' & \rightarrow 0 \end{array}$$

which exists because P is projective. From this diagram we deduce the commutative square

$$\begin{array}{ccc} \text{Hom}\,(C, C) & \xrightarrow{\text{Hom}\,(\gamma', C)} & \text{Hom}\,(M, C) \\ \Theta(\mathfrak{e}') \downarrow & & \downarrow \vartheta \\ \text{Ext}\,(A', C) & \xrightarrow{\text{Ext}\,(f, C)} & \text{Ext}\,(A, C). \end{array}$$

In view of (1) and (2) we have $\chi(\mathfrak{e}) = \vartheta\gamma$. Now assuming $\chi(\mathfrak{e}) = \text{Ext}\,(f, C)\chi(\mathfrak{e}')$ we get $\vartheta\gamma = \text{Ext}\,(f, C)\chi(\mathfrak{e}') = \text{Ext}\,(f, C)\Theta(\mathfrak{e}')i = \vartheta\,\text{Hom}\,(\gamma', C)i = \vartheta\gamma'$. Thus $\vartheta\gamma = \vartheta\gamma'$ and there exists a map $\omega:P \rightarrow C$ such that $\gamma' = \gamma + \omega\beta$. We define $\Omega:C + P \rightarrow E'$ by

$$\Omega(c, p) = \phi'c - \phi'\omega p + \tau'p.$$

It is easily verified that $\Omega\mu = 0$, so that a homomorphism $h:E \rightarrow E'$ exists such that $h\eta = \Omega$. This is the required homomorphism.

THEOREM 1.2. *The diagram*

$$\begin{array}{llllllllll} (\mathfrak{e}) & 0 \rightarrow & C & \rightarrow & E & \rightarrow & A & \rightarrow 0 \\ & & g \downarrow & & & & \| & \\ (\mathfrak{e}') & 0 \rightarrow & C' & \rightarrow & E' & \rightarrow & A & \rightarrow 0 \end{array}$$

can be made commutative by a homomorphism $h:E \rightarrow E'$ *if and only if* $\chi(\mathfrak{e}') = \text{Ext}\,(A, g)\chi(\mathfrak{e})$.

The proof of this theorem is similar to that of the previous theorem and is omitted.

2. The only property of Dedekind rings used in this section is that they are hereditary: in particular $\mathrm{Tor}_n^R = 0 = \mathrm{Ext}_R^n$ for all $n > 1$, Ext_R^1 is right exact, and Tor_1^R is left exact.

Let R, S be Dedekind rings; let A be an R-module, C an S-module, and B an R-S-bimodule. There is a natural isomorphism

$$\mathrm{Hom}_S (A \otimes_R B, C) \approx \mathrm{Hom}_R (A, \mathrm{Hom}_S (B, C)) \tag{1}$$

which sends each $g: A \otimes_R B \to C$ into the $f: A \to \mathrm{Hom}_S (B, C)$ such that $f(a)b = g(a \otimes b)$.

Let Y be an S-injective resolution for C. We then have

$$\mathrm{Ext}_S (A \otimes_R B, C) = H(\mathrm{Hom}_S (A \otimes_R B, Y)) \approx H(\mathrm{Hom}_R (A, \mathrm{Hom}_S (B, Y))).$$

We apply the homomorphism α' of [2, IV, 6.1a] to get

$$H(\mathrm{Hom}_R (A, \mathrm{Hom}_S (B, Y))) \xrightarrow{\alpha'} \mathrm{Hom}_R (A, H(\mathrm{Hom}_S (B, Y)))$$
$$\approx \mathrm{Hom}_R (A, \mathrm{Ext}_S (B, C)).$$

Thus we obtain a homomorphism

$$\tau: \mathrm{Ext}_S (A \otimes_R B, C) \to \mathrm{Hom}_R (A, \mathrm{Ext}_S (B, C)) \tag{2}$$

which reduces to (1) in dimension 0. If A is R-projective, the functor $T(D) = \mathrm{Hom}_R (A, D)$ is exact. Then α', and hence τ, is an isomorphism.

Now define functors $T^n(A) = \mathrm{Ext}_S^1 (\mathrm{Tor}_n^R (A, B), C)$. For an exact sequence $0 \to A' \to A \to A'' \to 0$ of R-modules with connecting homomorphisms $\partial_n: \mathrm{Tor}_{n+1}^R (A'', B) \to \mathrm{Tor}_n^R (A', B)$, define the map $T^n(A') \to T^{n+1}(A'')$ to be $\mathrm{Ext}_S^1 (\partial_n, C)$. Then the sequence $T = \{T^n\}$ is a connected sequence of covariant functors. If we set $U(A) = \mathrm{Hom}_R (\mathrm{Ext}_S (B, C))$, (2) provides us with a natural transformation $\tau^1 = \mu^0: T^0 \to U$.

Since Ext_S^1 is right exact, it is easily verified that the conditions of [2, III, 5.2] are satisfied and μ^0 can be extended to a unique map $\mu: T \to SU$ defined for $n \geqq 0$, where $SU = \{S^n U\}$ is the sequence of satellites of U. In particular we have a natural homomorphism

$$\mu^1: \mathrm{Ext}_S^1 (\mathrm{Tor}_1^R (A, B), C) \to \mathrm{Ext}_R^1 (A, \mathrm{Ext}_S^1 (B, C)). \tag{3}$$

THEOREM 2.1. *The homomorphism μ^1 is an isomorphism. If $0 \to A' \to A \to A'' \to 0$ is an exact sequence of R-modules inducing a connecting homomorphism $\partial: \mathrm{Tor}_R^1 (A'', B) \to A' \otimes_R B$, then the diagram*

$$\begin{array}{ccc} \mathrm{Ext}_S^1 (A' \otimes_R B, C) & \xrightarrow{\mathrm{Ext}_S (\partial, C)} & \mathrm{Ext}_S^1 (\mathrm{Tor}_1^R (A'', B), C) \\ \downarrow \tau^1 & & \downarrow \mu^1 \\ \mathrm{Hom}_R (A', \mathrm{Ext}_S (B, C)) & \xrightarrow{\delta} & \mathrm{Ext}_R^1 (A'', \mathrm{Ext}_S^1 (B, C)) \end{array}$$

commutes, where δ is the appropriate connecting homomorphism.

Since $\mu: T \to SU$ is a map between connected sequences of functors and $\mu^0 = \tau^1$, the commutativity of the diagram is immediate. We have only to show that μ^1 is an isomorphism. Let $0 \to M \to P \to A \to 0$ be an exact sequence with P R-projective. Since R is a Dedekind ring, M is also R-projective. We have a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Ext}_S^1(P \otimes_R B, C) & \to & \mathrm{Ext}_S^1(M \otimes_R B, C) & \to & \mathrm{Ext}_S^1(\mathrm{Tor}_1^R(A, B), C) & \to & 0 \\
\downarrow \tau_P & & \downarrow \tau_M & & \downarrow \mu^1 & & \\
\mathrm{Hom}_R(P, \mathrm{Ext}_S^1(B, C)) & \to & \mathrm{Hom}_R(M, \mathrm{Ext}_S^1(B, C)) & \to & \mathrm{Ext}_R^1(A, \mathrm{Ext}_S^1(B, C)) & \to & 0.
\end{array}$$

The top row is exact because $\mathrm{Ext}_S^1(\ \ , C)$ is right exact. Since P is R-projective, $\mathrm{Ext}_R^1(P, \mathrm{Ext}_S^1(B, C)) = 0$, hence the bottom row is exact. Since M is also R-projective, τ_P and τ_M are isomorphisms. Therefore μ^1 is an isomorphism by the 5-lemma.

Suppose that A is an R-module, B is an S-module and C is an R-S-bimodule. We get similar results if we start from the natural isomorphism

$$\mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)) \approx \mathrm{Hom}_S(B, \mathrm{Hom}_R(A, C)) \tag{4}$$

which sends $f: A \to \mathrm{Hom}_S(B, C)$ into the $g: B \to \mathrm{Hom}_R(A, C)$ such that $g(b)a = f(a)b$.

THEOREM 2.2. *There is a natural homomorphism*

$$\sigma: \mathrm{Ext}_R(A, \mathrm{Hom}_S(B, C)) \to \mathrm{Hom}_S(B, \mathrm{Ext}_R(A, C))$$

which reduces to (4) *in dimension* 0 *and is an isomorphism when B is S-projective; and an isomorphism*

$$\nu: \mathrm{Ext}_R^1(A, \mathrm{Ext}_S^1(B, C)) \to \mathrm{Ext}_S^1(B, \mathrm{Ext}_R^1(A, C))$$

such that, for any exact sequence $0 \to B' \to B \to B'' \to 0$, *commutativity holds in the diagram*

$$\begin{array}{ccc}
\mathrm{Ext}_R^1(A, \mathrm{Hom}_S(B', C)) & \xrightarrow{\mathrm{Ext}(A, \delta)} & \mathrm{Ext}_R^1(A, \mathrm{Ext}_S^1(B'', C)) \\
\downarrow \sigma^1 & & \downarrow \nu \\
\mathrm{Hom}_S(B', \mathrm{Ext}_R^1(A, C)) & \xrightarrow{\Delta} & \mathrm{Ext}_S^1(B'', \mathrm{Ext}_R^1(A, C))
\end{array}$$

where $\delta: \mathrm{Hom}_S(B', C) \to \mathrm{Ext}_S^1(B'', C)$ *and* Δ *are the appropriate connecting homomorphisms.*

The proof is similar to that of Theorem 2.1 using first an R-projective resolution of A and then an S-injective resolution of C.

3. Throughout the remainder of the paper R is a fixed Dedekind ring; all modules are R-modules. Since $\mathrm{Ext}_R^n = 0$ for $n \geqq 2$, and R is fixed, we shall use Ext for Ext_R^1 and Tor for Tor_1^R.

Let Q denote the field of quotients of R. If I is any nonzero ideal of R, then I^{-1} is the set of all elements q in Q such that $qI \subseteq R$. Since R is a Dedekind ring $II^{-1} = R$ and both I and I^{-1} are projective. It follows that the sequence

$$0 \to R \to I^{-1} \to I^{-1}/R \to 0$$

is a projective resolution for I^{-1}/R; hence Tor $(I^{-1}/R, C) = \text{Ker } \phi$ and Ext $(I^{-1}/R, C) = \text{Coker } \psi$ where $\phi: C \to I^{-1} \otimes C$ and $\psi: \text{Hom } (I^{-1}, C) \to C$ are defined by $\phi c = 1 \otimes c$ and $\psi f = f(1)$ respectively. Let An (I, C) denote the set of elements c in C such that $Ic = 0$.

THEOREM 3.1. *If I is a nonzero ideal of R, then*

$$\text{Tor } (I^{-1}/R, C) = \text{An } (I, C) \quad \textit{and} \quad \text{Ext } (I^{-1}/R, C) = C/IC.$$

Since $II^{-1} = R$, there are q_i in I^{-1} and r_i in I such that $\sum_i r_i q_i = 1$. For any c in An (I, C) we have $\phi c = \sum_i r_i q_i \otimes c = \sum_i q_i \otimes r_i c = 0$. Hence An $(I, C) \subseteq \text{Ker } \phi$. Suppose that c is in Ker ϕ and r is any element of I. Define $h: I^{-1} \otimes C \to C$ by $h(q \otimes c) = (rq)c$. Then $0 = h\phi c = h(1 \otimes c) = rc$. Since r was arbitrary in I, we have Ker $\phi \subseteq$ An (I, C). This shows that Tor $(I^{-1}/R, C) = \text{Ker } \phi = \text{An } (I, C)$.

To show that Coker $\psi = C/IC$ we must show that Im $\psi = IC$. Let $f: I^{-1} \to C$. Then $\psi f = f(1) = \sum_i r_i f(q_i)$ which belongs to IC. Thus Im $\psi \subseteq IC$. If, on the other hand, c is in IC, there exist elements c_i in C such that $c = \sum_i r_i c_i$ because the r_i generate I. The homomorphism $f: I^{-1} \to C$ defined by $fq = \sum_i (r_i q) c_i$ satisfies $\psi f = c$; hence $IC \subseteq \text{Im } \psi$.

Let P be a prime ideal in R. We use $t_P C$ to denote the P-primary component of C and tC to denote the torsion submodule of C. The module Q/R will be denoted by K, and $t_P K$ will be denoted by K_P. The union of the submodules P^{-n} of Q for all $n \geqq 0$ will be denoted by Q_P. Then $K_P = Q_P/R$. A module C is *P-divisible* if $PC = C$. The union of all the P-divisible submodules of C is itself P-divisible and will be denoted by $d_P C$; if $d_P C = 0$, then C will be said to be *P-reduced.* The maximal divisible submodule of C will be denoted by dC; if $dC = 0$ then C is *reduced.* To simplify the statements of some of the theorems we shall use $t_R C$ for tC, $d_R C$ for dC, and K_R for K. Now let P be a prime ideal of R. Since the sequence

$$0 \to R \to Q_P \to K_P \to 0$$

is exact and Q_P is torsion-free, we derive exact sequences

$$0 \to \text{Tor } (K_P, C) \xrightarrow{\partial} C \xrightarrow{\alpha} Q_R \otimes C,$$

$$\text{Hom } (K_P, C) \to \text{Hom } (Q_P, C) \xrightarrow{\beta} C \to \text{Ext } (K_P, C),$$

where $\alpha c = 1 \otimes c$ and $\beta f = f(1)$.

THEOREM 3.2. *Let P be a prime ideal or R. Then*
(a) $\operatorname{Ker} \alpha = t_P C$; *hence ∂ induces an isomorphism*

$$\operatorname{Tor}(K_P, C) \approx t_P C.$$

(b) $\operatorname{Im} \beta = d_P C$. *If $t_P C = 0$, then β induces an isomorphism*

$$\operatorname{Hom}(Q_P, C) \approx d_P C.$$

The module C is P-reduced if and only if $\operatorname{Hom}(Q_P, C) = 0$.

To prove (a) assume first that P is a prime ideal. Since Q_P is the direct limit of the submodules P^{-n} of Q and commutes with direct limits, $\operatorname{Ker} \alpha$ is the union of the kernels of $\alpha_n : C \rightarrow P^{-n} \otimes C$ where $\alpha_n c = 1 \otimes c$. By Theorem 3.1, $\operatorname{Ker} \alpha_n = \operatorname{An}(P^n, C)$; hence $\operatorname{Ker} \alpha = t_P C$. The proof for the case $P = R$ is similar.

To prove (b) assume first that $P = R$. If $f: Q \rightarrow C$, then $\operatorname{Im} f$ is divisible. Therefore βf is in dC. On the other hand suppose c is any element of dC. Since dC is injective, the homomorphism $f_0: R \rightarrow dC$ sending 1 into c has an extension $f: Q \rightarrow dC$. The composition of f with the inclusion $dC \subseteq C$ is mapped by β into c. Hence c is in $\operatorname{Im} \beta$. This shows that $\operatorname{Im} \beta = dC$. The remaining parts of (b) are then easily proved.

Now suppose that P is a prime ideal. The only part of the proof requiring different treatment is the proof of the statement $d_P C \subseteq \operatorname{Im} \beta$. To show this let c be any element of $d_P C$. We want a homomorphism $f: Q_P \rightarrow C$ with $f(1) = c$. Since Q_P is the union of the modules $R = P^0, P^{-1}, P^{-2}, \cdots$, it suffices to define a sequence of homomorphisms $f_n: P^{-n} \rightarrow C$ such that (i) f_{n+1} extends f_n, (ii) $\operatorname{Im} f_n \subseteq d_P C$, and (iii) $f_0(r) = rc$ for every r in R. We use (iii) to define f_0 and proceed by induction. Suppose $f_0, \cdots, f_n$ have been defined satisfying (i) and (ii). There exists elements r_i in P, q_i in P^{-1}, s_j in P^n, t_j in P^{-n} such that

$$1 = \sum_i r_i q_i = \sum_j s_j t_j .$$

Since f_n satisfies (ii) and the r_i generate P, we have elements c_{ij} in $d_p C$ such that $f_n(t_j) = \sum_i r_i c_{ij}$. If u is any element of P^{-n-1}, then $r_i s_j u$ is in R; hence we define f_{n+1} by

$$f_{n+1}(u) = \sum_{ij} (r_i s_j u) c_{ij} .$$

Clearly f_{n+1} satisfies (ii). As for (i) we have, for any t_k,

$$\begin{aligned} f_{n+1}(t_k) &= \sum_{ij} (r_i s_j t_k) c_{ij} = \sum_j (s_j t_k) \sum_i r_i c_{ij} \\ &= \sum_j (s_j t_k) f_n(t_j) = f_n\left(\sum_j s_j t_j t_k\right) = f_n(t_k). \end{aligned}$$

Since the t_k generate P^{-n}, (i) is established.

4. The results of the last two sections will now be used to investigate divisibility in $\operatorname{Ext}(A, C)$.

THEOREM 4.1. *If I is a nonzero ideal and if f_I:An $(I, A) \to A$ and $g_I: C \to C/IC$ are the inclusion and quotient homomorphisms respectively, then the homomorphisms*

$$\delta : \text{Ext}\,(A, C) \to \text{Ext}\,(I^{-1}/R, \text{Ext}\,(A, C))$$

$$\text{Ext}\,(f_I, C) : \text{Ext}\,(A, C) \to \text{Ext}\,(\text{An}\,(I, A), C)$$

$$\text{Ext}\,(A, g_I) : \text{Ext}\,(A, C) \to \text{Ext}\,(A, C/IC)$$

are all epimorphisms with kernel I Ext (A, C). *Hence they induce isomorphisms*

$$\begin{aligned} \text{Ext}\,(A, C)/I\,\text{Ext}\,(A, C) &\approx \text{Ext}\,(I^{-1}/R, \text{Ext}\,(A, C)) \\ &\approx \text{Ext}\,(\text{An}\,(I, A), C) \\ &\approx \text{Ext}\,(A, C/IC). \end{aligned}$$

The homomorphism δ has already been considered in Theorem 3.1. If we apply Theorem 2.1 to the sequence $0 \to R \to I^{-1} \to I^{-1}/R \to 0$, we get the commutative diagram

$$\begin{array}{ccc} \text{Ext}\,(A, C) & \xrightarrow{\text{Ext}\,(f_I,\, C)} & \text{Ext}\,(\text{An}\,(I, A), C) \\ & \searrow^{\delta} & \downarrow \mu \\ & & \text{Ext}\,(I^{-1}/R, \text{Ext}\,(A, C)) \end{array}$$

where An (I, A) has been identified with Tor $(I^{-1}/R, A)$ according to Theorem 3.1. Since Ext (f_I, C) is an epimorphism and μ is an isomorphism, Ker Ext (f_I, C) = Ker δ = I Ext (A, C). The remaining part of the theorem is proved similarly using Theorem 2.2.

A similar description of the submodule d_P Ext (A, C) is contained in the following theorem.

THEOREM 4.2. *If P is a prime ideal or R and $j_P : t_P\,A \to A$ is the inclusion homomorphism, then*

$$\text{Ext}\,(j_P, C) : \text{Ext}\,(A, C) \to \text{Ext}\,(t_P\,A, C)$$

is an epimorphism with kernel d_P Ext (A, C). *If $k_P : A \to A/t_P\,A$ is the quotient homomorphism, d_P* Ext (A, C) *is the image of* Ext (k_P, C).

The first part of this theorem is proved in the same manner as the previous theorem using Theorem 3.2 instead of Theorem 3.1. The second statement of the theorem follows from the first statement and the exactness of the sequence Ext $(A/t_P\,A, C) \to$ Ext $(A, C) \to$ Ext $(t_P\,A, C) \to 0$.

The following corollary is an immediate consequence of the theorem.

COROLLARY 4.3. *If A is a torsion (P-primary) module, then* Ext (A, C) *is reduced (P-reduced).*

We are now in a position to give necessary and sufficient conditions on the modules A and C so that Ext (A, C) will be divisible. Since each ideal of R can be expressed as a product of prime ideals it is sufficient to investigate P-divisibility for each prime P.

We will need the equality of I^{-1}/R and R/I. This is proved in

LEMMA 4.4. *If I is any nonzero ideal, then*

$$I^{-1}/R \approx R/I.$$

It is sufficient to find an element q in I^{-1} which, together with 1, generates I^{-1}. Let r be any nonzero element of I. Then rI^{-1} is an ideal of R containing r. Since R is a Dedekind ring, there is an element s of R such that r and s generate rI^{-1}. Then the element $q = s/r$ has the required properties.

THEOREM 4.5. *If P is a prime ideal, then* Ext (A, C) *is P-divisible if and only if either A has no P-torsion or C is P-divisible.*

Suppose Ext (A, C) is P-divisible and A has P-torsion (i.e., $t_P A \neq 0$). Since P is a prime ideal, A contains an element a with order ideal P. The submodule generated by a is isomorphic to R/P, hence to P^{-1}/R by the lemma. Hence there exists a monomorphism $P^{-1}/R \rightarrow A$. Passing to Ext we get an epimorphism Ext $(A, C) \rightarrow$ Ext $(P^{-1}/R, C) = C/PC$. Since Ext (A, C) is P-divisible, so is C/PC. Hence $C/PC = 0$ or, equivalently, $C = PC$.

On the other hand $t_P A = 0$ or $C/PC = 0$ implies Ext (A, C) is P-divisible in view of the isomorphisms established in Theorem 4.1.

5. Suppose A' is a submodule of A. An extension $(\mathfrak{e}): 0 \rightarrow C \rightarrow E \rightarrow A \rightarrow 0$ is *trivial on* A' if there is a homomorphism $A' \rightarrow E$ such that the composite $A' \rightarrow E \rightarrow A$ is the identity on A'. In view of Theorem 1.1, this is equivalent to the statement Ext $(f, C)\chi(\mathfrak{e}) = 0$ where f is the inclusion of A' in A. If C' is a quotient module of C, then $(\mathfrak{e})$ is *trivial over* C' if a homomorphism $E \rightarrow C'$ exists such that the composite $C \rightarrow E \rightarrow C'$ is the quotient homomorphism $g: C \rightarrow C'$. According to Theorem 1.2 this is equivalent to Ext $(A, g)\chi(\mathfrak{e}) = 0$.

Now suppose I is a nontrivial ideal of R. Applying the above paragraph to Theorem 4.1 we find that the statements (i), (ii), (iii) in the following theorem are equivalent.

THEOREM 5.1. *For I a proper ideal of R the following statements about an extension $(\mathfrak{e})$ are equivalent:*

(i) $\chi(\mathfrak{e})$ *is in I* Ext (A, C).

(ii) $(\mathfrak{e})$ *is trivial on* An (I, A).

(iii) $(\mathfrak{e})$ *is trivial over* C/IC.

(iv) $JC = C \cap JE$ *for every proper ideal J containing I.*

In order to prove (iv) equivalent to the others we need a lemma.

LEMMA 5.2. *If J is a proper ideal of R, then $JC = C \cap JE$ if and only if every homomorphism $J^{-1}/R \to A$ can be lifted into a homomorphism $J^{-1}/R \to E$.*

Consider a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & R & \to & J^{-1} & \to & J^{-1}/R & \to & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \to & C & \to & E & \xrightarrow{\psi} & A & \to & 0. \end{array} \tag{1}$$

It is easily shown that the existence of $\eta : J^{-1}/R \to E$ such that $\gamma = \psi\eta$ is equivalent to the existence of $\mu : J^{-1} \to C$ such that $\alpha = \mu \mid R$. Now suppose that $JC = C \cap JE$ and that $\gamma : J^{-1}/R \to A$ is given. Since J^{-1} is projective, a diagram (1) exists. Then $\alpha(1)$ is in $C \cap JE$, hence in JC. This implies the existence of $\mu : J^{-1} \to C$ such that $\mu(1) = c$. Thus $\mu \mid R = \alpha$ and γ can be lifted to η. On the other hand, suppose every such γ can be lifted, and suppose c is any element of $C \cap IE$. This implies the existence of $\beta : J^{-1} \to E$ such that $c = \beta(1)$. Then a diagram (1) is obtained. Since γ can be lifted, there is a $\mu : J^{-1} \to C$ such that $\mu(1) = c$; i.e., c is in JC. Thus $JC = C \cap JE$.

Returning to the proof of the theorem, assume that ($\mathfrak{e}$) is trivial on An (I, A). There is a homomorphism ω:An $(I, A) \to E$ such that $\psi\omega$ is the identity where $\psi : E \to A$ is the epimorphism of ($\mathfrak{e}$). If $\gamma : J^{-1}/R \to A$ with J containing I, then the image of γ is contained in An (I, A). The homormorphism $\omega\gamma$ is a lifting of γ into E. By Lemma 5.2 this implies $JC = C \cap JE$. Thus (iv) is proved.

Conversely, suppose (iv) true. Since An (I, A) has bounded order, it is a direct sum of cyclic modules each of which is isomorphic to J^{-1}/R for some J containing I. In view of the lemma, (iv) implies that ($\mathfrak{e}$) is trivial on each of these direct summands. Since Ext commutes with direct sums in the first factor, this implies that ($\mathfrak{e}$) is trivial on An (I, A).

An extension ($\mathfrak{e}$) is a *pure extension* if $rC = C \cap rE$ for every r in R. Kaplansky [4] has shown that purity is equivalent to the following: for each element a of A with order ideal I, there is an element e in E mapping onto a and having order ideal I. If a in A has order ideal I, then the submodule of A generated by a is isomorphic to I^{-1}/R. It follows from Lemma 5.2 that purity is equivalent to the relation $IC = C \cap IE$ holding for every ideal I. For any module M let $R^{\omega}M$ denote the intersection of all the modules IM with I ranging over the proper ideals of R. We can then state

COROLLARY 5.3. *($\mathfrak{e}$) is a pure extension if and only if $\chi(\mathfrak{e})$ is in R^{ω}* Ext (A, C).

In [3] Eilenberg and Mac Lane have considered the group of abelian group extensions which are trivial on finitely generated subgroups. These are evidently equivalent to the pure extensions.

6. The further discussion of the module Ext (A, C) will be facilitated by the introduction of the P-adic topology. If P is a prime ideal and M is any R-module, the *P-adic topology* on M has, as a base at 0, all submodules P^nM where n ranges over the positive integers. Addition is continuous in both variables jointly, and the maps $x \to rx$, r in R, are continuous. The topology is, in general, not Hausdorff, the closure of zero being the submodule $P^\omega M = \bigcap_n P^nM$. If we say that M is complete in the P-adic topology, we shall also mean that M is Hausdorff, i.e. $P^\omega M = 0$.

For any M, the modules M/P^nM form an inverse system in a natural fashion. Its limit will be denoted by M_P^*. The natural homomorphism $M \to M_P^*$ induced by the projections $M \to M/P^nM$ has $P^\omega M$ as its kernel. The discrete topology on each M/P^nM induces a complete topology on M_P^* whose base at zero consists of the kernels of the various projections $M_P^* \to M/P^nM$. It will be proved presently that this topology on M_P^* is the P-adic topology; hence M_P^* is complete in the P-adic topology. Thus M_P^* will be called the P-adic completion of M.

Let A be the direct limit of modules A_α with maps $\phi_{\alpha\alpha'}: A_\alpha \to A_{\alpha'}$. Then, for any module C, the modules Ext (A_α, C) form an inverse system of modules with maps Ext $(\phi_{\alpha\alpha'}, C)$. The homomorphisms Ext (ϕ_α, C):Ext $(A, C) \to$ Ext (A_α, C) induce a map

$$\rho: \text{Ext}\,(A, C) \to \varprojlim \text{Ext}\,(A_\alpha, C).$$

If the system A_α is indexed by the positive integers in their natural order, the maps of the system are completely determined by the maps $\phi_{i,i+1}: A_i \to A_{i+1}$. In this case we will call the system A_i a *direct sequence* of modules.

LEMMA 6.1. *If A is the direct limit of a sequence of modules A_i, then the map ρ is an epimorphism.*

Let y be any element of $\varprojlim$ Ext (A_i, C) and let y_i be its component in Ext (A_i, C). Then $y_i =$ Ext $(\phi_{i+1}, C)y_{i+1}$ where $\phi_{i,i+1}: A_{i+1} \to A_i$ is a map of the sequence A_i. If $(\mathfrak{e}_i): 0 \to C \to E_i \to A_i \to 0$ is such that $y_i = \chi(\mathfrak{e}_i)$ we can find $(\mathfrak{e}_{i+1}): 0 \to C \to E_{i+1} \to A_{i+1} \to 0$ such that the diagram

$$\begin{array}{llllllll} (\mathfrak{e}_i) & 0 \to & C & \to & E_i & \longrightarrow & A_i & \longrightarrow 0 \\ & & \| & & \downarrow \psi_{i,i+1} & & \downarrow \phi_{i,i+1} & \\ (\mathfrak{e}_{i+1}) & 0 \to & C & \to & E_{i+1} & \longrightarrow & A_{i+1} & \longrightarrow 0 \end{array}$$

commutes. Starting with any $(\mathfrak{e}_1)$ such that $y_1 = \chi(\mathfrak{e}_1)$, we define by recursion a direct sequence $(\mathfrak{e}_i)$ of extensions of C by A_i. Since the taking of direct limits commutes with exact sequences, we have, setting $E = \varinjlim E_i$, an extension $(\mathfrak{e}): 0 \to C \to E \to A \to 0$ as the direct limit of the extensions

$(\mathfrak{e}_i)$. Therefore, for every i, a commutative diagram

$$\begin{array}{llllllllll} (\mathfrak{e}_i) & 0 \to & C & \to & E_i & \to & A_i & \to 0 \\ & & \| & & \downarrow \psi_i & & \downarrow \phi_i & \\ (\mathfrak{e}) & 0 \to & C & \to & E & \to & A & \to 0 \end{array}$$

exists. Thus $y_i = \chi(\mathfrak{e}_i) = \text{Ext}(\phi_i, C)\chi(\mathfrak{e})$ for every i and $y = \chi(\mathfrak{e})$ by the definition of ρ.

THEOREM 6.2. *If P is a prime ideal and A, C are any modules, then the natural map $\iota_P:\text{Ext}(A, C) \to \text{Ext}(A, C)^*_P$ is an epimorphism with kernel $P^\omega \text{Ext}(A, C)$, and the module $\text{Ext}(A, C)^*_P$ is complete in the P-adic topology.*

Since $t_P A = \varinjlim \text{An}(P^n, A)$, the map

$$\rho:\text{Ext}(t_P A, C) \to \varprojlim \text{Ext}(\text{An}(P^n, A), C)$$

is an epimorphism. Since the map $\text{Ext}(A, C) \to \text{Ext}(t_P A, C)$ is an epimorphism, their composite $\rho':\text{Ext}(A, C) \to \varprojlim \text{Ext}(\text{An}(P^n, A), C)$ is also epimorphic. From Theorem 4.1 we have identifications $\text{Ext}(\text{An}(P^n, A), C) = \text{Ext}(A, C)/P^n \text{Ext}(A, C)$, and therefore $\varprojlim \text{Ext}(\text{An}(P^n, A), C) = \text{Ext}(A, C)^*_P$. The map ρ' then becomes identified with the map ι_P.

The module $\text{Ext}(A, C)^*_P$ is complete in the topology having as a base at 0 the kernels of the maps $f_n^*:\text{Ext}(A, C)^*_P \to \text{Ext}(A, C)/P^n \text{Ext}(A, C)$. To show that the module is complete in the P-adic topology we need only show that $\text{Ker} f_n^* = P^n \text{Ext}(A, C)^*_P$. It is clear that $P^n \text{Ext}(A, C)^*_P \subseteq \text{Ker} f_n^*$. To show the converse inclusion we use the fact that ι_P is epimorphic. If y is any element of $\text{Ker} f_n^*$, there is an element z of $\text{Ext}(A, C)$ such that $y = \iota_P z$; hence $0 = f_n^* \iota_P z$. Now $f_n^* \iota_P:\text{Ext}(A, C) \to \text{Ext}(\text{An}(P^n, A), C)$ is induced by the inclusion $\text{An}(P^n, A) \to A$. It therefore follows from Theorem 4.1 that z is in $P^n \text{Ext}(A, C)$, and therefore $y = \iota_P z$ is in $P^n \text{Ext}(A, C)^*_P$.

THEOREM 6.3. *If P is a prime ideal, there exists a natural isomorphism*

$$C^*_P \approx \text{Ext}(K_P, C)^*_P$$

such that the diagram

$$\begin{array}{ccc} C & \xrightarrow{\delta} & \text{Ext}(K_P, C) \\ \downarrow & & \downarrow \\ C^*_P & \approx & \text{Ext}(K_P, C)^*_P \end{array}$$

commutes, where δ is the connecting homomorphism associated with the sequence $0 \to R \to Q_P \to K_P \to 0$. Thus C^*_P is always complete in the P-adic topology.

For each positive integer n there is a commutative diagram

$$
(*) \quad \begin{array}{ccccc}
C & \xrightarrow{\delta} & \operatorname{Ext}(K_P, C) & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow \\
C/P^nC & \approx & \operatorname{Ext}(P^n/R, C) & \approx & \operatorname{Ext}(K_P, C)/P^n \operatorname{Ext}(K_P, C).
\end{array}
$$

The two isomorphisms are provided by Theorems 3.1 and 4.1 respectively. The outside vertical maps are quotient maps, while the central one is induced by the inclusion $P^{-n}/R \to K_P$. Composing the two isomorphisms gives, for each n, $C/P^nC \approx \operatorname{Ext}(K_P, C)/P^n \operatorname{Ext}(K_P, C)$. Thus

$$C_P^* = \varprojlim C/P^nC \approx \varprojlim \operatorname{Ext}(K_P, C)/P^n \operatorname{Ext}(K_P, C) = \operatorname{Ext}(K_P, C)_P^* .$$

This is the required isomorphism. The commutativity of the diagram follows from (*). The remainder of the proposition is then an immediate consequence of the preceding theorem.

The preceding discussion considered each prime ideal separately. There remains the task of fitting the pieces together. This is quite simple to do. The epimorphisms ι_P mentioned in Theorem 6.2 define an epimorphism $\iota_R:\operatorname{Ext}(A, C) \to \prod_P \operatorname{Ext}(A, C)_P^*$, the product ranging over the prime ideals of R. The kernel of ι_R is $\bigcap_P P^\omega \operatorname{Ext}(A, C) = R^\omega \operatorname{Ext}(A, C)$. Thus $\operatorname{Ext}(A, C)$ is, modulo $R^\omega \operatorname{Ext}(A, C)$, the direct product of its P-adic completions.

By introducing yet another topology one can give the above discussion a little more unity. The *R-topology*, on a module M, has all the submodules IM (for I a nonzero ideal of R) as a base at zero. The closure of 0 is now $R^\omega M$. We denote by M_R^* the completion of M in the R-topology. Using Theorem 2 of [5] and the fact that each module M/IM is the direct product of its P-primary components, one can show that $M_R^* \approx \prod_P M_P^*$. In M_P^* one has the relation $IM_P^* = P^nM_P^*$, for any ideal I, where P^n is the highest power of P dividing I. It is then easy to show that the cartesian product topology on $\prod_P M_P^*$ is the R-topology. Hence M_R^* is complete in the R-topology. Now Theorem 5.2 is still true if P is everywhere replaced by R.

7. A module M will be called *realizable* if there are modules A and C such that $M \approx \operatorname{Ext}(A, C)$. The basic theorem for the study of realizable modules is the following:

THEOREM 7.1. *The following four statements about M are equivalent:*

(a) $\operatorname{Ext}(A, M) = 0$ *for every torsion-free module A.*

(b) $\operatorname{Ext}(Q, M) = 0$.

(c) $M/dM \approx \operatorname{Ext}(K, M)$.

(d) M/dM *is realizable.*

COROLLARY 7.2. *If M is reduced, then it is realizable if and only if* $\operatorname{Ext}(Q, M) = 0$.

It is clear that (a) implies (b) and (c) implies (d). To show that (b) implies (c), we consider the exact sequence

$$\text{Hom}\,(Q,\,M) \to M \to \text{Ext}\,(K,\,M) \to \text{Ext}\,(Q,\,M).$$

The image of the leftmost homomorphism is dM, while (b) gives $\text{Ext}\,(Q,\,M) = 0$. This proves (c). Now suppose (d) is true so that $M/dM \approx \text{Ext}\,(B,\,C)$ for some modules B and C. Since $\text{Ext}\,(A,\,M) \approx \text{Ext}\,(A,\,M/dM)$, we have, by Theorem 2.1, $\text{Ext}\,(A, M) \approx \text{Ext}\,(\text{Tor}\,(A, B), C)$. If A is torsion-free, $\text{Tor}\,(A,\,B) = 0$, so $\text{Ext}\,(A,\,M) = 0$. This proves (a).

The corollary is an immediate consequence of the theorem.

THEOREM 7.3. *Let M be any module, and let L be a submodule of M. Then the following statements are true:*

(a) *If M is reduced and realizable, then $(M/L)/d(M/L)$ is realizable and*

$$\text{Hom}\,(Q,\,M/L) \approx \text{Ext}\,(Q,\,L). \tag{1}$$

(b) *If M is reduced and realizable and either M/L is reduced or L is realizable, then both L and M/L are reduced and realizable.*

(c) *If L and M/L are both reduced and realizable, then M is reduced and realizable.*

(d) *If M is reduced and realizable, so is every direct summand of M.*

(e) *The direct product of a family of reduced realizable modules is also reduced and realizable.*

We consider the exact sequence

$$\begin{aligned} &\text{Hom}\,(Q,\,M) \to \text{Hom}\,(Q,\,M/L) \to \text{Ext}\,(Q,\,L) \\ &\qquad\qquad \to \text{Ext}\,(Q,\,M) \to \text{Ext}\,(Q,\,M/L) \to 0. \end{aligned} \tag{2}$$

As stated in Theorem 3.2b, M is reduced if and only if $\text{Hom}\,(Q,\,M) = 0$. If M is reduced and realizable, then $\text{Hom}\,(Q,\,M) = 0 = \text{Ext}\,(Q,\,M)$. This and the exactness of (2) give the isomorphism (1) and $\text{Ext}\,(Q,\,M/L) = 0$. Then (a) is an immediate consequence of Theorem 7.1.

To prove (b) we note first that, in view of the isomorphism (1), the conditions M/L reduced and L realizable are equivalent. If either condition is true, then so is the other, and M/L is reduced and realizable by (a). But $\text{Ext}\,(Q,\,L) = 0$ follows from consideration of the sequence (2); hence L, being reduced, is realizable according to Corollary 7.2.

The proof of (c) is similar to the foregoing, using the exactness of (2).

A direct summand of M is a submodule, hence reduced if M is, and a factor module. Thus (d) is a direct consequence of (b).

As for (e), suppose that $M = \prod_\alpha M_\alpha$ with each M_α reduced and realizable. Then $\text{Hom}\,(Q,\,M) = \prod_\alpha \text{Hom}\,(Q,\,M_\alpha) = 0$ so that M is reduced; and $\text{Ext}\,(Q,\,M) = \prod_\alpha \text{Ext}\,(Q,\,M_\alpha) = 0$ so that M is realizable.

THEOREM 7.4. *Every module complete in the R-topology is realizable.*

If M is complete in the R-topology, then $M = \prod_P M_P^*$. In view of Theorem 7.3e we can assume M is complete in a P-adic topology. Since M is reduced, the sequence $0 \to M \to \text{Ext}(K_P, M) \to \text{Ext}(Q_P, M) \to 0$ is exact. Since $M = M_P^*$, Theorem 6.3 shows that the homomorphism

$$M \to \text{Ext}(K_P, M)$$

has a left inverse, and hence the above sequence splits. Since $\text{Ext}(K_P, M)$ is reduced and $\text{Ext}(Q_P, M)$ is divisible (K_P is a torsion module and Q_P is torsion-free), this implies that $\text{Ext}(Q_P, M) = 0$. Thus $M \approx \text{Ext}(K_P, M)$ and is realizable.

LEMMA 7.5. *If A is a torsion module and C is torsion-free, then $R^\omega \text{Ext}(A, C) = 0$.*

Since the elements of $R^\omega \text{Ext}(A, C)$ are just the pure extensions of C by A, it suffices to show that if E is any module containing C as a pure submodule and E/C is a torsion module, then C is a direct summand of E. Since $C \cap tE = 0$, it is sufficient to show that $C + tE = E$. Let x be any element of E and suppose it maps onto y in E/C where $ry = 0$ with r a nonzero element of R. Then $rx = z$ is in C. By purity, $z = rw$ with w in C. Then $x = w + (x - w)$ is the desired expression showing that x is in $C + tE$.

LEMMA 7.6. *A torsion module is complete in the R-topology if and only if it has bounded order.*

If M is complete in the R-topology, then $M = \prod_P M_P^*$ where the M_P^* range over the P-adic completions of M. Each M_P^* is P-primary for the corresponding prime ideal. Hence the number of ideals P for which $M_P^* \neq 0$ is finite because an element of M with a nonzero component in each M_P^* has order ideal 0. This reduces the problem to showing that a P-primary module complete in the P-adic topology has bounded order. Suppose M is such a module. For each positive integer n, $\text{An}(P^n, M)$ is closed and $M = \bigcup_n \text{An}(P^n, M)$. Since M is complete, it is of the second category. Therefore $\text{An}(P^n, M)$ has an interior for some n, and being a submodule, it is open. Then an integer exists for which $P^m M \subseteq \text{An}(P^n, M)$. Therefore $P^{m+n} M = 0$, showing that M has bounded order. On the other hand, suppose M is a module such that $IM = 0$ for some $I \neq 0$. Then the R-topology on M is discrete; hence M is complete.

THEOREM 7.7. *If M is reduced, realizable, and either a torsion module or a torsion-free module, then M is complete in the R-topology.*

Since M is reduced and realizable, $M = \text{Ext}(K, M)$; hence $M/R^\omega M$ is complete in the R-topology by Theorem 6.2. If M is torsion-free, $R^\omega M = R^\omega \text{Ext}(K, M) = 0$, by Lemma 7.5. Hence M is complete in this case. If M is a torsion module, $M/R^\omega M$ has bounded order, by Lemma 7.6.

Suppose $0 \neq r$ in R annihilates $M/R^\omega M$ so that $rM \subsetneq R^\omega M$. If u is any element of $R^\omega M$ and s is any nonzero element of R, then $u = srv$ for some v in M. But rv is in $R^\omega M$; hence u is divisible by s in $R^\omega M$. Hence $R^\omega M$ is divisible. Since M is reduced, this implies $R^\omega M = 0$ and M is complete.

COROLLARY 7.8. *A reduced realizable torsion module has bounded order. If C is a torsion module, then* Ext $(A, C) = 0$ *for every torsion-free module A if and only if C is the direct sum of a divisible module and a module with bounded order.*

The second part of the corollary was first proved by Baer in [1] for R the ring of rational integers.

COROLLARY 7.9. *The ring R is realizable as an R-module if and only if it is a complete discrete valuation ring.*

Since R is torsion-free it is realizable if and only if it is complete in the R-topology. If it is complete in this topology, then it is ring isomorphic to the direct product of its P-adic completions (see [5]). If there is more than one factor present, R has zero-divisors. Hence R has only one prime ideal.

THEOREM 7.10. *If any nonzero projective R-module is realizable, then R is realizable. If R is realizable, then every finitely generated module is realizable. If R is not realizable, then a finitely generated module is realizable if and only if it is a torsion module.*

In view of [4, Theorem 4], every projective module is a direct sum of ideals. Hence the existence of a nonzero realizable projective module implies, by Theorem 7.3d, the existence of a nonzero realizable ideal I. Since R/I has bounded order, it is reduced and realizable. Thus both I and R/I are reduced and realizable; hence R is realizable by Theorem 7.3c.

It is shown in [4, Theorem 1] that a finitely generated module A is the direct sum of its torsion submodule tA and a finitely generated projective module B. Suppose R is realizable. Every ideal of R is realizable by Theorem 7.3b. Then B, being a direct product of ideals, is realizable by Theorem 7.3e. The module tA has bounded order, and hence is realizable in any case. Thus $A = tA + B$ is realizable by Theorem 7.3e. If, on the other hand, B is not zero and A is realizable, then B is also realizable; hence R is realizable by the first part of the theorem. This proves the last statement of the theorem.

The preceding results concerned the realizability of reduced modules. The final result of this section goes to the opposite extreme.

THEOREM 7.11. *Every torsion-free divisible module is realizable.*

Let C be a torsion-free divisible module. Suppose the existence of a reduced module M and a submodule L such that $C \approx M/L$. By Theorem 7.3a, $C \approx$ Hom $(Q, C) \approx$ Ext (L, C); hence C is realizable. To find such an

M and L we proceed as follows. The module C, being torsion-free divisible, is a direct sum of copies of Q. The number of copies is the rank of C. Let F be a torsion-free module not complete in the R-topology, and let F^* be its completion. If R is not complete, then F can be taken to be R; in any event, F can be taken to be a countable direct sum of copies of R. Then F^* is reduced realizable, and F^*/F is torsion-free divisible. By taking the direct product of sufficiently many copies of the sequence $0 \to F \to F^* \to F^*/F \to 0$, we get a sequence $0 \to L \to M \to M/L' \to 0$ in which M is reduced realizable and M/L' is torsion-free divisible with rank $\geqq$ rank C. Then $M/L' \approx C + D$ for some module D; hence there exists an epimorphism $M/L' \to C$. Let L be the kernel of the composite map $M \to M/L' \to C$. The modules M and L have the required properties.

8. In view of Theorem 7.1 the reduced realizable modules have the following property: *A module M is a direct summand of every module containing it as a submodule closed under division if and only if M is the direct sum of a divisible module and a reduced realizable module.* A submodule M of a module E is closed under division if E/M is torsion-free.

Corollary 7.8 states that a torsion module satisfies the first half of the above equivalence if and only if it is the direct sum of a divisible module and a module with bounded order. This result is due, for R the ring of integers, to Baer [1].

Baer also attempts to determine all those abelian groups A which are a direct summand of every abelian group E such that $A = E/tE$. He finds that if A has countable rank, then it is free. This result, generalized to Dedekind rings, can be rephrased as follows: *If* Ext $(A, C) = 0$ *for every torsion module C and A has countable rank, then A is projective.* This will be proved in Theorem 8.4.

LEMMA 8.1. *If I is a nonzero ideal of R, I* Ext $(A, C) = 0$, *and A has an element with nonzero order ideal J, then $IC \subseteq JC$.*

The hypothesis states the existence of a monomorphism $J^{-1}/R \to A$. This produces an epimorphism Ext $(A, C) \to$ Ext $(J^{-1}/R, C) = C/JC$. If I Ext $(A, C) = 0$, we have $I(C/JC) = 0$, hence $IC \subseteq JC$.

LEMMA 8.2. *If R is not realizable and* Ext (A, R) *is finitely generated, then tA has bounded order.*

Since R is not realizable, Ext (A, R) is a torsion module by Theorem 7.10. Hence it has bounded order. Suppose r Ext $(A, R) = 0$ for $r \neq 0$. Let a be any nonzero element of tA, and let J be its order ideal. In view of Lemma 8.1 we have $rR \subseteq J$, i.e., r is in I. Hence $ra = 0$. Since a was arbitrary in tA, we have $r(tA) = 0$.

LEMMA 8.3. *If A is a torsion-free module for which every submodule with finite rank is projective, then every submodule of A with countable rank is projective.*

We may as well assume that A itself has countable rank. We build a sequence $\{K_j\}$ of submodules of A with the properties:
(i) $K_1 \subsetneqq K_2 \subsetneqq K_3 \subsetneqq \cdots$,
(ii) each K_j is projective,
(iii) K_{j+1} is the direct sum of K_j and a projective module,
(iv) A is the union of the K's.
It is clear that the existence of such a sequence implies that A is projective.

Since A has countable rank, there exists a countable maximal independent subset $\{x_j\}$. Let K_j be the pure submodule of A generated by $x_1, \cdots, x_j$. Then (i) is satisfied and (ii) holds by hypothesis, since K_j has finite rank. To prove (iii) note that each K_j is pure in K_{j+1} so that K_{j+1}/K_j is torsion-free. Since K_{j+1} is projective and has finite rank, it is finitely generated. Thus K_{j+1}/K_j is finitely generated and, hence, projective by [4, Theorem 1]. We then have $K_{j+1} = K_j \oplus (K_{j+1}/K_j)$ so that (iii) holds. Since $\{x_j\}$ is a maximal independent subset of A, we have, for each a in A, a nonzero r in R and a natural number j such that ra is a linear combination of $x_1, \cdots, x_j$. Thus a is in K_j, because K_j is closed under division. This proves (iv).

THEOREM 8.4. *If A satisfies any one of the hypotheses:*
(i) $\operatorname{Ext}(A, C) = 0$ *for every torsion module C,*
(ii) $\operatorname{Ext}(A, C) = 0$ *for $C = \sum_I (I^{-1}/R)$ where I ranges over the nonzero ideals of R,*
(iii) $\operatorname{Ext}(A, R) = 0$ *and R is not realizable,*
then A is torsion-free and every submodule of A with countable rank is projective.

If A satisfies hypothesis (i), it also satisfies hypothesis (ii). We therefore assume that $\operatorname{Ext}(A, C) = 0$, where either $C = \sum_I (I^{-1}/R)$ or $C = R$ with R not realizable. In either alternative C is divisible by no prime ideal. For each prime ideal P, $\operatorname{Ext}(A, C)$ is P-divisible; hence A has no P-torsion by Theorem 4.5. Thus A is torsion-free.

In view of Lemma 8.3 we need only show that every submodule of A with finite rank is projective. Since $\operatorname{Ext}(A, C) = 0$ implies $\operatorname{Ext}(B, C) = 0$ for every submodule B of A, we can assume that A itself has finite rank and show that it is projective.

If A has finite rank, there is a positive integer n and a torsion module T such that

$$0 \to R^n \to A \to T \to 0 \tag{1}$$

is exact. If T has bounded order, then A is projective. Indeed, suppose $rT = 0$ for $r \neq 0$, and embed A in $A \otimes Q$. Let $a_1, \cdots, a_n$ be elements of A

generating R^n as a submodule of A. The a_i are linearly independent, and, for any a in A, ra is a linear combination of them. Then $r^{-1}a_1, \cdots, r^{-1}a_n$ generates a free submodule of $A \otimes Q$ containing A. Hence A as a submodule of a projective module is itself projective. Therefore, we have only to show that the module T in (1) has bounded order.

If we apply Hom (, C) to the sequence (1) and identify Hom (R^n, C) with C^n we get an exact sequence

$$C^n \to \text{Ext}(T, C) \to 0, \tag{2}$$

where the 0 occurs because Ext $(A, C) = 0$. If $C = R$, then Ext (T, C) is finitely generated. If R is not realizable, then T has bounded order by Lemma 8.2. If $C = \sum_I (I^{-1}/R)$, then C is a torsion module. From (2) we deduce that Ext (T, C) is a torsion module. Since T is a torsion module, Ext (T, C) is reduced; hence it has bounded order by Corollary 7.8. Suppose r Ext $(T, C) = 0$ for some $r \neq 0$ in R, suppose a is any nonzero element of T, and let its order ideal be I. Then Lemma 8.1 gives us $rC \subsetneq IC$. Since I^{-1}/R is a direct summand of C, we have $r(I^{-1}/R) \subsetneq I(I^{-1}/R) = 0$. Therefore r is in I. This means that $ra = 0$. Since a was arbitrary in T, $rT = 0$. Thus in either case T has bounded order as required.

THEOREM 8.5. *If* Hom $(A, R) = 0 =$ Ext (A, R), *then A is a divisible torsion-free module. If, in addition, R is not realizable, then $A = 0$.*

Since R is P-divisible for no prime ideal, Theorem 4.5 and the hypothesis Ext $(A, R) = 0$ imply that A is torsion-free. Now let r be any nonzero element of R. The sequence

$$0 \to A \xrightarrow{r} A \to A/rA \to 0$$

is exact. Applying Ext (, R) we get an exact sequence:

$$\text{Hom}(A, R) \to \text{Ext}(A/rA, R) \to \text{Ext}(A, R).$$

The hypotheses of the theorem imply Ext $(A/rA, R) = 0$. Hence A/rA is torsion-free. Since, on the other hand, A/rA is a torsion module, it is 0; hence $rA = A$. Since r was arbitrary, it follows that A is divisible.

If R is realizable, no more can be said; every torsion-free divisible module satisfies the hypotheses of the theorem. If R is not realizable and A is a nonzero torsion-free divisible module, A contains Q as a direct summand. Since Ext $(Q, R) \neq 0$, it follows that Ext $(A, R) \neq 0$. This proves the last statement of the theorem.

BIBLIOGRAPHY

1. REINHOLD BAER, *The subgroup of the elements of finite order of an abelian group*, Ann. of Math. (2), vol. 37 (1936), pp. 766–781.

2. H. CARTAN AND S. EILENBERG, *Homological algebra*, Princeton, Princeton University Press, 1956.
3. S. EILENBERG AND S. MAC LANE, *Group extensions and homology*, Ann. of Math. (2), vol. 43 (1942), pp. 757–831.
4. I. KAPLANSKY, *Modules over Dedekind rings and valuation rings*, Trans. Amer. Math. Soc., vol. 72 (1952), pp. 327–340.
5. DANIEL ZELINSKY, *Rings with ideal nuclei*, Duke Math. J., vol. 18 (1951), pp. 431–442.

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