NONALTERNATING LINKS

BY
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1. Introduction

A link projection[1] is said to be *alternating* if and only if it is connected and, as one follows along any component of the link, undercrossings and overcrossings alternate. A connected projection with no crossings, is included as alternating. Figure 1 below shows an alternating and a nonalternating projection. A *link type is alternating* according as it possesses or does not possess an alternating projection. For example, it is obvious that both projections of Figure 1 represent alternating types. Several interesting and important theorems have been proved for link types of this class. In particular the conjectures of asphericity and of the knot-theoretic formulation of Dehn's lemma, both now known to hold for all tame knot types [7], were first proved for alternating link types [1, 3]. The question naturally arises as to whether tame, nonalternating types exist. An affirmative answer was given by Bankwitz in 1930 [2]. Unfortunately, his paper contains an error. However, a proof of his principal result by a completely different method is included in the present paper (Theorem 5.5).

After the problem of existence comes the question of recognition: Can one decide from a given link projection whether or not it represents an alternating type? The general answer is unknown; but a good place to begin is with examples and a good place to find them is the Knot Table in the back of Reidemeister's *Knotentheorie* [8]. This Knot Table consists of 84 knot

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[1] A *link* L of *multiplicity* μ is the union of μ ordered, oriented, and pairwise disjoint topological circles (1-spheres) L_i imbedded in the 3-sphere S^3. Two links L and L' are *equivalent* if and only if $\mu = \mu'$ and there exists an orientation preserving homeomorphism f of S^3 on itself such that $fL_i = L_i'$ and $f \mid L_i$ is also orientation preserving, $i = 1, \cdots, \mu$. An equivalence class of links is a *link type*. A *knot* is a link of multiplicity $\mu = 1$. For any link L, we may select a "point at infinity" $\infty \in S^3 - L$ and consider a Cartesian coordinate system $R \times R \times R = S^3 - \infty$. The projection

$$p: S^3 \to S^2$$

defined by $p(\infty) = \infty$ and $p(x, y, z) = (x, y)$ is said to be *regular* if and only if (i) $p \mid L$ is a homeomorphism except for at most a finite number of double points, and (ii) for each double point $p(a) = p(b)$, $a, b \in L$, $a \neq b$, L is linear in every sufficiently small neighborhood of a and of b (the one of a and b with the larger z-coordinate is the *overcrossing* and the other is the *undercrossing*). Condition (ii) is just one of several ways of insuring that each double point describes a genuine crossing. By the *link type* of p is meant, of course, the link type of L. A given link type is *tame* if and only if it possesses a regular projection. The projection p is *connected* if and only if the image $P = p(L)$ is connected. Finally, in this paper all link projections are assumed to be regular and all link types, tame.

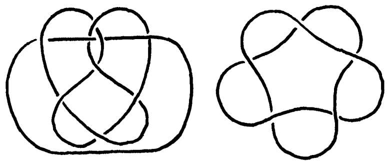


Figure 1

projections representing the 84 simplest, distinct, prime knot types. Of these 84 projections, 73 are alternating and 11 are not. We shall prove that 7 of these 11 nonalternating projections represent nonalternating types.

It may be remarked that our statement of the Bankwitz theorem is a generalization of the original to the extent that the basic inequality is here proved for alternating types of arbitrary multiplicity. We also note that we have used the phrase "spherical graph" throughout instead of "planar graph." For example, the image of a link projection obviously possesses a decomposition as a spherical graph (cf. opening sentence of Section 3).

I should like to express my thanks to Professor R. H. Fox of Princeton University, who supervised my doctoral thesis, for his encouragement and interest in this research, which grew out of and is partially included in the results of the thesis. I am also indebted to Dr. J. P. Mayberry, now of the Radio Corporation of America, who first suggested that the matrix-tree theorem (cf. (4.6) and preceding paragraph; also [6]) could be used to prove the Bankwitz theorem.

2. Graph theory

The results of this section are based on Hassler Whitney's important paper *Non-separable and planar graphs* [11] which is hereafter referred to as HW.

By a *graph* we shall mean a finite set of points, called vertices, and open arcs, called edges, together with incidence relations which make the collection a cell-complex. (Thus, formally, a graph is a finite CW complex of dimension less than 2.) The underlying space of a graph G is denoted by $|G|$ and the i^{th} Betti number by $p_i(G)$. A tree T is a connected graph such that $p_1(T) = 0$. A graph G possesses a *nontrivial factorization* into subgraphs G_1 and G_2 if and only if $G = G_1 \cup G_2$, neither of G_1 and G_2 is contained in the other, and G_1 and G_2 intersect in a single vertex of G (called a *cut vertex*). A graph G is *nonseparable* if and only if it is connected and possesses no nontrivial factorization (possesses no cut vertex); otherwise, G is *separable*. A *loop* is a graph containing exactly one edge and one vertex. The *order* of a vertex v of a graph G is the number of edges of G to which v is incident plus the number of loops containing v (thus, it is locally the number of edges to which v is incident). An *n-circuit* is a connected graph containing exactly n vertices each of which is of order 2. Thus, a 1-circuit is the same thing as a loop. A graph is *cyclic* if and only if every edge lies in a circuit and *strongly cyclic* if and only if it is cyclic and, in addition, contains no loops. An n-

bridge of a graph G is a subset of G consisting of $n + 1$ edges and n vertices each of which is incident to 2 edges of the bridge and, furthermore, is of order 2 in G. Intuitively a bridge is "an edge which may happen to be subdivided by vertices"; a 0-bridge is just an edge. A *spherical graph* G is one imbedded in a 2-sphere, i.e., $|G| \subset S^2$, and the *regions* of a spherical graph are the components of $S^2 - |G|$.

The following result is essentially Theorem 18 of HW. The present formulation is a trivial generalization and uses Theorem 10 of HW which states that a nonseparable graph G with $p_1(G) = 1$ is a circuit.[2]

(2.1) *If G is a nonseparable graph with $p_1(G) > 0$, there exists a bridge S of G such that $G - S$ is a nonseparable graph.*

From (2.1) it is a straightforward matter to obtain

(2.2) *If G is a nonseparable graph with $p_1(G) > 1$ and S is any bridge of of G, there exists a bridge S' of G disjoint from S such that $G - S'$ is a nonseparable graph.*

Let G be a graph, V the set of vertices of G, and H any subset of G. We define ex (G, H) to be the number of trees in G which contain $H \cup V$. For connected graphs, ex (G, H) is simply the number of extensions of H to maximal trees of G. The number of trees containing V, equal to ex (G, V) or ex $(G, \emptyset)$, will be written tr (G).

(2.3) *If G is a nonseparable graph, ν the number of vertices of G, and S any n-bridge of G, then*

$$\text{ex}\,(G, S) \geqq \nu - n - 1.$$

Proof (by induction on $p_1(G)$). If $p_1(G) = 0$, G is a tree. Obviously, then, G is either void, or consists of a single point, or consists of a single edge and two distinct endpoints (See HW, Theorem 8). In the first two cases (2.3) holds vacuously. In the last, S is the single edge and $n = 0$ and $\nu = 2$. Thus,

$$\text{ex}\,(G, S) = 1 = 2 - 0 - 1.$$

If $p_1(G) = 1$, G is a ν-circuit (HW, Theorem 10). Obviously,

$$\text{ex}\,(G, S) = \nu - (n + 1).$$

We next assume that $p_1(G) > 1$. By (2.2) there exists an m-bridge S' of G such that $G - S'$ is nonseparable and contains S. Since $p_1(G - S') = p_1(G) - 1$, we have by hypothesis of induction that

$$\text{ex}\,(G - S', S) \geqq \nu - m - n - 1.$$

[2] Notice the difference between Whitney's definition of suspended chain and our definition of bridge.

Since each maximal tree of $G - S'$ can be extended to a maximal tree of G in exactly $m + 1$ ways,

$$\begin{aligned} \operatorname{ex}(G, S) &\geqq (m + 1) \operatorname{ex}(G - S', S) \\ &\geqq (m + 1)(\nu - m - n - 1) \qquad (1) \\ &= \nu - n - 1 + m(\nu - m - n - 2). \end{aligned}$$

It is obvious that any bridge of a nonseparable graph whose first Betti number is greater than 1 must have distinct endpoints. Moreover, since the bridge S is contained in $G - S'$, we may conclude that its two endpoints are included in $(G - S') - S$. Hence,

$$\nu - m - n \geqq 2, \qquad (2)$$

and (1) and (2) complete the proof.

(2.4) THEOREM. *If G is a nonseparable spherical graph with $\nu \geqq 1$ vertices and ρ regions, then*

$$\operatorname{tr}(G) - 1 \geqq (\nu - 1)(\rho - 1).$$

Proof (by induction on $p_1(G)$). If $p_1(G) = 0$, G is a tree, and

$$\operatorname{tr}(G) = \rho = 1.$$

If $p_1(G) = 1$, G is a ν-circuit (HW, Theorem 10), $\rho = 2$, and $\operatorname{tr}(G) = \nu$. Thus, for $p_1(G) \leqq 1$, the contended inequality holds as an equality. We next assume that $p_1(G) > 1$. By (2.1) there exists an m-bridge S of G such that $G - S$ is a nonseparable graph. As we remarked in the proof of (2.3), it is obvious that any bridge of a nonseparable graph whose first Betti number is greater than 1 must have distinct endpoints. Hence, we have that

$$\nu - m \geqq 2. \qquad (3)$$

Since $p_1(G - S) = p_1(G) - 1$, the number of regions of $G - S$ is $\rho - 1$ (Alexander duality). By hypothesis of induction, therefore,

$$\operatorname{tr}(G - S) \geqq (\nu - m - 1)(\rho - 2) + 1.$$

Clearly,

$$\operatorname{tr}(G) = (m + 1) \operatorname{tr}(G - S) + \operatorname{ex}(G, S).$$

Combining these results with (2.3), we have

$$\operatorname{tr}(G) \geqq (m + 1)((\nu - m - 1)(\rho - 2) + 1) + \nu - m - 1.$$

Then,

$$\begin{aligned} (\operatorname{tr}(G) - 1) - (\nu - 1)(\rho - 1) &\geqq (m + 1)((\nu - m - 1)(\rho - 2) + 1) \\ &\quad + \nu - m - 1 - (\nu - 1)(\rho - 1) - 1 \\ &= (m + 1)(\nu\rho - m\rho - \rho - 2\nu + 2m + 3) + \nu - m - 3 - \nu\rho + \rho + v \\ &= m(\nu\rho - m\rho - \rho - 2\nu + 2m + 3 - \rho + 2 - 1) \\ &= m(\rho(\nu - m - 2) - 2(\nu - m - 2)) \\ &= m(\rho - 2)(\nu - m - 2). \end{aligned}$$

Since $\rho = p_1(G) + 1$ (Alexander duality), we have that $\rho > 2$. Thus, using (3), we may conclude that

$$m(\rho - 2)(\nu - m - 2) \geqq 0,$$

and the proof is complete.

The preceding theorem is the principal result of this section. However, it is also interesting to obtain a variant of (2.4) from which the Bankwitz inequality follows. Hence, we prove that

(2.5) *If G is a nonseparable spherical graph with $\nu \geqq 1$ vertices, d edges, and ρ regions, then*

$$(\rho - 1)(\nu - 1) + 1 \geqq d.$$

Proof. G determines a cellular decomposition of the 2-sphere; so

$$\nu - d + \rho = 2.$$

Hence

$$\begin{aligned}(\rho - 1)(\nu - 1) + 1 - d &= \nu\rho - \nu - \rho + 2 - \nu - \rho + 2 \\ &= \rho(\nu - 2) - 2(\nu - 2) \\ &= (\rho - 2)(\nu - 2).\end{aligned}$$

If $p_1(G) = 0$, then G consists either of a single point or of a single edge and two distinct endpoints (HW, Theorem 8). Thus, $\rho = 1$ and $\nu = 1$ or 2. If $p_1(G) = 1$, then $\rho = 2$. If $p_1(G) > 1$, then $\rho > 2$ and G contains a circuit which cannot be a loop; so $\nu \geqq 2$. In all cases, therefore,

$$(\rho - 2)(\nu - 2) \geqq 0,$$

and the proof is complete.

Consider a nontrivial factorization of a graph G into subgraphs G_1 and G_2. It is easy to check that

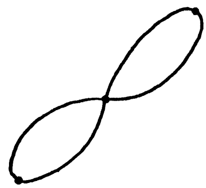
(2.6) *If H is any nonseparable subgraph of G, then either $H \subset G_1$ or $H \subset G_2$.*

(2.7) *G is connected if and only if both G_1 and G_2 are connected.*

(2.8) $\operatorname{tr}(G) = \operatorname{tr}(G_1)\operatorname{tr}(G_2)$.

Notice also that, if G is connected, each of G_1 and G_2 must contain at least one edge.

Starting with any graph G, we may obtain by successive factorization a decomposition of G into subgraphs $H_1, \cdots, H_m$ each of which possesses no nontrivial factorization. It is a consequence of (2.6) and induction that any nonseparable subgraph of G lies wholly in one of $H_1, \cdots, H_m$. Furthermore, it is easy to check that no one of $H_1, \cdots, H_m$ is contained in any other. Let us assume, for the moment, that G is connected. By (2.7) and induction, we may then conclude that each one of $H_1, \cdots, H_m$



$\nu = 3$
$d = 4$
$\rho = 3$
$\text{tr}(G) = 4$

Figure 2

is nonseparable. It follows immediately that the decomposition is unique, and we therefore call $H_1, \cdots, H_m$ the *nonseparable factors*[3] of the connected graph G. The nonseparable factors of an arbitrary graph (not necessarily connected) are defined to be those of its components. A consequence of the last sentence of the preceding paragraph is that if G is connected and $m \geqq 2$, then each of $H_1, \cdots, H_m$ contains an edge. Since any nonseparable subgraph lies wholly in one of the nonseparable factors, we have the further result that

(2.9) *If $m \geqq 2$, each of the nonseparable factors $H_1, \cdots, H_m$ of a connected, strongly cyclic graph must contain at least two edges.*

The above mentioned variant of (2.4) is now

(2.10) THEOREM. *If a spherical graph G containing d edges is connected and strongly cyclic, then*

$$\text{tr}(G) \geqq d.$$

Proof. We may assume immediately that G is nonvoid since otherwise $\text{tr}(G) = d = 0$. Let us denote the nonseparable components of G by

$$H_1, \cdots, H_m$$

and the number of edges of H_i by d_i, $i = 1, \cdots, m$. As a consequence of (2.4) and (2.5), we may conclude that

$$\text{tr}(H_i) \geqq d_i. \tag{4}$$

Thus, if $m = 1$, we have $H_1 = G$ and $d_1 = d$, and the proof is complete. So we next assume that $m \geqq 2$. From (2.8) and induction, we obtain

$$\text{tr}(G) = \textstyle\prod_{i=1}^{m} \text{tr}(H_i) \tag{5}$$

and, from (2.9),

$$d_i \geqq 2. \tag{6}$$

Since the product of integers greater than 2 is greater than or equal to their sum, we may combine (4), (5), and (6) to obtain

$$\text{tr}(G) \geqq \textstyle\sum_{i=1}^{m} d_i = d.$$

[3] Whitney calls them non-separable components.

It should be remarked that (2.10) has really nothing to do with spherical graphs as such; it is perfectly valid if the word "spherical" is dropped [3]. Notice also that the stronger inequality of Theorem (2.4) is false for separable graphs. For an example, see Figure 2.

3. The graphs of a link projection

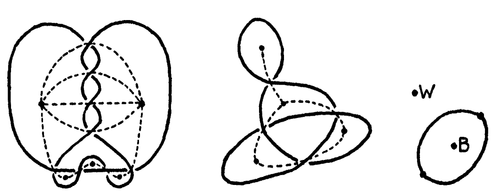
The image $P = p(L)$ of a link projection may be characterized as a nonvoid spherical graph, each of whose vertices is of order 4 or 2. The vertices of order 4 are the *crossings* of P, i.e., the double points of the projection. Those of order 2 serve only to provide a true cellular decomposition of P when some component of L contains no over- or undercrossing. It is a consequence of the fact that no vertex is of odd order that the regions of P may be shaded black and white so that adjacent regions are never of the same color. This construction was used by Reidemeister [8], and we call a selection of one of the two possible shadings a Reidemeister shading.

Consider a connected link projection and a Reidemeister shading of the image P. In each black region we distinguish one point and denote the set of points so chosen by V. Through each crossing of P we next draw an open arc subject to conditions: (i) each arc, with the exception of the crossing through which it is drawn, is contained in the union of the black regions, (ii) the endpoints of each arc belong to V, (iii) distinct arcs are disjoint. We denote the set of arcs so constructed by E. It is obvious that the union $V \cup E$ is a nonvoid spherical graph which always exists and, to within isomorphism, is unique. We define the graph $B = B(P) = V \cup E$. Similarly, starting with the white regions of P, we form the graph $W = W(P)$. A few examples are given in Figure 3.

If P is not connected and has components $P_1, \cdots, P_r$, the graphs $B(P_i)$, $i = 1, \cdots, r$, are chosen to be pairwise disjoint, and the same goes for $W(P_i)$, $i = 1, \cdots, r$. We then define

$$(3.1) \qquad \begin{aligned} B &= B(P) = \textstyle\bigcup_{i=1}^{r} B(P_i), \\ W &= W(P) = \textstyle\bigcup_{i=1}^{r} W(P_i). \end{aligned}$$

Thus, an n-tuply connected black or white region contains n vertices of B or W, respectively. The graphs B and W are what we call *the graphs of a link projection* with image P.



(a) (b) (c)

Figure 3

We observe that

(3.2) *A cut vertex of P is a crossing, i.e., must be a vertex of order* 4.

Furthermore, it is a straightforward matter to check that

(3.3) *The graphs of a link projection are nonvoid spherical graphs. They are connected if and only if P is connected, and they are strongly cyclic if P is nonseparable.*

It is also true that, if P is connected, the graphs B and W are dual to each other in the sense of determining dual cellular subdivisions of the 2-sphere in which they lie. Incidentally, in what follows we shall generally make use of only one of the two graphs of a given projection. However, the choice of shading will be immaterial; it will not matter which graph we choose.

4. The quadratic form and determinant of a link projection

The quadratic form of a link projection p is defined in this section to be a certain symmetric integral matrix $Q = Q(p)$. Our definition differs slightly from, but is fully equivalent to, the usual one (see Section 1 of Kyle's paper [5]; $Q(p)$ is his $\tilde{A}$—although his definition, as it stands, is presumably intended to apply only to connected projections). We select a Reidemeister shading of the image P and an orientation of the 2-sphere containing P and construct the graph $B = B(P)$, whose vertices we denote by $(1), \cdots, (n)$. For each edge e of B, the index $\eta(e) = \pm 1$ is defined as is shown in Figure 4.

We set E_{ij}, $i, j = 1, \cdots, n$, equal to the set of all edges of B whose endpoints are the vertices (i) and (j) of B. *The quadratic form*

$$Q = Q(p) = \| q_{ij} \|, \qquad i, j, = 1, \cdots, n,$$

is then defined by the formula:

$$(4.1) \qquad \begin{aligned} q_{ij} &= \textstyle\sum_{e \epsilon E_{ij}} \eta(e), & i \neq j, \\ q_{jj} &= -\textstyle\sum_{i=1, i \neq j}^{n} q_{ij} . \end{aligned}$$

Notice that the definition of the index η, and hence of the matrix Q, depends on distinguishing at each crossing the overcrossing from the undercrossing. It is for this reason that we write $Q(p)$ instead of $Q(P)$.

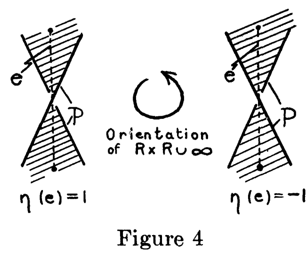


Figure 4

Since the matrix Q has row and column sums equal to zero, it follows that the determinants of all $(n - 1) \times (n - 1)$ minors of Q have the same absolute value. This value is called the *determinant* of the link projection p and is denoted by $\det(p)$ (it is understood that if $n = 1$, then $\det(p) = 1$). The specific matrix Q constructed from a projection p depends on the choice of the Reidemeister shading, the orientation of $R \times R \cup \infty$, and the ordering of the vertices of the graph $B(P)$. The determinant $\det(p)$, however, is independent of these choices. In fact, it can be proved (cf. footnote 6) that

(4.2) $\det(p)$ *is an invariant of the link type of* p.

By the *trivial knot type* we mean the link type of a connected projection p_0 having no crossings (obviously, p_0 is the projection of a knot, i.e., multiplicity $\mu = 1$). Since the graph of p_0 possesses just one vertex, i.e., $n = 1$, we have $\det(p_0) = 1$. By (4.2), then

(4.3) *The determinant of the trivial knot type equals* 1.

If the image P has components $P_1, \cdots, P_r$, there obviously exist connected projections $p_1, \cdots, p_r$ with images $P_1, \cdots, P_r$, respectively. It follows from the definition of the graph $B(P)$ that the quadratic form is the direct sum

$$(4.4) \qquad Q(p) = \dot{\sum}_{i=1}^{r} Q(p_i) = \begin{Vmatrix} Q(p_1) & 0 & \cdots \\ 0 & Q(p_2) & \cdots \\ \cdot & \cdot & \cdot \\ \cdots & & Q(p_r) \end{Vmatrix}.$$

Hence

(4.5) *The determinant of a nonconnected link projection equals zero.*

That is, a link which can be "pulled apart" has zero determinant.

The bridge between the determinant of a link projection and the theorems of graph theory in the preceding section is provided by an important combinatorial theorem which expresses certain minor determinants of matrices in terms of trees of graphs. This theorem, which is thus the keystone of this paper, is apparently due to Kirchhoff; in any event it has been around for a long time, and in [6] we have included some references. Its application to a link projection p is as follows: We select the graph B of p and an index η. Let the set of all trees of B which contain all the vertices of B be denoted by Tr and the set of all edges of B by E. Then,

(4.6) THE MATRIX-TREE THEOREM.

$$\det(p) = \left| \sum_{T \in \mathrm{Tr}} \prod_{e \in T \cap E} \eta(e) \right|.$$

It is obvious, cf. Figure 5 and (3.3), that

(4.7) *A link projection is alternating if and only if it is connected and any index* η *is a constant.*

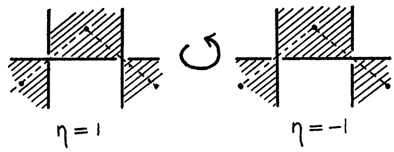


Figure 5

In fact, Aumann [1] uses (4.7) as the definition of an alternating projection. Since all maximal trees of a connected graph have the same number of edges, we conclude from (4.6) and (4.7) that

(4.8) *If p is alternating,* $\det(p) = \operatorname{tr}(B)$.

We recall from Section 2 that $\operatorname{tr}(B)$ is the number of trees of B which contain all of the vertices of B.

The following lemma, with which we conclude this section, is the essence of the Bankwitz theorem.

(4.9) *If p is an alternating link projection whose image P is nonseparable and has d crossings, then* $\det(p) \geqq d$.

Proof. Construct the graph B of P. By (4.8),

$$\det(p) = \operatorname{tr}(B).$$

By (3.3), the graph B is spherical, connected, and strongly cyclic. The number of edges of B is d. Hence, by (2.10)

$$\operatorname{tr}(B) \geqq d,$$

and we are done.

5. The principal theorems

For any link type $\mathfrak{K}$, we define its *minimal crossing number* $\delta = \delta(\mathfrak{K})$ to be the smallest integer d for which there exists a link projection of type $\mathfrak{K}$ having d overcrossings (i.e., d = number of overcrossings = number of undercrossings = number of crossings of the image of the projection). It is obvious that δ is an invariant of type and that

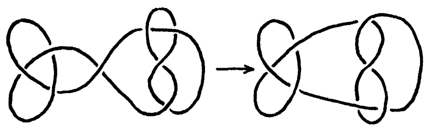
(5.1) *A knot type K is trivial if and only if $\delta(K) = 0$.*

If the image of a link projection p has a cut vertex, we may obtain by a single twist, cf. Figure 6, another projection p' of the same type as p whose image has one less cut vertex. By iterating this procedure, we obtain finally a projection p_* of type p whose image contains no cut vertex. Clearly,

(5.2) *p_* is connected if and only if p is,*

and, thus,

(5.3) *If p is connected, the image of p_* is nonseparable.*



p p'

Figure 6

Furthermore,

(5.4) *If p is alternating, then so is p_* .*

We now have the machinery assembled for proving the first of our principal theorems.

(5.5) BANKWITZ THEOREM. *The determinant of any alternating link type $\mathfrak{L}$ is greater than or equal to its minimal crossing number.*

Proof. There exists an alternating projection p of type $\mathfrak{L}$. By (5.3) and (5.4), p_* is alternating, and its image is nonseparable (an alternating projection is connected by definition). Let d be the number of overcrossings of p_* , which equals the number of crossings of its image. By (4.9),

$$\det (p_*) \geqq d.$$

Since p_* is of type $\mathfrak{L}$, we have, by (4.2) and the definition of minimal crossing number,

$$\det \mathfrak{L} = \det (p_*) \geqq d \geqq \delta(\mathfrak{L}),$$

and the proof is complete.

The following three corollaries of the proof of the Bankwitz theorem are interesting theorems in their own right. The assertion that (5.6) holds for all (tame) link types is the result referred to in the introduction as the knot-theoretic form of Dehn's lemma.

(5.6) THEOREM. *If the group of an alternating link type $\mathfrak{L}$ is infinite cyclic, then $\mathfrak{L}$ is the trivial knot type.*

Proof. By the group of $\mathfrak{L}$ we mean the fundamental group $\pi_1(S^3 - L)$ for any representative link $L \in \mathfrak{L}$. It can be shown (cf. footnote 6) that if this group is infinite cyclic, then the determinant of $\mathfrak{L}$ equals 1. But we know (cf. proof of (5.5)) that there exists a connected projection p_* of type $\mathfrak{L}$ having d overcrossings for which $d \leqq \det \mathfrak{L} = 1$. Since the image of p_* is nonseparable, we may conclude that $d = 0$ and, therefore, $\mathfrak{L}$ is the trivial knot type.

(5.7) THEOREM. *An alternating link type cannot be pulled apart (does not have a disconnected projection).*[4]

[4] There are at least two other proofs of this theorem [1, 3].

Proof. This theorem, by virtue of (4.5), is equivalent to the assertion that the determinant of an alternating link type is never zero. But if an alternating link type has zero determinant, it also has (proof of (5.5)) a connected projection with no crossings and, therefore, is the trivial knot type. Since, by (4.3), the trivial knot type has determinant equal 1, the proof is complete. The third corollary, observed by Reidemeister in [8], is an immediate consequence of (4.3) and the same inequality det $(p_*) \geqq d$.

(5.8) THEOREM. *Any alternating projection p of the trivial knot type can be effectively untwisted; in fact, the algorithm is simply the reduction $p \rightarrow p_*$.*

Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be two arbitrary link types of multiplicity μ_1 and μ_2, respectively. We shall describe the construction of a link type of multiplicity $\mu_1 + \mu_2 - 1$ which we denote by $\mathfrak{L}_1 * \mathfrak{L}_2$ and call the product of $\mathfrak{L}_1$ and $\mathfrak{L}_2$. It is clear that we may select representative links $L_1 \epsilon \mathfrak{L}_1$ and $L_2 \epsilon \mathfrak{L}_2$, whose components are $L_1^1, \cdots, L_1^{\mu_1}$ and $L_2^1, \cdots, L_2^{\mu_2}$, respectively, and a regular projection p of the union $L_1 \cup L_2$ which is such that the images $P_1 = p(L_1)$ and $P_2 = p(L_2)$ are disjoint and such that the boundary of one of the regions of $P_1 \cup P_2$ contains points of both $p(L_1^{\mu_1})$ and $p(L_2^1)$. We then join the components $L_1^{\mu_1}$ and L_2^1 as shown in Figure 7 to form a single component $L_1^{\mu_1} * L_2^1$. The resulting link, whose components are $L_1^1, \cdots, L_1^{\mu_1-1}, L_1^{\mu_1} * L_2^1, \cdots, L_2^{\mu_2}$, is denoted by $L_1 * L_2$ and its link type, by $\mathfrak{L}_1 * \mathfrak{L}_2$. It is not hard to prove that the product $\mathfrak{L}_1 * \mathfrak{L}_2$ is uniquely determined by $\mathfrak{L}_1$ and $\mathfrak{L}_2$. Furthermore, multiplication is associative, and the trivial knot type is the identity. It is an interesting fact that the knot types form a commutative, cancellation subsemigroup with a homomorphism onto the semigroup of nonnegative integers [9].

A link type is prime if and only if it is not the trivial knot type and is not the product of two link types neither of which is the trivial knot type. We have introduced this concept in order to prove

(5.9) *If an alternating link projection of prime type has a nonseparable image P, then the graph $B(P)$ is also nonseparable.*

Proof. We contend, first of all, that if a connected spherical graph G has a cut vertex v, then there exist a topological circle C which contains v and is contained in $(S^2 - G) \cup v$, and a nontrivial factorization of G at v into subgraphs G_1 and G_2 such that $G_1 - v$ is contained in the interior, and

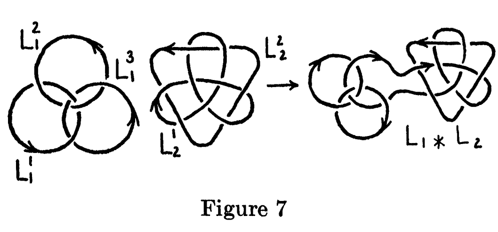


Figure 7

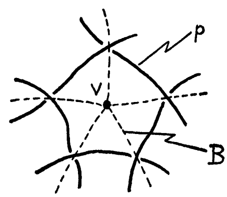


Figure 8

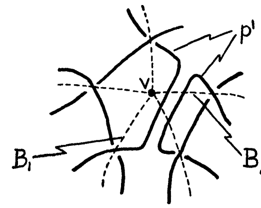


Figure 9

$G_2 - v$ in the exterior, of the circle C.[5] To prove this contention, it is obviously sufficient to prove that the number of regions of G which contain v on their boundary is less than the order of v. But if these numbers were equal, the boundary modulo 2 of every such region would constitute a circuit containing v. Since this conclusion clearly contradicts the assertion that v is a cut vertex, we are done.

Suppose that B is separable. Since P is nonseparable, it and, by (3.3), also B are connected. Hence B possesses a cut vertex v. The vertex v belongs to one of the black regions of P, whose boundary, since P is nonseparable, is a simple closed curve containing m crossings. Since v is a cut vertex of B, we have $m \geqq 2$. Finally, because p is alternating, we may conclude that the example shown in Figure 8, or its mirror image, illustrates the projection in the neighborhood of v in complete generality (except that in the picture $m = 5$). By virtue of our preliminary contention, we know that B possesses a nontrivial factorization at v into subgraphs B_1 and B_2 such that $B_1 - v$ and $B_2 - v$ can be separated by a Jordan curve passing through v. Consider now a link projection p' obtained by changing p as is shown in Figure 9. It follows from the construction of B_1 and B_2 that the image of p' is disconnected. Thus, p' determines link projections p_1 and p_2 the product of whose types is the link type of p. It only remains to show that neither p_1 nor p_2 is of trivial knot type, and, by (4.3), this conclusion will follow if neither has determinant equal to 1. But it is obvious from Figure 9 that both projections are alternating and that their images are nonseparable. The number of crossings of the image of p_i, $i = 1, 2$, which we denote by d_i, is the number of edges of B_i. Since B is strongly cyclic, we may conclude that $d_i \geqq 2$. Hence, by (4.9),

$$\det(p_i) \geqq d_i \geqq 2.$$

Consequently, p is not of prime type, and the proof is complete.

The next theorem is the central result of this paper. In it we obtain, for prime link types, a considerable improvement of the Bankwitz inequality.

(5.10) THEOREM. *Any prime, alternating link type $\mathfrak{L}$ has an alternating*

[5] We assumed this result in describing the algorithm $p \to p^*$; cf. Figure 6.

projection whose image P and graph B(P) are both nonseparable and for which

$$\det \mathfrak{K} - 1 \geqq (\rho - 1)(\nu - 1),$$

where ρ *and* ν *are the number of regions and vertices, respectively, of* B.

Proof. There exists an alternating projection p of type $\mathfrak{K}$. Since an alternating projection is ipso facto connected, the projection p_*, also of type $\mathfrak{K}$, is alternating, and its image P is nonseparable (cf. (5.2), (5.3), (5.4)). Thus, since $\mathfrak{K}$ is prime, the graph $B(P)$ is also nonseparable (cf. (5.9)). We know B is spherical and nonvoid (cf. (3.3)). Hence (cf. (2.4)),

$$\operatorname{tr}(B) - 1 \geqq (\rho - 1)(\nu - 1).$$

Since

$$\det \mathfrak{K} = \det (p_*) = \operatorname{tr}(B)$$

(cf. (4.2), (4.8)), the proof is complete.

6. Examples of nonalternating knots

In this section we shall apply Theorem (5.10) to the eleven nonalternating projections which appear in the Knot Table in an attempt to prove that the knot types which they determine are nonalternating. The attack succeeds in seven out of the eleven examples. Of the seven, only two are nonalternating by virtue of the Bankwitz theorem. It is stated in the *Introduction* that the projections which appear in the Knot Table represent distinct, prime knot types. We shall use the additional fact that the number of crossings of each projection in the Table is the minimal crossing number δ of the knot type it represents. Incidentally, the Table includes representatives of all prime knot types for which $\delta \leqq 9$.

It is convenient to eliminate one easily recognizable possibility from the outset. To this end, we define a link type $\mathfrak{K}$ to be an *elementary torus type* if and only if it has a projection one of whose graphs (i.e., either B or W) is a circuit. Examples are shown in Figure 10. It is obvious that

(6.1) *The trivial knot type is an elementary torus type.*

(6.2) *Any elementary torus type is alternating.*

Furthermore,

(6.3) *If* $\mathfrak{K}$ *is not an elementary torus type, then the graph* B *of any nonseparable image of a link projection of* $\mathfrak{K}$ *has at least three regions and three vertices.*

Proof. Suppose the number of regions of B, denoted by ρ, is less than 3. Since B is nonvoid and connected (cf. (3.3)) we know (Alexander duality) that $\rho = p_1(B) + 1$. Thus, we are assuming that $p_1(B) \leqq 1$. (3.3) also tells us that B is strongly cyclic. Hence, either B is a point and $\mathfrak{K}$ is the

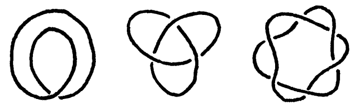


Figure 10

trivial knot type, or B is a circuit and $\mathfrak{L}$ is an elementary torus type. Since both possibilities are contrary to hypothesis, we conclude that $\rho \geqq 3$. Finally, since the two graphs B and W are dual to each other, the number of vertices of one is the number of regions of the other. Since that graph which is W for one Reidemeister shading is B for the other, we conclude that the number of vertices of each is also at least three, and the proof is complete.

(6.4) Theorem. *If $\mathfrak{L}$ and B are as in* (6.3), *then the minimal crossing number δ of $\mathfrak{L}$ and the numbers ρ and ν of regions and vertices, respectively, of B satisfy the inequality*

$$(\rho - 1)(\nu - 1) \geqq 2(\delta - 2).$$

Proof. Let the number of edges of B, which equals the number of overcrossings of the projection, be denoted by d. The graph B, being connected (cf. (3.3)), determines a cellular decomposition of the 2-sphere; hence by the Euler-Poincaré formula and the definition of δ, we have

$$(\rho - 1) + (\nu - 1) = d \geqq \delta.$$

By (6.3)

$$(\rho - 1) \geqq 2 \quad \text{and} \quad (\nu - 1) \geqq 2.$$

Hence, for some nonnegative ε and η,

$$(\rho - 1) = 2 + \varepsilon, \qquad (\nu - 1) = 2 + \eta, \qquad (\varepsilon + \eta) \geqq \delta - 4.$$

Thus,

$$\begin{aligned}(\rho - 1)(\nu - 1) &= 4 + 2(\varepsilon + \eta) + \varepsilon\eta \\ &\geqq 4 + 2(\delta - 4) \\ &= 2(\delta - 2),\end{aligned}$$

and the proof is complete.

The next theorem, which is the result of combining Theorems (5.10) and (6.4), provides the principal inequality used in our subsequent applications to the nonalternating projections of the Knot Table.

(6.5) Theorem. *If $\mathfrak{L}$ is any prime, alternating link type which is not an elementary torus type, then the determinant* $\det \mathfrak{L}$ *and the minimal crossing number δ of $\mathfrak{L}$ satisfy the inequality* $\delta \leqq (\det \mathfrak{L} + 3)/2$.

Proof. The theorem is a direct corollary of (5.10) and (6.4).

We now consider the problem of recognizing whether or not a particular $\mathfrak{L}$

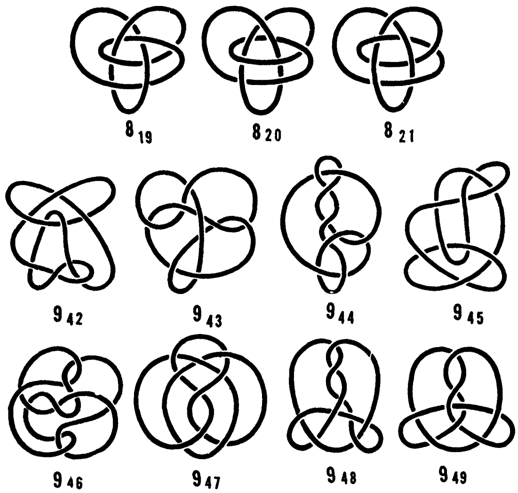


Figure 11[6]

is an elementary torus type. For this purpose, we mention briefly another invariant of link type, the *Alexander polynomial* $\Delta(t)$ of $\mathfrak{L}$.[7] This polynomial is an element of the ring of Laurent polynomials in one variable t over the integers (by "Laurent" we mean that negative powers of t are permitted), and it is defined only up to a unit factor of $\pm t^n$, where n is arbitrary. It is sufficient for our purposes to know that $\Delta(t)$ is effectively calculable from any link projection of $\mathfrak{L}$ and that

(6.6) *The Alexander polynomial* $\Delta(t)$ *of an elementary torus type of multiplicity one is of the form* $1 - t + t^2 - t^3 + \cdots + (-1)^n t^n$ *for some integers* n.

We come finally to the nonalternating projections of the Knot Table. These are pictured in Figure 11, and the notation for each projection is that

[6] This figure is reproduced from [8] with the kind permission of Springer-Verlag and Professor Reidemeister.

[7] $\Delta(t)$ is actually what I have called in [3] the *reduced Alexander polynomial* of $\mathfrak{L}$. If the multiplicity μ of $\mathfrak{L}$ is one, $\Delta(t)$ is the customary Alexander polynomial [4, 8, 10]. If $\mu > 1$, it is shown in [3] that $\Delta(t) = (1 - t)\Delta(t, \cdots, t)$, where $\Delta(t_1, \cdots, t_\mu)$ is the usual Alexander polynomial of $\mathfrak{L}$. The invariance of the determinant of a link type $\mathfrak{L}$ follows from the invariance of its polynomial $\Delta(t)$ and the fact that $\det \mathfrak{L} = |\Delta(-1)|$. That this equation is valid was first pointed out to me by J. P. Mayberry. His doctoral thesis contains a proof that $|\Delta(-1)|$ is the order of the first homology group of the two-sheeted branched covering space of any link of type $\mathfrak{L}$. That $\det \mathfrak{L}$ equals this order is a well known result for knots, and the proof is presumably the same for links. In addition, I have worked out a direct proof of the above equation which shows that the quadratic form of any link projection is the homomorphic image of an Alexander matrix [3, 4] of the given link under the group ring homomorphism determined by setting $t_i = -1$, $i = 1, \cdots, \mu$. (In lectures on knot theory at Princeton, R. H. Fox has also given similar calculations which imply this result.) Since, for an infinite cyclic group, $\Delta(t) = 1$ [4], it follows that $\det \mathfrak{L} = 1$ if the group of $\mathfrak{L}$ is infinite cyclic.

of the Table, e.g., 8_{21} is the 21st of those projections in the table which have 8 crossings. The relevant knot-theoretic properties are conveniently tabulated in Table 1.

As a result of the summary in Table 1, we have

(6.7) (Corollary of the Bankwitz theorem, (5.5)). *Projections* 8_{19} *and* 9_{42} *represent nonalternating knot types.*

All of the types represented in the Knot Table are prime. From (6.6) we see that none of the types represented in Table 1 is an elementary torus type. Hence, consideration of (6.5) and the last two columns of Table 1, yields

(6.8) (Corollary of Theorem (6.5)). *Projections* 8_{19}, 8_{20}, 9_{42}, 9_{43}, *and* 9_{46} *represent nonalternating knot types.*

Although 8_{21} and 9_{44} fail to satisfy the criterion provided by (6.5), it is still fairly easy to prove that they represent nonalternating types. Let us assume that they are in fact projections of alternating types. From (5.10)

TABLE 1

Knot Type	Alexander Polynomial Δ (t)	Determinant det	Min crossing no. δ	$\frac{\text{det} + 3}{2}$
8_{19}	$t^6 - t^5 + t^3 - t + 1$	3	8	3
8_{20}	$t^4 - 2t^3 + 3t^2 - 2t + 1$	9	8	6
8_{21}	$t^4 - 4t^3 + 5t^2 - 4t + 1$	15	8	9
9_{42}	$t^4 - 2t^3 + t^2 - 2t + 1$	7	9	5
9_{43}	$t^6 - 3t^5 + 2t^4 - t^3 + 2t^2 - 3t + 1$	13	9	8
9_{44}	$t^4 - 4t^3 + 7t^2 - 4t + 1$	17	9	10
9_{45}	$t^4 - 6t^3 + 9t^2 - 6t + 1$	23	9	13
9_{46}	$2t^2 - 5t + 2$	9	9	6
9_{47}	$t^4 - 7t^3 + 11t^2 - 7t + 1$	27	9	15
9_{48}	$t^6 - 4t^5 + 6t^4 - 5t^3 + 6t^2 - 4t + 1$	27	9	15
9_{49}	$3t^4 - 6t^3 + 7t^2 - 6t + 3$	25	9	14

TABLE 2

Knot Type	ρ	ν	$d = \rho + \nu - 2$
8_{21}	3	7	8
	3	8	9
	4	5	7X
9_{44}	3	8	9
	3	9	10
	4	6	8X
	5	5	8X

and (6.4), we may then conclude that each has an alternating projection whose graph is nonseparable and satisfies the inequality

(1) $$\det -1 \geqq (\rho - 1)(\nu - 1) \geqq 2(\delta - 2),$$

which becomes

(2) $$14 \geqq (\rho - 1)(\nu - 1) \geqq 12$$

and

(3) $$16 \geqq (\rho - 1)(\nu - 1) \geqq 14$$

for 8_{21} and 9_{44}, respectively. We denote, as usual, by d the number of edges of the graph of each projection, which is thus the number of crossings of each projection. Recalling the duality between ρ and ν and that (by (6.3))

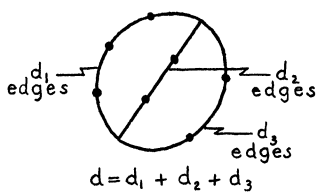


Figure 12

TABLE 3

$d = d_1 + d_2 + d_3$	d_1	d_2	d_3	$\det = d_1d_2 + d_2d_3 + d_3d_1$
8	1	1	6	13
	1	2	5	17
	1	3	4	19
	2	2	4	20
	2	3	3	21
9	1	1	7	15
	1	2	6	20
	1	3	5	23
	1	4	4	24
	2	2	5	24
	2	3	4	26
	3	3	3	27
10	1	1	8	17
	1	2	7	23
	1	3	6	27
	1	4	5	29
	2	2	6	28
	2	3	5	31
	2	4	4	32
	3	3	4	33

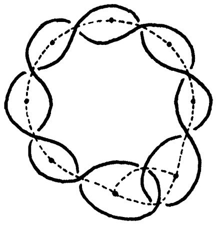


Figure 13

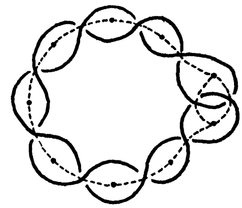


Figure 14

ρ, $\nu \geqq 3$, we have that the only possibilities satisfying (2) and (3) are shown in Table 2. The requirement $d \geqq \delta$ eliminates the cases marked with an X.

The generic spherical, nonseparable graph with 3 regions, d edges, and ν vertices ($\nu = d - \rho + 2 = d - 1$) is pictured in Figure 12. It is easy to see that the number of maximal trees of such a graph is given by

$$d_1 d_2 + d_2 d_3 + d_3 d_1 .$$

Thus, we obtain the simultaneous Diophantine equations

$$\text{det} = d_1 d_2 + d_2 d_3 + d_3 d_1 , \tag{4}$$

$$d = d_1 + d_2 + d_3 , \tag{5}$$

to be solved for positive d_1, d_2, d_3 in the four cases indicated in Table 2: (i) for 8_{21}, det $= 15$, $d = 8$; (ii) for 8_{21}, det $= 15$, $d = 9$; (iii) for 9_{44}, det $= 17$, $d = 9$; (iv) for 9_{44}, det $= 17$, $d = 10$. Notice that the ordering of d_1, d_2, d_3 is immaterial; $(d_1, d_2, d_3) = (1, 2, 6)$ yields the same type as $(d_1, d_2, d_3) = (6, 1, 2)$. Thus, we may assume that

$$d_1 \leqq d_2 \leqq d_3 . \tag{6}$$

The solutions to (5) and (6) are given for $d = 8, 9, 10$ in Table 3. As a result we conclude that $d_1 = 1$, $d_2 = 1$, $d_3 = 7$ is the only possibility for 8_{21} and $d_1 = 1$, $d_2 = 1$, $d_3 = 8$ is the only possibility for 9_{44}. Thus, if 8_{21} and 9_{44} represent alternating types, they are of the knot types of the projections shown in Figures 13 and 14, respectively. It is, however, a simple matter to check that the projections of Figures 13 and 14 are not of the types of 8_{21} and 9_{44}. One way is to calculate their Alexander polynomials. The projections of Figures 13 and 14 both have polynomials of degree 2 while three of 8_{21} and 9_{44} are of degree 4.[8] We conclude that

(6.9) *Projections* 8_{21} *and* 9_{44} *represent nonalternating knot types.*

The inequalities with which our results have been obtained are apparently not good enough to handle the remaining four projections easily. Whether or not they can be sharpened to do the job is an open question. At present,

[8] Since $\Delta(t)$ is defined only to within a factor $\pm t^n$, we define the *degree* of $\Delta(t)$ to be the difference between the highest and the lowest power of t.

however, it seems rather unlikely that the methods of this paper will yield any general criterion for deciding whether or not an arbitrary projection represents an alternating type. We conjecture, naturally enough, that the remaining 9_{45}, 9_{47}, 9_{48}, and 9_{49} are nonalternating. It seems probable that any projection which does not obviously represent an alternating type is in fact of nonalternating type.

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