MINIMAL FREE C.S.S. GROUPS

BY

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1. Introduction

It has been shown by Eilenberg-Zilber [3] that the total singular complex of a topological space contains a minimal subcomplex (i.e., roughly speaking, a smallest subcomplex of the same homotopy type). By the same method it can be shown [8] that every c.s.s. complex which satisfies the extension condition has a minimal subcomplex (which also satisfies the extension condition). The importance of the notion of minimal subcomplex lies in the fact that two c.s.s. complexes which satisfy the extension condition have the same homotopy type if and only if they have isomorphic minimal subcomplexes.

It was shown in [4] that among c.s.s. groups the free c.s.s. groups play a role similar to that of the c.s.s. complexes which satisfy the extension condition among c.s.s. complexes. This suggests the problem of finding a notion of minimal free c.s.s. group such that two free c.s.s. groups have the same loop homotopy type (in the sense of [4]) if and only if they have isomorphic minimal subgroups. The present paper contains a solution of this problem for those free c.s.s. groups which have finitely generated homotopy groups and are trivial in dimension 0. In fact it will be shown that every such free c.s.s. group is the free product of a contractible free c.s.s. group and a minimal one.

Free use will be made of the definitions and notation of [4], with one exception: Because the *loop homotopy* relation is the natural homotopy relation for c.s.s. groups (see [5]), the word *homotopy* will be used throughout instead of loop homotopy.

All groups will be written multiplicatively.

2. Statement of results

DEFINITION (2.1). Let F be a free c.s.s. group ([4], Definition (5.1)). A subset $\mathfrak{F} \subset F$ will be called a *basis* of F if

(a) $\mathfrak{F}_n = \mathfrak{F} \cap F_n$ freely generates F_n for all $n \geqq 0$,

(b) $\mathfrak{F}$ is stable under degeneracies, i.e., $\sigma \in \mathfrak{F}_n$ implies $\sigma\eta^i \in \mathfrak{F}_{n+1}$ for $0 \leqq i \leqq n$.

DEFINITION (2.2). Let F be a free c.s.s. group, and let $G, H \subset F$ be subgroups. Then F is called the *free product* of G and H if there exists a basis $\mathfrak{F}$ of F such that $(\mathfrak{F} \cap G) \cup (\mathfrak{F} \cap H) = \mathfrak{F}$, $(\mathfrak{F} \cap G) \cap (\mathfrak{F} \cap H) = \emptyset$, and $\mathfrak{F} \cap G$ and

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$\mathfrak{F} \cap H$ are bases of G and H. The subgroups G and H then are called free *factors* of F (Notation: $F = G * H$).

DEFINITION (2.3). A free c.s.s. group F is called of *finite type* if
(a) F_0 is trivial,
(b) $\pi_n(F)$ is finitely generated for all $n > 0$.

DEFINITION (2.4.) A free c.s.s. group of finite type is called *minimal* if it has no contractible free factors (except the trivial one).

Several equivalent definitions of minimality will be given in an appendix. We now state the main result.

THEOREM (2.5). *Let F be a free c.s.s. group of finite type. Then F has a free factorization $F = C * M$ such that C is contractible and M is minimal. Let $j_M : M \to F$ be the inclusion map and $p_M : F \to M$ the projection. Then j_M and p_M are homotopy equivalences.*

THEOREM (2.6). *Let F and F' be free c.s.s. groups of finite type, and let $f: F \to F'$ be a homotopy equivalence. Let $F = C * M$ and $F' = C' * M'$ where C and C' are contractible and M and M' are minimal. Then there exists an isomorphism $m: M \approx M'$ such that $m \simeq p_{M'} \circ f \circ j_M$.*

COROLLARY (2.7). *Let F be a free c.s.s. group of finite type. Let $F = C * M$ and $F = C' * M'$ where C and C' are contractible and M and M' are minimal. Then there exists an isomorphism $m: M \approx M'$ such that $m \simeq p_{M'} \circ j_M$.*

It follows from the construction of a minimal free c.s.s. group (see §6) that the rank of M_n (which is finite) is a homotopy invariant of F. Hence Theorem (2.6) is an immediate consequence of

THEOREM (2.8). *Let F and G be free c.s.s. groups of finite type. Let rank F_n = rank $G_n < \infty$ for each n, and let $h: F \to G$ be a homotopy equivalence. Then there exists an isomorphism $h': F \approx G$ such that $h' \simeq h$.*

Remark (2.9). Let F be a free c.s.s. group of finite type the basis of which contains only a finite number of nondegenerate elements. Then it follows from the proof of Theorem (2.5) that a free factorization $F = C * M$ such that C is contractible and M is minimal may be constructed in a finite number of steps.

Remark (2.10). The following example shows that Theorem (2.8) does not hold if the ranks of F_n and G_n are allowed to be infinite. Let F and G be such that a basis is formed by elements $\phi_{i,j}$ and $\chi_{i,j}$ ($1 \leqq i \leqq 4$, $j = 1, 2, \cdots$) and their degeneracies, where $\phi_{i,j} \in F_i$ and $\chi_{i,j} \in G_i$, and let the face homomorphisms be defined by

$$\begin{aligned}
&\phi_{2,j}\, \varepsilon^2 = \phi_{1,j}\,, && \chi_{2,2j}\, \varepsilon^2 = \chi_{1,j}\,, && j = 1, 2, \cdots,\\
&\phi_{3,j}\, \varepsilon^3 = e_2\,, && \chi_{3,2j}\, \varepsilon^3 = \chi_{2,2j-1}\,, && j = 1, 2, \cdots,\\
&\phi_{4,j}\, \varepsilon^4 = \phi_{3,j}\,, && \chi_{4,j}\, \varepsilon^4 = \chi_{3,2j-1}\,, && j = 1, 2, \cdots,\\
&\phi_{i,j}\, \varepsilon^k = e_{i-1}\,, && \chi_{i,j}\, \varepsilon^k = e_{i-1}\,, && \text{otherwise.}
\end{aligned}$$

Clearly F and G are contractible, and hence there exists a homotopy equivalence $h:F \to G$. However it is obvious that F and G are not isomorphic.

3. Organization of the proofs of Theorems (2.5) and (2.8)

I. With a free c.s.s. group F one may associate a free c.s.s. abelian group A and an epimorphism $p:F \to A$, where $A = F/[F, F]$, i.e., "*F made abelian,*" and $p:F \to A$ is the *projection.* The proofs of Theorems (2.5) and (2.8) then may be divided in the following two steps:

(a) Obtaining results for free c.s.s. abelian groups analogous to Theorems (2.5) and (2.8). This will be done in §4.

(b) Lifting the results obtained for free c.s.s. abelian groups into free c.s.s. groups. For Theorem (2.5) this will be done in §6, and for Theorem (2.8) in §7.

II. In lifting the results for free c.s.s. abelian groups into free c.s.s. groups an important role will be played by Lemmas (3.1), (3.3), and (3.5) below.

In the proofs of Theorems (2.5) and (2.8), use will be made of the following c.s.s. group version of a theorem of J. H. C. Whitehead ([10], Theorem 3). A proof may be found in [5].

LEMMA (3.1). *Let F and G be connected free c.s.s. groups, and let $h:F \to G$ be a c.s.s. homomorphism. Let $A = F/[F, F]$ and $B = G/[G, G]$, and let $c:A \to B$ be the map induced by h. Then h is a homotopy equivalence if and only if c is so.*

DEFINITION (3.2). Let F be a free c.s.s. group. A subgroup $G \subset F$ will be called *proper* if there exists a basis $\mathfrak{F}$ of F such that $\mathfrak{F} \cap G$ is a basis of G.

In the proof of Theorem (2.8) the following homotopy extension covering lemma will be needed, which is a special case of [6], Corollary (6.2).

LEMMA (3.3). *Let F be a free c.s.s. group, and let $G \subset F$ be a proper subgroup. Let K and L be c.s.s. groups, and let $p:K \to L$ be an epimorphism. Let $f_0:F \to K$ be a c.s.s. homomorphism, let $g_I:g_0 \simeq g_1$ where $g_0 = f_0 \mid G$ (the restriction of f_0 to G), and let $h_I : p \circ f_0 \simeq h_1$ be such that $p \circ g_I = h_I \mid (I \otimes G)$. Then there exists a homotopy $f_I : f_0 \simeq f_1$ such that $f_I \mid (I \otimes G) = g_I$ and $p \circ f_I = h_I$.*

DEFINITION (3.4). For a c.s.s. group F let F^n denote the n-skeleton, i.e., the subgroup generated by F_n. Clearly, if F is free, then F^n is a proper subgroup of F.

The following lemma will be proved in §8.

LEMMA (3.5). *Let F be a c.s.s. group such that F_0 is trivial, and let $\sigma \in [F^n, F^n]_n$. Then there exist an element $\rho \in [F^n, F^n]_{n+1}$ and an element $\phi \in [F^{n-1}, F^{n-1}]_n$ such that $\rho\varepsilon^{n+1} = \sigma \cdot \phi^{-1}$ and $\rho\varepsilon^i = e_n$ for $0 \leqq i < n + 1$.*

III. A simplification in the proofs may be obtained by using only a special kind of bases, namely those for which every nondegenerate element has at most one nontrivial face. They will be called CW-bases and are described in §5.

It is sometimes necessary to go from one basis to another. A method frequently used is the following:

Let F be a free c.s.s. group, and let $\mathfrak{F}$ be a basis of F. For every $\sigma \in \mathfrak{F}_n$ and $\tau \in F_n$ let $\{\mathfrak{F} - \sigma + \sigma \cdot \tau\}$ denote the set obtained from $\mathfrak{F}$ by omitting σ and its degeneracies and adding $\sigma \cdot \tau$ and its degeneracies. Then it is readily verified that

LEMMA (3.6). *If σ is nondegenerate and τ is in the subgroup of F_n generated by $\{\mathfrak{F}_n - \sigma\}$, then $\{\mathfrak{F} - \sigma + \sigma \cdot \tau\}$ is a basis of F.*

4. The abelian case

DEFINITION (4.1). A c.s.s. abelian group A is called *free*, if the groups A_n are free for all n.

DEFINITION (4.2). Let A be a free c.s.s. abelian group. A sub*set* $\mathfrak{A} \subset A$ is called a *basis* of A if

(a) A_n is the free abelian group freely generated by $\mathfrak{A}_n = \mathfrak{A} \cap A_n$,

(b) $\mathfrak{A}$ is stable under degeneracies (see Definition (2.1)).

PROPOSITION (4.3). *Every free c.s.s. abelian group has a basis.*

Proof. This follows easily from the facts that

(i) every subgroup of a free abelian group is free abelian,

(ii) for every pair of integers (i, n) with $0 \leqq i \leqq n$, the map $\eta^i: A_n \rightarrow A_{n+1}$ maps A_n isomorphically onto a direct summand of A_{n+1}.

DEFINITION (4.4). Let A be a free c.s.s. abelian group and let $B, C \subset A$ be subgroups. Then A is called the *direct sum* of B and C if there exists a basis $\mathfrak{A}$ of A such that $(\mathfrak{A} \cap B) \cup (\mathfrak{A} \cap C) = \mathfrak{A}$, $(\mathfrak{A} \cap B) \cap (\mathfrak{A} \cap C) = \emptyset$, and $\mathfrak{A} \cap B$ and $\mathfrak{A} \cap C$ are bases of B and C. The subgroups B and C are then called *direct summands* of A (Notation: $A = B + C$).

DEFINITION (4.5). A free c.s.s. abelian group is called *of finite type* if $\pi_n(A)$ is finitely generated for all n.

DEFINITION (4.6). A free c.s.s. abelian group of finite type is called *minimal* if it contains no contractible direct summand.

We now may state the abelian analogues of Theorems (2.5) and (2.8).

THEOREM (4.7). *Let A be a free c.s.s. abelian group of finite type. Then A has a direct summation $A = D + N$ such that D is contractible and N is minimal. Let $j_N: N \rightarrow A$ be the inclusion map and $p_N: A \rightarrow N$ the projection. Then j_N and p_N are homotopy equivalences.*

THEOREM (4.8). *Let A and B be free c.s.s. abelian groups of finite type such that rank $A_n =$ rank $B_n < \infty$ for each n, and let $c: A \rightarrow B$ be a homotopy equivalence. Then there exists an isomorphism $c': A \approx B$ such that $c' \simeq c$.*

In view of the equivalence of the concepts of c.s.s. abelian group and chain complex (see [1]), it follows that Theorems (4.7) and (4.8) are equivalent to certain theorems on chain complexes, which may be readily proved using the well known properties of finitely generated abelian groups (see for instance [2]). To Theorem (4.7), for instance, corresponds the theorem that every free abelian chain complex which has finitely generated homology and is trivial in dimension < 0 is the direct sum of an acyclic chain complex and a chain complex which is minimal in the obvious sense. The details are left to the reader.

5. CW-bases

DEFINITION (5.1). Let F be a free c.s.s. group. A basis $\mathfrak{F}$ of F is called a *CW-basis* if for every integer $n \geqq 0$ and every nondegenerate element $\sigma \in \mathfrak{F}_n$ we have $\sigma \overset{i}{\varepsilon} = e_{n-1}$ for $0 \leqq i < n$. The element $\sigma \overset{n}{\varepsilon} \in F_{n-1}$ will be called the *attaching element* of σ.

PROPOSITION (5.2). *Every free c.s.s. group has a CW-basis.*

In order to prove Proposition (5.2) we need the following lemma which is due to J. C. Moore ([7]).

LEMMA (5.3). *Let F be any c.s.s. group, and let $\sigma_0, \cdots, \sigma_{n-1} \in G_{n-1}$ be such that $\sigma_i \overset{j-1}{\varepsilon} = \sigma_j \overset{i}{\varepsilon}$ for $0 \leqq i < j < n$.* Let

$$\tau_0 = \sigma_0 \overset{0}{\eta},$$

$$\tau_i = \sigma_i \overset{i}{\eta} \cdot \tau_{i-1}^{-1} \overset{i}{\varepsilon} \overset{i}{\eta} \cdot \tau_{i-1}, \qquad 0 < i < n.$$

Then $\tau_{n-1} \overset{i}{\varepsilon} = \sigma_i$ for $0 \leqq i < n$.

Proof of Proposition (5.2). The element τ_{n-1} of Lemma (3.3) is obtained from the elements $\sigma_0, \cdots, \sigma_{n-1}$ by application of the following operations only: $\overset{i}{\varepsilon}$, $\overset{i}{\eta}$, multiplication, and taking inverses. Denote the element so obtained by $m(\sigma_0, \cdots, \sigma_{n-1})$.

Let F be a free c.s.s. group, and let $\mathfrak{F}$ be a basis of F. Let $\mathfrak{F}' \subset F$ denote the subset consisting of all elements $\sigma \cdot m(\sigma \overset{0}{\varepsilon}, \cdots, \sigma \overset{n-1}{\varepsilon})$ for which σ is a nondegenerate element of $\mathfrak{F}$, together with all their degeneracies. Then iterated application of Lemma (3.6) yields that $\mathfrak{F}'$ is a basis of F.

An immediate consequence of Lemma (3.6) is

LEMMA (5.4). *Let F be a free c.s.s. group, and let $\mathfrak{F}$ be a CW-basis of F. If $\sigma \in \mathfrak{F}_n$ is nondegenerate and τ is in the subgroup of F_n generated by $\mathfrak{F}_n - \sigma$ and $\tau \overset{i}{\varepsilon} = e_{n-1}$ for $0 \leqq i < n$, then $\{\mathfrak{F} - \sigma + \sigma \cdot \tau\}$ is a CW-basis of F.*

COROLLARY (5.5). *If $\sigma \in \mathfrak{F}_n$ is nondegenerate and $\tau \in [F^{n-1}, F^{n-1}]_n$ is such that $\tau \overset{i}{\varepsilon} = e_{n-1}$ for $0 \leqq i < n$, then $\{\mathfrak{F} - \sigma + \sigma \cdot \tau\}$ is a CW-basis of F.*

Remark (5.6). Free c.s.s. groups with a CW-basis may be considered as a kind of algebraization of the CW-complexes of J. H. C. Whitehead (see [10]). In fact it may be shown using the properties of the constructions G and $\bar{W}$

(see [4]) that for every connected CW-complex K one may construct a free c.s.s. group F with a CW-basis $\mathfrak{F}$ (or that for every free c.s.s. group F with a CW-basis $\mathfrak{F}$ one may construct a CW-complex K) which are related as follows:

(i) F^{n-1} has the homotopy type of the loops on K^n for $1 \leqq n \leqq \infty$; in particular, $\partial_* : \pi_i(K^n) \approx \pi_{i-1}(F^{n-1})$ for all i and n.

(ii) The nondegenerate elements of $\mathfrak{F}_n$ are in one-to-one correspondence with the $(n + 1)$-cells of K in such a manner that the element $\alpha \in \pi_n(K^n)$ containing the attaching map of an $(n + 1)$-cell of K is such that $\partial_*\alpha \in \pi_{n-1}(F^{n-1})$ contains the attaching element of the corresponding nondegenerate element of $\mathfrak{F}_n$.

6. Proof of Theorem (2.5)

Let $A = F/[F, F]$, and let $p:F \to A$ be the projection. As $\pi_n(F) \approx \pi_{n+1}(\bar{W}F)$ and $\pi_n(A) \approx H_{n+1}(\bar{W}F)$ ([5], §6), it follows from [9] that A is also of finite type. By Theorem (4.7), $A = D + N$ where D is contractible and N is minimal, and hence there exists a basis $\mathfrak{A}$ of A such that $\mathfrak{D} = \mathfrak{A} \cap D$ and $\mathfrak{N} = \mathfrak{A} \cap N$ are bases of D and N. Clearly (compare the proof of Proposition (5.2)) $\mathfrak{A}$ may be chosen such that for every nondegenerate element $\alpha \in \mathfrak{A}_n$ we have $\alpha\varepsilon^i = e_{n-1}$ for $0 \leqq i < n$. It also follows readily from the contractibility of D that in addition $\mathfrak{A}$ may be chosen such that the nondegenerate elements of $\mathfrak{D}$ may be divided into two types, having the following properties:

(i) Let $\alpha \in \mathfrak{D}_n$ be of type I. Then there is a unique $\beta \in \mathfrak{D}_{n+1}$ of type II such that $\beta\varepsilon^{n+1} = \alpha$.

(ii) Let $\beta \in \mathfrak{D}_{n+1}$ be of type II. Then $\beta\varepsilon^{n+1} \in \mathfrak{D}_n$ and is of type I.

Now suppose we have already shown that there exists a CW-basis $\mathfrak{F}$ of F such that

(a) $p\mathfrak{F} = \mathfrak{A}$;

(b) if $C \subset F$ is the subgroup generated by $p^{-1}\mathfrak{D} \cap \mathfrak{F}$, then $p^{-1}\mathfrak{D} \cap \mathfrak{F}$ is a basis of C;

(c) if $M \subset F$ is the subgroup generated by $p^{-1}\mathfrak{N} \cap \mathfrak{F}$, then $p^{-1}\mathfrak{N} \cap \mathfrak{F}$ is a basis of M.

Then clearly $F = C * M$.

The contractibility of C is proved as follows. Let P denote a c.s.s. group which contains exactly one element in every dimension. The contractibility of D then implies that the unique map $D \to P$ is a homotopy equivalence. As $D \approx C/[C, C]$ and $P \approx P/[P, P]$, it follows from Lemma (3.1) that the unique map $C \to P$ is also a homotopy equivalence, and hence C is contractible.

The minimality of M may be shown as follows. Suppose $M = C' * M'$ where C' is contractible. Then $N \approx C'/[C', C'] + M'/[M', M']$ where (by an argument similar to the one above) $C'/[C', C']$ is contractible. This contradicts the minimality of N, and hence M is minimal.

That the maps j_M and p_M are homotopy equivalences follows (using Lemma (3.1)) from the fact that the maps j_N and p_N are so.

It thus remains to show that there exists a CW-basis $\mathfrak{F}$ of F satisfying conditions (a), (b), and (c).

Condition (a). Let $\mathcal{B} \subset \mathcal{A}_n$ contain at least all nondegenerate elements; suppose that a CW-basis $\mathfrak{F}^0$ of F has already been defined such that $\mathcal{B} \subset p\mathfrak{F}_n^0$. Let $\alpha \in \{\mathcal{A}_n - \mathcal{B}\}$. Then there exist nondegenerate elements $\sigma_1, \cdots, \sigma_s, \tau_1, \cdots, \tau_t \in \mathfrak{F}_n^0$ with $p\sigma_i \not\subset \mathcal{B}$ $(0 \leqq i \leqq s)$ and $p\tau_i \in \mathcal{B}$ $(0 \leqq i \leqq t)$ and such that

$$\alpha = \prod_{i=1}^{s} p\sigma_i^{q_i} \cdot \prod_{i=1}^{t} p\tau_i^{r_i},$$

where q_i and r_i are suitable integers. Let $S \subset A_n$ be generated by the set $\mathcal{S} = \{p\sigma_1, \cdots, p\sigma_s\}$, and let $B \subset A_n$ be the subgroup generated by $\mathcal{B}$. Let

$$\alpha_1 = \prod_{i=1}^{s} p\sigma_i^{q_i},$$

and let $Q \subset A_n$ be generated by $\{\mathcal{B} + \alpha_1\}$. Then Q is a direct summand of $S + B$, and it is readily seen that there exists a set $\mathcal{S}' = \{\alpha_1, \alpha_2, \cdots, \alpha_s\} \subset S$ which also generates S. The basis $\mathcal{S}'$ then may be obtained from $\mathcal{S}$ in a finite number of steps

$$\mathcal{S} = \mathcal{S}^0, \quad \mathcal{S}^1, \quad \cdots, \quad \mathcal{S}^{c-1}, \quad \mathcal{S}^c = \mathcal{S}',$$

where each $\mathcal{S}^i$ generates S, and $\mathcal{S}^{i+1}$ is related to $\mathcal{S}^i$ by

$$\mathcal{S}^{i+1} = \{\mathcal{S}^i - \beta_i + \beta_i \cdot \gamma_i^{\varepsilon_i}\},$$

where $\varepsilon_i = \pm 1$ and β_i and γ_i are distinct elements of $\mathcal{S}^i$. Let

$$\mathcal{S}'' = \{\mathcal{S}' - \alpha_1 + \alpha\}.$$

Then similarly $\mathcal{S}''$ may be obtained from $\mathcal{S}'$ in a finite number of steps

$$\mathcal{S}' = \mathcal{S}^c, \quad \mathcal{S}^{c+1}, \quad \cdots, \quad \mathcal{S}^{d-1}, \quad \mathcal{S}^d = \mathcal{S}'',$$

where each $\mathcal{S}^{i+1}$ is related to $\mathcal{S}^i$ by

$$\mathcal{S}^{i+1} = \{\mathcal{S}^i - \beta_i + \beta_i \cdot \gamma_i^{\varepsilon_i}\},$$

where $\varepsilon_i = \pm 1$, $\beta_c = \alpha_1$, $\beta_i = \beta_{i-1} \cdot \gamma_{i-1}^{\varepsilon_{i-1}}$ and $\gamma_i \in \mathcal{B}$ is nondegenerate. By Lemma (5.4) we now may form a sequence

$$\mathfrak{F}^0, \quad \mathfrak{F}^1, \quad \cdots, \quad \mathfrak{F}^{d-1}, \quad \mathfrak{F}^d$$

of CW-bases of F by defining

$$\mathfrak{F}^{i+1} = \{\mathfrak{F}^i - \phi_i + \phi_i \cdot \chi_i^{\varepsilon_i}\},$$

where $\phi_i = p^{-1}\beta_i \cap \mathfrak{F}^i$ and $\chi_i = p^{-1}\gamma_i \cap \mathfrak{F}^i$. Then it is readily verified that $\{\mathcal{B} + \alpha\} \subset p\mathfrak{F}_n^d$.

The existence of a CW-basis of F satisfying (a) now follows by (possibly transfinite) induction on the elements of $\{\mathfrak{A}_n - \mathfrak{B}\}$ and by induction on n.

Condition (b). Suppose there has already been constructed a CW-basis $\mathfrak{F}'$ of F satisfying condition (a) and the following conditions:

(i) Let $\alpha \in \mathfrak{D}$ be of type I, let $\dim \alpha < n$, and let $\sigma = p^{-1}\alpha \cap \mathfrak{F}'$. Then $\sigma\varepsilon^{\dim \alpha} = e_{n-1}$.

(ii) Let $\beta \in \mathfrak{D}$ be of type II, let $\dim \beta \leqq n$, let $\tau = p^{-1}\beta \cap \mathfrak{F}'$, and let $\sigma = p^{-1}(\beta\varepsilon^{\dim \beta}) \cap \mathfrak{F}'$. Then $\tau\varepsilon^{\dim \beta} = \sigma$.

Let $\alpha_1 \in \mathfrak{D}_n$ be of type I, and $\beta_1 \in \mathfrak{D}_{n+1}$ of type II and such that $\beta_1 \varepsilon^{n+1} = \alpha_1$. Let $\sigma_1 = p^{-1}\alpha_1 \cap \mathfrak{F}'$ and $\tau_1 = p^{-1}\beta_1 \cap \mathfrak{F}'$. Then $p(\tau_1 \varepsilon^{n+1} \cdot \sigma_1^{-1}) = e_n$, i.e., $\tau_1 \varepsilon^{n+1} \cdot \sigma_1^{-1} \in [F^n, F^n]_n$. Hence by Lemma (3.5) there exist a $\rho \in [F^n, F^n]_{n+1}$ and a $\phi \in [F^{n-1}, F^{n-1}]_n$ such that $\rho\varepsilon^i = e_n$ for $0 \leqq i < n + 1$ and $\rho\varepsilon^{n+1} = \tau_1 \varepsilon^{n+1} \cdot \sigma_1^{-1} \cdot \phi^{-1}$. By Corollary (5.5) the sets $\mathfrak{F}'' = \{\mathfrak{F}' - \sigma_1 + \sigma_1 \cdot \phi^{-1}\}$ and $\mathfrak{F}''' = \{\mathfrak{F}'' - \tau_1 + \tau_1 \cdot \rho^{-1}\}$ are CW-bases of F, and it is now readily verified that by repeating this procedure for the other elements of $\mathfrak{D}_n$ of type I one may obtain a CW-basis of F which satisfies condition (a) and conditions (i) and (ii) above with $n + 1$ instead of n.

It now follows by induction on n that there exists a CW-basis of F satisfying conditions (a) and (b).

Condition (c). For every integer $n \geqq 0$ let $\mathfrak{N}^n = N^n \cap \mathfrak{A}$. Suppose there has already been constructed a CW-basis $\mathfrak{F}'$ of F satisfying conditions (a) and (b) and the following condition:

(iii) Let $M^{n-1} \subset F$ be the subgroup generated by the set $p^{-1}\mathfrak{N}^{n-1} \cap \mathfrak{F}'$. Then $p^{-1}\mathfrak{N}^{n-1} \cap \mathfrak{F}'$ is a basis of M^{n-1}.

Let $\beta \in \mathfrak{N}_n$ be nondegenerate. Then $\beta\varepsilon^n = \prod \alpha_i^{p_i}$ where $\alpha_i \in \mathfrak{N}_{n-1}$ and the p_i are integers. Let $\sigma_i = p^{-1}\alpha_i \cap \mathfrak{F}'$ and $\tau = p^{-1}\beta \cap \mathfrak{F}'$. Then there exists a $\sigma \in [F^{n-1}, F^{n-1}]_{n-1}$ such that $\tau\varepsilon^n = (\prod \sigma_i^{p_i}) \cdot \sigma$. Because D is contractible, the inclusion map $N^{n-1} \to D + N^{n-1}$ is a homotopy equivalence. As $N^{n-1} \approx M^{n-1}/[M^{n-1}, M^{n-1}]$, it follows from Lemma (3.1) that the inclusion map $M^{n-1} \to C * M^{n-1}$ is also a homotopy equivalence. Hence there exists a $\rho \in [F^{n-1}, F^{n-1}]_n$ such that

$$\rho\varepsilon^n \cdot \sigma^{-1} \in [M^{n-1}, M^{n-1}]_{n-1} \quad \text{and} \quad \rho\varepsilon^i = e_{n-1} \text{ for } 0 \leqq i < n.$$

By Corollary (5.5) $\mathfrak{F}'' = \{\mathfrak{F}' - \tau + \tau \cdot \rho^{-1}\}$ is again a CW-basis of F. Furthermore $(\tau \cdot \rho^{-1})\varepsilon^n \in M^{n-1}$. It is now readily verified that by repeating this procedure for the other nondegenerate elements of $\mathfrak{N}_n$ one obtains a CW-basis of F satisfying conditions (a) and (b) and condition (iii) with n instead of $n - 1$.

It now follows by induction on n that there exists a CW-basis of F satisfying conditions (a), (b), and (c).

This completes the proof of Theorem (2.5).

7. Proof of Theorem (2.8)

Let $A = F/[F, F]$, let $B = G/[G, G]$, and let $c: A \to B$ be the map induced by h. By Lemma (3.1) the map c is a homotopy equivalence, and hence by Theorem (4.8) there exists an isomorphism $c': A \approx B$ such that $c' \simeq c$. Lemma (3.3) then yields a map $h'': F \to G$ such that $h'' \simeq h$ and such that c' = "h'' made abelian." Hence identifying A with B under the isomorphism c' we get a commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\quad h'' \quad} & G \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & A & \end{array}$$

where p and q denote the projections.

Now suppose we have already obtained a map $h^{n-1}: F \to G$ such that

(i) $h^{n-1} \simeq h''$,

(ii) h^{n-1} is an isomorphism in dimension $< n$,

(iii) commutativity holds in the diagram

$$\begin{array}{ccc} F & \xrightarrow{\quad h^{n-1} \quad} & G \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & A & \end{array}$$

Let $\mathfrak{F}$ be a CW-basis of F, let $\mathfrak{A} = p\mathfrak{F}$, and let $\mathfrak{G}$ be a CW-basis of G such that $q\mathfrak{G} = \mathfrak{A}$ (see §6, condition (a)). Let $\sigma \in \mathfrak{F}_n$ be nondegenerate, and let $\tau_\sigma \in \mathfrak{G}_n$ be such that $p\sigma = q\tau_\sigma$. Then $(h^{n-1}\sigma)\cdot\tau_\sigma^{-1} \in [G^n, G^n]_n$. Hence, by Lemma (3.5) there exist a $\rho_\sigma \in [G^n, G^n]_{n+1}$ and a $\phi_\sigma \in [G^{n-1}, G^{n-1}]_n$ such that $\rho_\sigma \varepsilon^{n+1} = \phi_\sigma \cdot \tau_\sigma \cdot (h^{n-1}\sigma)^{-1}$ and $\rho_\sigma \varepsilon^i = e_n$ for $0 \leqq i < n + 1$. Do this for all nondegenerate $\sigma \in \mathfrak{F}_n$, and define a homotopy $g_I : I \otimes F^n \to G$ by

$$\begin{aligned}
g_I(\psi \otimes \sigma) &= h^{n-1}\sigma, && \psi \in I_q, \quad \sigma \in \mathfrak{F}_q, \quad q < n, \\
g_I(\varepsilon_1^0 \eta^0 \cdots \eta^{n-1} \otimes \sigma) &= h^{n-1}\sigma, && \sigma \in \mathfrak{F}_n, \\
g_I(\varepsilon_1^1 \eta^0 \cdots \eta^{n-1} \otimes \sigma) &= \phi_\sigma \cdot \tau_\sigma, && \sigma \in \mathfrak{F}_n, \\
g_I(\varepsilon_1 \eta^0 \cdots \eta^{i-1}\eta^{i+1} \cdots \eta^n \otimes \sigma\eta^i) &= (h^{n-1}\sigma)\eta^i, && \sigma \in \mathfrak{F}_n, \quad i < n, \\
g_I(\varepsilon_1 \eta^0 \cdots \eta^{n-1} \otimes \sigma\eta^n) &= \rho_\sigma \cdot (h^{n-1}\sigma)\eta^n, && \sigma \in \mathfrak{F}_n.
\end{aligned}$$

By Lemma (3.3) this homotopy may be extended to a homotopy $h_I : h^{n-1} \simeq h^n$ such that condition (iii) holds with n instead of $n - 1$. Clearly $h^n \simeq h''$, and a simple computation yields that h^n is an isomorphism in dimension $< n$.

The theorem now follows by induction on n.

8. Proof of Lemma (3.5)

Use will be made of the following lemma.

LEMMA $(8.1)_k$. *Let F be a c.s.s. group, let F_0 be trivial, and let $k \leqq n$. Let $\sigma \in F_n$ and let $\tau \in F_k$. Then there exist an element $\rho \in [F^n, F^n]_{n+1}$ and an element $\phi \in [F^{n-1}, F^{n-1}]_n$ such that $\rho\varepsilon^{n+1} = [\sigma, \tau\eta^k \cdots \eta^{n-1}]\cdot\phi^{-1}$ and $\rho\varepsilon^i = e_n$ for $0 \leqq i < n+1$.*

Proof of Lemma (3.5). As $\sigma \in [F^n, F^n]_n$, it follows that there exist an integer q and elements $\sigma_1, \cdots, \sigma_q, \tau_1, \cdots, \tau_q \in F^n$ such that

$$\sigma = \textstyle\prod_{j=1}^q [\sigma_j, \tau_j].$$

Application of Lemma $(8.1)_n$ yields elements $\rho_j \in [F^n, F^n]_{n+1}$ and $\phi_j \in [F^{n-1}, F^{n-1}]_n$ such that $\rho_j\varepsilon^{n+1} = [\sigma_j, \tau_j]\cdot\phi_j^{-1}$ and $\rho_j\varepsilon^i = e_n$ for $0 \leqq i < n+1$. Let

$$\phi = \textstyle\prod_{j=1}^q \phi_j \qquad \in [F^{n-1}, F^{n-1}]_n,$$

$$\rho = \textstyle\prod_{j=1}^q (\rho_j\cdot\phi_j\,\eta^n)\cdot\phi^{-1}\eta^n \qquad \in [F^n, F^n]_{n+1}.$$

A simple computation then yields that $\rho\varepsilon^{n+1} = \sigma\cdot\phi^{-1}$ and $\rho\varepsilon^i = e_n$ for $0 \leqq i < n+1$.

Proof of Lemma $(8.1)_k$. For $k = 0$, the lemma is obvious.

Now let $k > 0$, and suppose that Lemma $(8.1)_{k-1}$ has already been proved. Let

$$\alpha = \textstyle\prod_{i=k-1}^n [\sigma\eta^i, \tau\eta^k \cdots \eta^n]^{(-1)^{n+i+1}},$$

$$\beta = \textstyle\prod_{i=k-1}^n [\sigma\eta^i, \tau\varepsilon^{k-1}\eta^{k-1} \cdots \eta^n]^{(-1)^{n+i+1}};$$

then clearly

(i) $(\alpha^{-1}\cdot\beta)\varepsilon^i \in [F^{n-1}, F^{n-1}]_n$ for $0 \leqq i < n+1$,

(ii) there exists an element $\gamma \in [F^{n-1}, F^{n-1}]_n$ such that

$$\alpha^{-1}\varepsilon^{n+1} = [\sigma, \tau\eta^k \cdots \eta^{n-1}]\cdot\gamma\cdot[\sigma\varepsilon^n\eta^{k-1}, \tau\eta^k \cdots \eta^{n-1}]^{(-1)^{n+k+1}},$$

(iii) there exists an element $\delta \in [F^{n-1}, F^{n-1}]_n$ such that

$$\beta\varepsilon^{n+1} = \delta\cdot[\sigma, \tau\varepsilon^{k-1}\eta^{k-1} \cdots \eta^{n-1}]^{-1}.$$

By the induction hypothesis there exist $\phi_0, \phi_1 \in [F^{n-1}, F^{n-1}]_n$ and $\rho_0, \rho_1 \in [F^n, F^n]_{n+1}$ such that

$$\rho_0\,\varepsilon^{n+1} = [\sigma\varepsilon^n\eta^{k-1}, \tau\eta^k \cdots \eta^{n-1}]^{(-1)^{n+k}}\cdot\phi_0^{-1},$$

$$\rho_1\,\varepsilon^{n+1} = [\sigma, \tau\varepsilon^{k-1}\eta^{k-1} \cdots \eta^{n-1}]\cdot\phi_1^{-1},$$

$$\rho_0\,\varepsilon^i = \rho_1\,\varepsilon^i = e_n \qquad \text{for } 0 \leqq i < n+1.$$

Let $\mu = m((\alpha^{-1}\cdot\beta)\varepsilon^0, \cdots, (\alpha^{-1}\cdot\beta)\varepsilon^n)$ (see Lemma (5.3)), and let

$$\rho = \alpha^{-1}\cdot\rho_0\cdot\beta\cdot\rho_1\cdot\mu^{-1} \qquad \in [F^n, F^n]_{n+1},$$

$$\phi = \mu\varepsilon^{n+1}\cdot\phi_1\cdot\delta^{-1}\cdot\phi_0\cdot\gamma^{-1} \qquad \in [F^{n-1}, F^{n-1}]_n.$$

Then a straightforward computation yields that $\rho\varepsilon^{n+1} = [\sigma, \tau\eta^{k} \cdots \eta^{n-1}]\cdot\phi^{-1}$ and $\rho\varepsilon^{i} = e_n$ for $0 \leqq i < n + 1$.

Appendix

9. Equivalent definitions of minimality

It follows immediately from the proof of Theorem (2.5) that

PROPOSITION (9.1). *Let F be a free c.s.s. group of finite type. Then the following statements are equivalent*:

(a) *F contains no contractible free factor* (*except the trivial one*).

(b) *F contains no contractible proper subgroup* (*except the trivial one*).

(c) *F contains no free factor of the same homotopy type* (*except F itself*).

(d) *F contains no proper subgroup of the same homotopy type* (*except F itself*).

(*e*) *If $h{:}F \to G$ is an epimorphism, where G is free and h is a homotopy equivalence, then h is an isomorphism.*

For minimal complexes in the sense of Eilenberg-Zilber ([3]) it is possible to give a "local" definition (see [8], Lemma 1.20). The corresponding statement for free c.s.s. groups is contained in Proposition (9.2). The proof is straightforward and is left to the reader.

PROPOSITION (9.2). *Let F be a free c.s.s. group of finite type. Then F is minimal if for each pair of elements $\sigma, \tau \in F_n$ such that*

(i) *σ and τ are compatible and homotopic,*

(ii) *there exists a basis $\mathfrak{F}$ of F containing σ and τ,*

we have $\sigma = \tau$.

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