PRIME IDEALS IN RINGS OF CONTINUOUS FUNCTIONS[1]

BY

CARL W. KOHLS

Let $C = C(X)$ denote the ring of all continuous real-valued functions on a completely regular Hausdorff space X. The maximal ideals of $C(X)$ were characterized by Gelfand and Kolmogoroff [3]. The existence of prime ideals of $C(X)$ other than the maximal ideals was considered by Gillman and Henriksen [5]. In general, nonmaximal prime ideals do occur. Specifically, whenever the maximal ideal M^p $(p \in \beta X)$ is distinct from the ideal $N^p = \{f \in C(X): f = 0 \text{ on an } X\text{-neighborhood of } p\}$, there exists a nonmaximal prime ideal containing N^p. Except for a few results in [5], [6], and [9], there seems to be nothing in the literature on the prime ideals of $C(X)$. This paper initiates an investigation of the prime ideal structure of $C(X)$.

Most of the results obtained are stated for homomorphic images of $C(X)$. Thus, any structural properties that are not reflected in the images will not become apparent from this discussion. The quotient rings of $C(X)$ considered are usually those obtained from the prime ideals themselves.

In §1, we present some preliminary concepts and results. The main results are contained in §2. Let P be an arbitrary prime ideal of C. The key theorem in the whole study is that C/P is a totally ordered ring. The order in these totally ordered rings is used throughout to obtain information about prime ideals. For example, a corollary to the above theorem is that the prime ideals of C that contain P form a chain under set inclusion. A useful tool is the result that each integral domain C/P has an order property related to those of the η_1-sets of Hausdorff.

Much of the investigation centers about two related types of prime ideals of C/P, which we call upper ideals and lower ideals. An upper ideal turns out to be the McCoy radical of a principal ideal (a); and the associated lower ideal, the (unique) ideal that is maximal with respect to disjointness from the powers of a. Between any pair of upper ideals there is another (in the sense of set inclusion), and correspondingly for lower ideals. Each upper ideal is countably generated, but no lower ideal is. A consequence of the last statement is that there exist uncountably many upper (and lower) ideals in C/P (provided P is not maximal). On the other hand, not every prime ideal of C/P need be an upper or a lower ideal.

Received August 1, 1957.

[1] Presented to the Society, December 29, 1956. This paper is based on part of the author's doctoral dissertation, written under the supervision of Professor Leonard Gillman. The author wishes to thank Professor Gillman for his advice and encouragement. Support for this research was provided by the Purdue Research Foundation, and by the National Science Foundation.

Let I be a $\mathfrak{z}$-ideal of C (that is, I is maximal with respect to its family of zero-sets), and suppose that I contains some N^p. The concluding theorem of §2 states that the following are equivalent: (1) I is prime. (2) The prime ideals of C containing I form a chain. (3) Every $\mathfrak{z}$-ideal of C containing I is a prime ideal. (4) C/I is a totally ordered ring. (5) Zero-sets are comparable on zero-sets of I.

In §3, the concept of totally ordered valuation ring is introduced. A totally ordered ring that is a valuation ring is called a totally ordered valuation ring if the valuation is monotone decreasing on the set of positive elements. These rings arise naturally in the study of quotient rings of $C(X)$. In fact, we show that for any $\mathfrak{z}$-ideal I containing some N^p, the ring C/I is a totally ordered valuation ring if and only if it is a valuation ring, and if and only if either of the following statements holds: (1) The ideals of C containing I form a chain. (2) Every finitely generated ideal of C/I is a principal ideal. Associated with any value group are certain archimedean quotient groups. In the case of the value group of a totally ordered valuation ring of the form C/P, where P is a prime $\mathfrak{z}$-ideal of C, each such quotient group is associated with some upper ideal of C/P. A characterization of the elements in this upper ideal but not in the lower ideal paired with it enables us to show that the corresponding quotient group is the additive group of all real numbers.

The last section continues the study of the βF-points introduced in [9]. A point $p \in \beta X$ is a βF-point if and only if the ideal N^p is prime. The main theorem of this section gives sufficient conditions that C/N^p be a totally ordered valuation ring. (A *necessary* condition is that N^p be prime.)

1. Preliminary remarks

We shall use the results of [5], [6], [7], and [9] freely, and assume familiarity with background material given in these papers whenever this is convenient.

It is assumed throughout that the space X is a completely regular Hausdorff space. The letter R is reserved for the field of real numbers. $C(X)$ denotes the ring of all continuous real-valued functions on X; and $C^*(X)$, the subring of all bounded functions in $C(X)$. Since the underlying space X is fixed in many discussions, we shall often abbreviate $C(X)$ and $C^*(X)$ to C and C^* respectively. As usual, βX denotes the Stone-Čech compactification of X; and vX, the largest subspace of βX over which *every* function in $C(X)$ (whether bounded or not) has a continuous extension. Furthermore, if $f \in C(X)$ is regarded as a function from X to the one-point compactification of R, designated by $R \cup \{\infty\}$, then f may be extended to a continuous function $\hat{f}$ from βX to $R \cup \{\infty\}$.

For every $f \in C(X)$, the set $Z(f) = \{x \in X : f(x) = 0\}$ is called the *zero-set* of f. For any subset I of $C(X)$, we let $\mathfrak{Z}(I) = \{Z(f) : f \in I\}$. We recall that an ideal I of C is called a *$\mathfrak{z}$-ideal* if $Z(f) \in \mathfrak{Z}(I)$ implies that $f \in I$; in other words, if I contains every function whose zero-set belongs to the family of zero-sets $\mathcal{Z}(I)$. Every $\mathfrak{Z}$-ideal of C is an intersection of prime ideals [9, Lemma 2.2].

An ideal of $C(X)$ of particular interest in the study of prime ideals is the ideal N^p, introduced in [5]. This is defined, for $p \in \beta X$, to be the set of all $f \in C(X)$ such that $Z(f)$ contains the intersection of X with a neighborhood of p in βX. The corresponding ideal $N^p \cap C^*(X)$ of $C^*(X)$ will be denoted by N^{*p}. The only maximal ideal containing N^p is M^p. If P is a prime ideal of C contained in M^p, then P contains N^p [5, Lemma 3.2]. Consequently, M^p is the only maximal ideal containing P [5, Theorem 3.3].

A *partially ordered group*[2] G is a commutative group on which is defined a partial ordering relation $\geqq$ that is invariant under translation, that is, for all $a, b, c \in G$, we have

(1) (i) $a \geqq a$; (ii) if $a \geqq b$ and $b \geqq a$, then $a = b$; (iii) if $a \geqq b$ and $b \geqq c$, then $a \geqq c$.

(2) If $a \geqq b$, then $a + c \geqq b + c$.

It is evident that (2) is equivalent both to

$$a \geqq b \text{ if and only if } a - b \geqq 0,$$

and to

$$\text{If } a \geqq b \text{ and } c \geqq d, \text{ then } a + c \geqq b + d.$$

A *partially ordered ring*[2] A is a commutative ring whose additive group is a partially ordered group, and such that, for all $a, b \in A$,

$$\text{If } a \geqq 0 \text{ and } b \geqq 0, \text{ then } ab \geqq 0.$$

The set of all nonzero elements a such that $a \geqq 0$ will be called the *positive cone* of A.

A subset S of A is called *order-convex*, or simply *convex*, if $s \geqq u \geqq t$ and $s, t \in S$ imply $u \in S$. If S is a subgroup of A, this is equivalent to

$$a \geqq b \geqq 0 \text{ and } a \in S \text{ imply } b \in S.$$

Let I be an ideal of A. For $a \in A$, the image of a in the quotient ring A/I (under the natural homomorphism) will be denoted by a_I. It is well known (see, for example, [1, p. 23, Exercise 4]) that the ideal I is convex if and only if A/I is a partially ordered ring under the following definition: $a_I \geqq 0$ if and only if there exists $x \in A$ with $x \geqq 0$ and $x_I = a_I$. Whenever we speak of a partially ordered quotient ring of a partially ordered ring A, the order on A/I will be understood to be this induced order.

If A is a partially ordered ring, and the relation $\geqq$ is a total ordering relation, that is, a partial ordering relation satisfying

(3) For any $a, b \in A$, either $a \geqq b$ or $b \geqq a$,

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[2] For further material on partially ordered groups and rings, see [1] and [4].

then A is a *totally ordered ring.* Evidently, (3) is equivalent to

For any $a \in A$, either $a \geqq 0$ or $0 \geqq a$.

Let A be a totally ordered ring with identity (so that A contains a subring that may be identified with the integers). We say that a positive element a of A is *infinitely large* if $a \geqq n$ for every positive integer n, and that a is *infinitely small* if $1 \geqq na$ for every positive integer n.

The symbol $|a|$ denotes an element b satisfying $b \geqq a$ and $b \geqq -a$, and such that whenever $x \geqq a$ and $x \geqq -a$, then $x \geqq b$. By (1) (ii), $|a|$ is unique whenever it exists. It is clear that in a totally ordered ring A, the element $|a|$ exists for all $a \in A$; in fact, $|a| = a$ if $a \geqq 0$, and $|a| = -a$ if $0 \geqq a$; thus $|a| \geqq 0$. And if a partially ordered ring A is a lattice under $\geqq$, then $|a|$ exists, and $|a| \geqq 0$ for all $a \in A$ (see [1, p. 15, Proposition 9]).

We now give a sufficient condition for A/I to be a totally ordered ring, where I is an ideal of A. In this case, the induced order can be defined in a manner different from, but equivalent to, that discussed above.

LEMMA 1.1. *Let A be a partially ordered ring such that $|a|$ exists and $|a| \geqq 0$ for every $a \in A$; and let I be a convex ideal of A such that $a \equiv |a|$ (mod I) or $a \equiv -|a|$ (mod I) for every $a \in A$. Then*

(1) *The quotient ring A/I is totally ordered.*

(2) *$a_I \geqq 0$ if and only if $a \equiv |a|$ (mod I).*

Proof. (1) is evident, and so is the sufficiency in (2). Suppose that $a_I \geqq 0$ and $a \equiv -|a|$ (mod I). Then $-(a_I) \geqq 0$, so $a \in I$, and $|a| \in I$.

Let A be a totally ordered ring. It is clear that any convex subset of A is an interval; and any convex ideal I of A is symmetric about 0, that is, I is a union of intervals of the form $\{x \in A : |a| \geqq x \geqq -|a|\}$. Thus, any collection of convex ideals of A forms a chain (under set inclusion).

Much of the preceding discussion is also valid for groups, with "subgroup" in place of "ideal". The next remark, however, applies only to *rings.*

Let A be a partially ordered ring such that $|a|$ exists, $|a| \geqq 0$, and $|a|^2 = a^2$ for every $a \in A$. Then any convex ideal I of A that contains a prime ideal of A satisfies the condition of 1.1. In particular, these requirements are fulfilled in $C(X)$ and $C^*(X)$ for arbitrary completely regular Hausdorff spaces X.

A subset U of the space X is called an *X-neighborhood* of $p \in \beta X$ if U has the form $X \cap \Omega$, where Ω is a neighborhood of p in βX. Thus when $p \in X$, the set of X-neighborhoods of p coincides with the set of neighborhoods of p in X.

Let $f \in C(X)$, and let Y be a subset of X. If a statement about $f(x)$ is true for each $x \in Y$, we shall say the statement is true for *f on Y.*

For any maximal ideal M^p of $C(X)$, $f \in M^p$ implies $\hat{f}(p) = 0$. But the converse is false in general, as can be seen from the Gelfand-Kolmogoroff Theorem (see, e. g., [7, Theorem 1]).

If σ is a function on a set S, and T is a subset of S, then the restriction of σ to T will be denoted by $\sigma \mid T$.

Finally, we describe the space E, first given in [6, §8.5], which will be used several times to construct examples. (The letter E will always denote this space.) Let $N = \{e_1, e_2, \cdots\}$ be the denumerable discrete space, and $e \in \beta N \sim N$. Set $E = N \cup \{e\}$. Thus every e_n is an isolated point, while deleted neighborhoods of e are the members of some free ultrafilter on N (i. e., maximal filter on N with total intersection void). The point e is called the β-point of E.

2. General results on prime ideals

Let X be an arbitrary completely regular Hausdorff space. If M^* is any maximal ideal of $C^*(X)$, the quotient ring $C^*(X)/M^*$ is isomorphic to the real field R. This well-known result was proved by Stone [12, Theorem 76]. Hewitt considered the case of a maximal ideal M in the ring $C(X)$; he showed that $C(X)/M$ is a totally ordered field containing a subfield isomorphic to R [8, Theorem 41]. In each case the order is induced by the order on C^* or C as in §1. It is possible for the field to coincide with R, as in the case of fixed ideals, or to contain R properly, as in the case of at least some free ideals when $C \neq C^*$. Precisely, C/M^p is the real field if and only if $p \in vX$. For $p \notin vX$, the field is non-archimedean, and is called "hyper-real".

The main result of this section is a generalization of these results to *arbitrary* prime ideals in C^* and C. Of course, to deduce Stone's Theorem from our statement requires additional facts about R; but the proof is easy.

THEOREM 2.1. *Every prime ideal P of C (C^*) is convex, and C/P (C^*/P) is a totally ordered integral domain containing a subset isomorphic to R. Let p be the unique point of βX such that $P \supseteqq N^p$ (N^{*p}). Then the image of M^p (M^{*p}) is the unique maximal ideal in C/P (C^*/P); and C/P has infinitely large elements if and only if $p \notin vX$ (C^*/P has no infinitely large elements).*

Proof. Suppose that $0 \leqq f \leqq g$ and $g \in P$. Define h as follows: $h(x) = 0$ for $x \in Z(g)$, $h(x) = (f(x))^2/g(x)$ for $x \in X \sim Z(g)$. It is evident that h is continuous on $X \sim Z(g)$. The continuity of h at each point of $Z(g)$ follows from the continuity of f and the relation $0 \leqq h \leqq f$. Hence $h \in C(X)$ (and if $f \in C^*(X)$, then $h \in C^*(X)$). Clearly $f^2 = hg$. It follows that $f^2 \in P$, and since P is prime, that $f \in P$. Hence P is convex. By the remarks in §1, C/P (C^*/P) is a totally ordered ring.

The ordering may be defined explicitly as follows: The image of $f \in C$ in C/P is positive if $f \equiv |f| \pmod P$ but $f \not\equiv 0 \pmod P$. Thus, it is clear that the constant functions map into a subset of C/P that is isomorphic to R, and that the mapping preserves the order on this set. Since M^p is the only maximal ideal of C containing P, it is evident that C/P has a unique maximal ideal, namely, the image of M^p. Corresponding remarks apply to C^*.

If $p \in vX$, then for each $f \in C$, there is an $r \in R$ such that $\hat{f}(p) = r$. Thus, setting $g = f - r$, we have $\hat{g}(p) = 0$; so for any positive integer n, there is an X-neighborhood U of p on which $|g| \leqq 1/n$. Hence

$$(1/n - |g|) - |1/n - |g|| \in N^p \subseteqq P.$$

It follows that the image of f differs in absolute value from the image of the constant function r by at most an infinitely small element. For C^*, this proof is valid for all $p \in \beta X$. Conversely, if $p \notin vX$, there is an $h \in C$, $h \geqq 0$, such that $\hat{h}(p) = \infty$. Hence h is unbounded on every X-neighborhood of p; so, in a similar way, we see that the image of h is an infinitely large element.

It is easy to verify that if P is not maximal, then the unique maximal ideal of C/P (C^*/P) consists of certain infinitely small elements, their negatives, and zero.

COROLLARY 2.2. *Let $p \in \beta X$ be arbitrary. Then C/N^p is isomorphic to a subdirect sum of totally ordered integral domains.*

Proof. By [6, Theorem 1.4], N^p is an intersection of prime ideals $\{P_\gamma\}_{\gamma \in G}$. Hence C/N^p is isomorphic to a subdirect sum of the rings

$$\{(C/N^p)/(P_\gamma/N^p)\}_{\gamma \in G}$$

(see, e. g., [10, corollary to Theorem 30]), which, by the second isomorphism theorem, is the same as $\{C/P_\gamma\}_{\gamma \in G}$. And by Theorem 2.1, each C/P_γ is a totally ordered integral domain.

The following lemma is simple but useful.

LEMMA 2.3. *Let P be a prime ideal of C, and let $f, g \in C$ be such that $f_P \leqq g_P$. Then there exist $f', g' \in C$ such that $f'_P = f_P$, $g'_P = g_P$, $f' \leqq g$ on X, and $f \leqq g'$ on X.*

Proof. By hypothesis, $g - f - |g - f| \in P$. Hence $g - |g - f|$ is a suitable choice for f', and $f + |g - f|$ is a suitable choice for g'.

THEOREM 2.4. *Every prime ideal of C/P (C^*/P) is an interval, symmetric about zero. The prime ideals of C/P (C^*/P) form a chain. Hence, the prime ideals of C (C^*) containing P form a chain.*

Proof. Let Q be any prime ideal of C/P, and let ν denote the natural homomorphism of C onto C/P. Then, as is well known, $\nu^{-1}(Q)$ is a prime ideal of C. By 2.1, $\nu^{-1}(Q)$ is convex. This implies that Q is convex, as follows immediately from Lemma 2.3. The conclusions about prime ideals of C/P are now a consequence of remarks in §1. The final statement follows from the fact that set inclusion is preserved under the correspondence between prime ideals containing a given ideal of a ring, and prime ideals in the quotient ring modulo this ideal.

The proof for C^*/P is identical.

The result used to obtain the final statement of 2.4 can be applied in many situations. In the sequel, we shall usually not point out explicitly when this method yields a conclusion about ideals of C (or C^*).

We note that an arbitrary ideal in a ring $C(X)$ need not be convex. For example, in the ring $C(R)$, let i be the identity function, defined by $i(x) = x$ for all $x \in R$. Then $(|i|)$ is not a convex ideal; for if $f \in C(R)$ is defined by $f(x) = |x \sin 1/x|$ for $x \neq 0$, $f(0) = 0$, then we have $0 \leqq f \leqq |i|$, but, evidently, $f \notin (|i|)$.

Let P be a prime ideal of $C(X)$; I, a convex ideal containing P; and ν, the natural homomorphism of C/P onto C/I. From §1, C/I is totally ordered; and since $f - |f| \in P$ implies $f - |f| \in I$, the mapping ν is order-preserving. It follows immediately that if $a \in C/P$ and $\nu(a) > 0$, then $a > 0$. The ideal I need not be prime, so that C/I need not be an integral domain simply because it is totally ordered. An example will be given in §4.

The next theorem clarifies to some extent the relationship between the integral domains which are homomorphic images of C and those which are homomorphic images of C^*. Not all of the latter have the form discussed in the theorem, however: if $p \notin \upsilon X$, then M^{*p} contains units of C (cf. [7, Theorem 3]), and hence cannot be obtained by intersecting C^* with a prime ideal of C.

THEOREM 2.5. *Let P be a prime ideal of C containing N^p, so that $P \cap C^*$ is a prime ideal of C^* containing N^{*p}. Denote the truncation of C/P obtained by removing all infinitely large elements and their negatives by $(C/P)_t$. Then there is a natural isomorphism of $(C/P)_t$ onto $C^*/P \cap C^*$, and this isomorphism is order-preserving. Furthermore, C/P is isomorphic to $C^*/P \cap C^*$ if and only if $p \in \upsilon X$.*

Proof. Define ν to be the natural homomorphism of C onto C/P. Then $\nu \mid C^*$, the restriction of ν to C^*, is a homomorphism of C^* into C/P; since the elements of $C/P \sim (C/P)_t$ come only from unbounded functions, $\nu \mid C^*$ is actually into $(C/P)_t$. Given $a \in (C/P)_t$, select any $g \in \nu^{-1}(a)$, and set $f = \sup \{\inf \{g, \hat{g}(p) + 1\}, \hat{g}(p) - 1\}$. Then $f \in C^*$, and $f - g \in N^p$, so $\nu(f) = a$. Hence $\nu \mid C^*$ is *onto* $(C/P)_t$. Now the kernel of $\nu \mid C^*$ is evidently $P \cap C^*$. Hence $C^*/P \cap C^*$ is isomorphic to $(C/P)_t$, and, as is easily seen, the isomorphism obtained from $\nu \mid C^*$ is order-preserving.

If $p \in \upsilon X$, it follows from 2.1 that C/P coincides with $(C/P)_t$, whence C/P is isomorphic to $C^*/P \cap C^*$ (under the natural mapping defined above). Conversely, suppose that there exists an isomorphism of C/P onto $C^*/P \cap C^*$. By 2.1, each of the rings C/P and $C^*/P \cap C^*$ has a unique maximal ideal, namely, the image of M^p and M^{*p}, respectively. Since any isomorphism takes maximal ideals onto maximal ideals, these two ideals must be isomorphic. Hence the quotient rings determined by them, $(C/P)/(M^p/P)$ and $(C^*/P \cap C^*)/(M^{*p}/P \cap C^*)$, are isomorphic. By the second isomorphism theorem, C/M^p and C^*/M^{*p} are isomorphic. Since the second ring is isomorphic to R, so is the first. Hence $p \in \upsilon X$.

We shall now investigate the properties of the integral domains C/P as totally ordered sets. We first recall a definition given by Hausdorff for abstract ordered sets.

A totally ordered set L is called an *η_1-set* provided that:

(i) if A, B are subsets of L of power less than $\aleph_1$, and such that $A < B$, then there is a $y \in L$ with $A < y < B$, and

(ii) no subset of L of power less than $\aleph_1$ is cofinal or coinitial with L.

Our results are similar to [2, Theorem 3.4], in both statement and proof. That theorem asserts that any hyper-real field (see remarks preceding 2.1) is an η_1-set.

THEOREM 2.6. *Let P be a prime ideal of C containing N^p. For each pair of countably infinite subsets D, D' of C/P with $D < D'$, of order type ω, ω^*, respectively, there is an element $\delta \in C/P$ such that $D < \delta < D'$. The ring C/P has a countable cofinal and coinitial subset if and only if $p \in vX$.*

Proof. The proof is modeled after the proof of [2, Theorem 3.4], but with significant modifications. For $f \in C$, we let $\bar{f}$ denote the image of f in C/P.

Let $\{f_n\}$, $\{g_n\}$ $(n = 1, 2, \cdots)$ be two sequences of elements of C/P such that

$$(1) \qquad \bar{f}_n < \bar{f}_{n+1} < \bar{g}_{m+1} < \bar{g}_m \qquad (m, n = 1, 2, \cdots).$$

We wish to find an $h \in C(X)$ such that $\bar{f}_n < \bar{h} < \bar{g}_m$ $(m, n = 1, 2, \cdots)$.

First we note that we may assume, without loss of generality, that

$$(2) \qquad f_n \leqq f_{n+1} \quad \text{and} \quad g_{m+1} \leqq g_m \quad \text{on} \quad X, \qquad \text{for all } m, n = 1, 2, \cdots.$$

For if we have defined $f'_1, \cdots, f'_n$ so that $f'_1 \leqq \cdots \leqq f'_n$ on X, then we obtain f'_{n+1} from 2.3 satisfying $f'_n \leqq f'_{n+1}$ on X, and $\overline{f'_{n+1}} = \bar{f}_{n+1}$. Similarly, if we have defined $g'_1, \cdots, g'_m$ so that $g'_m \leqq \cdots \leqq g'_1$ on X, then we obtain g'_{m+1} satisfying $g'_{m+1} \leqq g'_m$ on X and $\overline{g'_{m+1}} = \bar{g}_{m+1}$.

We now show that we may also assume that

$$(3) \qquad f_n \leqq g_n \quad \text{on} \quad X, \qquad \text{for all } n = 1, 2, \cdots.$$

Put $f''_1 = f'_1$, and $g''_1 = \sup \{f''_1, g'_1\}$. If we have defined $f''_1, \cdots, f''_n, g''_1, \cdots, g''_n$ so that $f''_1 \leqq \cdots \leqq f''_n \leqq g''_n \leqq \cdots \leqq g''_1$ on X, then we put

$$f''_{n+1} = \inf \{\sup \{f''_n, f'_{n+1}\}, g''_n\}, \qquad g''_{n+1} = \sup \{\inf \{g''_n, g'_{n+1}\}, f''_{n+1}\}.$$

Now for any functions $s, t \in C$ such that $\bar{s} \leqq \bar{t}$, the function $u = \sup \{s, t\}$ satisfies $\bar{u} = \bar{t}$. For, $t - s - |t - s| \in P$, so $2 \cdot \bar{u} = \overline{2u} = \bar{t} + \bar{s} + \overline{|t - s|} = \overline{2t}$. A corresponding statement holds for $\inf \{s, t\}$. It follows readily from these remarks that $\overline{f''_n} = \overline{f'_n}$ and $\overline{g''_n} = \overline{g'_n}$ for all $n = 1, 2, \cdots$. Moreover, we have $f''_n \leqq f''_{n+1} \leqq g''_{n+1} \leqq g''_n$ on X.

Resuming our original notation, we assume that (1), (2), and (3) hold.

The case in which $p \notin vX$. Since $p \notin vX$, there exists a function ϕ in C such that $\hat{\phi}(p) = \infty$ and $\phi \geqq 1$ on X. Then each of the sets $\Phi_n = \{x \in X : \phi(x) \geqq n\}$ is a zero-set in $\mathcal{Z}(N^p)$ $(n = 1, 2, \cdots)$. We now define a function h as follows:

$$h(x) = (n + 1 - \phi(x))f_n(x) + (\phi(x) - n)f_{n+1}(x)$$

whenever $n \leqq \phi(x) \leqq n + 1$ $(n = 1, 2, \cdots)$. Evidently, $h \in C$. Clearly, $f_n(x) \leqq h(x) \leqq f_{n+1}(x)$ whenever $n \leqq \phi(x) \leqq n + 1$. Since the sequence of functions $\{f_n\}$ is monotone increasing ((2)), we have $f_n \leqq h$ on Φ_n, so $Z(h - f_n - |h - f_n|) \supseteqq \Phi_n \in \mathcal{Z}(N^p)$; it follows that $h - f_n - |h - f_n| \in P$, so $\bar{f}_n \leqq \bar{h}$. From (1), we obtain $\bar{f}_n < \bar{h}$ for all $n = 1, 2, \cdots$.

Next, it follows from (2) and (3) that for each fixed m, $f_{m+k} \leqq g_m$ on X for all $k = 0, 1, \cdots$. It is easily seen from this that $h \leqq g_m$ on Φ_m. Thus $\bar{h} \leqq \bar{g}_m$, so again from (1), we obtain $\bar{h} < \bar{g}_m$ for $m = 1, 2, \cdots$. This proves the first statement of the theorem for $p \notin vX$, and half of the second statement.

We cannot conclude, as in [2], that C/P is an η_1-set.[3] For if $P \neq M^p$, there are nonunits other than zero in C/P; and the proof in [2] uses the fact that C/M^p is a *field*.

The case in which $p \in vX$. By Theorem 2.1, there are no infinitely large elements in C/P. Thus, the subset consisting of the images of the integer-valued constant functions is a countable cofinal and coinitial subset of C/P.

We show first that it suffices to consider the case where the $\bar{f}_n$ are infinitely small elements and the $\bar{g}_n$ are positive elements. Let $r = \sup_n \hat{f}_n(p)$ and $s = \inf_n \hat{g}_n(p)$; then $r \leqq s$. If $r < s$, we have $\bar{f}_n < \overline{(r + s)/2} < \bar{g}_n$ for all $n = 1, 2, \cdots$. We dismiss this trivial case, and assume that $r = s$. Now if there is a positive integer N such that $\bar{g}_n < \bar{r}$ (resp. $\bar{f}_n > \bar{r}$) for all $n > N$, but $\bar{g}_n \geqq \bar{r}$ (resp. $\bar{f}_n \leqq \bar{r}$) for some $n \leqq N$, we may discard $\{g_1, \cdots, g_N\}$ (resp. $\{f_1, \cdots, f_N\}$). So we may suppose that either $\bar{f}_n < \bar{r}$ and $\bar{g}_n < \bar{r}$ for all n, or $\bar{f}_n > \bar{r}$ and $\bar{g}_n > \bar{r}$ for all n. But the first situation may be reduced to the second by consideration of the negatives of all the elements involved. Finally, it is clearly no restriction to suppose that $r = 0$.

We may assume that the f_n and g_n satisfy $0 \leqq f_n \leqq 1$ and $0 \leqq g_n$ on X, and we may also require the f_n and g_n to satisfy conditions (2) and (3) above.

If there is a positive integer N such that $\hat{g}_n(p) = 0$ for all $n > N$, but $\hat{g}_n(p) > 0$ for some $n \leqq N$, we discard $\{g_1, \cdots, g_N\}$. Otherwise, there must be a subsequence $\{g_{n_1}, g_{n_2}, \cdots\}$ such that $\hat{g}_{n_i}(p) \neq \hat{g}_{n_j}(p)$ for $i \neq j$; we then discard all those g_n not belonging to this subsequence. Evidently, $\{\hat{g}_{n_1}(p), \hat{g}_{n_2}(p), \cdots\}$ is a strictly decreasing sequence of real numbers whose limit is zero. Resuming the original notation, we have two distinct cases: (I) $\hat{g}_n(p) = 0$ for all n, that is, $\bar{g}_n$ is infinitely small for all n. (II) $\hat{g}_n(p) > 0$ for all n, but $\lim_{n\to\infty} \hat{g}_n(p) = 0$, that is, no $\bar{g}_n$ is infinitely small, but there is no positive constant function t such that $\bar{g}_n \geqq \bar{t}$ for all n.

Case I. Since $\hat{g}_n(p) = 0$ for all n, we may suppose that $g_1 \leqq 1$ on X. We set $\phi = g_1$. For each n, $\Phi_n = \{x \in X : \phi(x) \leqq 1/n\}$ is a zero-set in $\mathcal{Z}(N^p)$. Define h as follows:

$$h(x) = \{n(n + 1)\phi(x) - n\}g_n(x) + \{1 + n - n(n + 1)\phi(x)\}\, g_{n+1}(x)$$

[3] In 4.2, we shall give an example of a space X and a βF-point $p \in \beta X \sim vX$, such that the positive cone of $C(X)/N^p$ has a countable coinitial subset. Thus, $C(X)/N^p$ cannot be an η_1-set.

whenever $1/(n+1) \leqq \phi(x) \leqq 1/n$, and $h(x) = 0$ whenever $\phi(x) = 0$. Then $g_{n+1}(x) \leqq h(x) \leqq g_n(x)$ whenever $1/(n+1) \leqq \phi(x) \leqq 1/n$. Let m be a fixed positive integer. For any $x \in \Phi_m$, either $\phi(x) = 0 = h(x) = g_m(x)$, or there is a positive integer $k \geqq m$ such that $1/(k+1) \leqq \phi(x) \leqq 1/k$, whence $h(x) \leqq g_k(x) \leqq g_m(x)$. Thus $h \leqq g_m$ on Φ_m. In particular, $h \leqq g_1$ on $\Phi_1 = X$. It is easily verified that h is continuous at each $x \notin Z(\phi)$. Now let $x \in Z(\phi) = Z(g_1)$. For a given $\varepsilon > 0$, there is a neighborhood U of x such that $g_1(y) < \varepsilon$, whence $0 \leqq h(y) < \varepsilon$, for all $y \in U$. Therefore $h \in C(X)$.

As shown above, for any positive integer m, we have $h \leqq g_m$ on Φ_m. Thus $\bar{h} \leqq \bar{g}_m$ for all $m = 1, 2, \cdots$. Now let n be a fixed positive integer, and let x be any point of X. If $\phi(x) \neq 0$, then $1/(k+1) \leqq \phi(x) \leqq 1/k$ for some positive integer k, so $f_n(x) \leqq g_{k+1}(x) \leqq h(x)$. If $\phi(x) = 0$, then $h(x) = f_n(x) = 0$. Thus, for each fixed $n, f_n \leqq h$ on X; so $\bar{f}_n \leqq \bar{h}$ for all $n = 1, 2, \cdots$. From (1), we obtain $\bar{f}_n < \bar{h} < \bar{g}_n$ for all $n = 1, 2, \cdots$.

Case II. For each n, $\bar{g}_{n+1} = \overline{g'_{n+1}}$. Since $\hat{g}_{n+1}(p) < \hat{g}_n(p)$, we have $g_{n+1} \leqq \hat{g}_n(p)$ on some zero-set in $\mathfrak{Z}(N^p)$. Hence $\bar{g}_{n+1} \leqq \overline{\hat{g}_n(p)}$; so $\overline{g'_{n+1}} \leqq \overline{\hat{g}_n(p)}$. Thus, g'_{n+1} can be replaced by $\inf \{g'_{n+1}, \hat{g}_n(p)\}$ (see remarks after (3)), so we may assume that $g'_{n+1} \leqq \hat{g}_n(p)$ on X, for all n. We define $\phi \in C(X)$ by $\phi(x) = \sum_{n=1}^{\infty} 2^{-n} f_n(x)$. Clearly $Z(\phi) = \bigcap_{n=1}^{\infty} Z(f_n)$. For any $x \in Z(\phi)$, one can easily see, by examining the definition of $f''_{n+1}(x)$ and $g''_{n+1}(x)$, that $f''_n(x) = 0$ and $g''_n(x) = g'_n(x)$, so that $g''_{n+1}(x) \leqq \hat{g}_n(p)$, for all n. Thus, we assume this condition is satisfied by the functions henceforth designated by $g_1, g_2, \cdots$. Since $\hat{f}_n(p) = 0$ for all n, it follows that $\hat{\phi}(p) = 0$; so $\Phi_n = \{x \in X : \phi(x) \leqq 1/n\}$ is a zero-set in $\mathfrak{Z}(N^p)$. The definition of h is formally the same as in Case I. We point out, however, that the function ϕ is now different from the function so designated in Case I, so that the function h is also different, although it still satisfies $g_{n+1}(x) \leqq h(x) \leqq g_n(x)$ whenever $1/(n+1) \leqq \phi(x) \leqq 1/n$. Let m be a fixed positive integer. For any $x \in \Phi_m$, either $\phi(x) = 0$, so that $h(x) = 0 \leqq g_m(x)$, or there is a positive integer $k \geqq m$ such that $1/(k+1) \leqq \phi(x) \leqq 1/k$, whence $h(x) \leqq g_k(x) \leqq g_m(x)$. Thus, $h \leqq g_m$ on Φ_m. It is easily verified that h is continuous at each $x \notin Z(\phi)$. Now let $x \in Z(\phi)$. Let $\varepsilon > 0$ be given, and let k be such that $\hat{g}_k(p) < \varepsilon/2$. Then, since $g_{k+1}(x) \leqq \hat{g}_k(p)$, there is a neighborhood W_1 of x such that $g_{k+1}(y) < 2\hat{g}_k(p) < \varepsilon$ for all $y \in W_1$. And, since $x \in Z(\phi)$, there is a neighborhood W_2 of x contained in Φ_{k+1}. Thus, $0 \leqq h(y) \leqq g_{k+1}(y) < \varepsilon$ for all $y \in W_1 \cap W_2$. Therefore $h \in C(X)$.

It has just been shown that for any positive integer m, we have $h \leqq g_m$ on Φ_m. Thus $\bar{h} \leqq \bar{g}_m$ for all $m = 1, 2, \cdots$. Now let n be a fixed positive integer, and let x be any point of X. If $\phi(x) \neq 0$, then $1/(k+1) \leqq \phi(x) \leqq 1/k$ for some positive integer k, so $f_n(x) \leqq g_{k+1}(x) \leqq h(x)$. If $\phi(x) = 0$, then $h(x) = f_n(x) = 0$. Thus, $f_n \leqq h$ on X; so $\bar{f}_n \leqq \bar{h}$ for all $n = 1, 2, \cdots$. As before, we conclude that $\bar{f}_n < \bar{h} < \bar{g}_n$ for all $n = 1, 2, \cdots$. This completes the proof of the theorem.

Remarks 2.7. An integral domain C/P with nontrivial prime ideals, considered as an ordered set, cannot be continuous, that is, Dedekind-complete.

In fact, if K denotes the positive cone of C/P and Q is any nontrivial prime ideal of C/P, then $K \cap Q$ and $K \sim Q$ define a gap in K (hence in C/P), as will now be shown.

First, if $a \in K \cap Q$, then $a^{1/2} \in K \cap Q$; and $a < a^{1/2}$, since $a^{1/2} < 1$. Hence $K \cap Q$ has no last element. Now consider any $b \in K \sim Q$ such that $b < 1$. (In searching for a first element of $K \sim Q$, one would soon arrive here.) Then $b^2 \notin Q$, and $b < 1$ implies $b^2 < b$. Hence $K \sim Q$ has no first element.

It will be seen in 2.19 that C/P contains nontrivial prime ideals whenever P is not a maximal ideal of C.

The observation that the square root of any positive element of Q is in Q can also be used to show that Q must be *infinitely generated.* For, suppose that $Q = (a_1, \cdots, a_n)$. We may assume that $0 < a_1 < a_2 < \cdots < a_n$. Now $a_n^{1/2} \in Q$, so for some $b_1, \cdots, b_n \in C/P$, we have $a_n^{1/2} = \sum_{i=1}^n b_i a_i$. Hence $a_n^{1/2} \leqq \sum_{i=1}^n | b_i | a_i \leqq (\sum_{i=1}^n | b_i |) a_n$, so $1 \leqq (\sum_{i=1}^n | b_i |) a_n^{1/2} \in Q$, contradicting the convexity of Q (2.1).

Finally, we note that if $a, b \in C/P$ satisfy $0 \leqq a \leqq b^2$, then a is a multiple of b. The proof is similar to the proof that any prime ideal of C is convex: If $f, g \in C$ map into a, b, respectively, we may assume that they satisfy $0 \leqq f \leqq g^2$ on X, by Lemma 2.3. Then the function h defined by $h(x) = 0$ for $x \in Z(g)$, $h(x) = f(x)/g(x)$ for $x \notin Z(g)$, is continuous, since $f(x)/g(x) = g(x)(f(x)/(g(x))^2)$ for $x \notin Z(g)$.

Before proceeding, we state some relevant facts about any commutative ring A whose prime ideals form a chain. These remarks are familiar in the theory of general valuation rings, assumed to be integral domains (cf. [11, pp. 43–44]).

Let $\{P_\gamma\}$ be the set of all prime ideals, including the zero ideal, of the ring A. The set $\{P_\gamma\}$ is totally ordered by $\supseteqq$. The usual terms for totally ordered sets, such as predecessor, successor and immediate predecessor, will be used for $(\{P_\gamma\}, \supseteqq)$. A prime ideal P is called a *limit prime ideal* if it has predecessors, but no immediate predecessor.

An ideal I of the ring A is a limit prime ideal if and only if it is the intersection of the prime ideals containing it properly. For, let P_1 be the intersection of all the prime ideals containing I properly. Then P_1 is a prime ideal, since the prime ideals of A form a chain (see, e. g., [4, Theorem 3.9]). Clearly $P_1 = I$ if I is a limit prime ideal, and P_1 contains I properly otherwise. In case I is a nonlimit prime ideal, P_1 is its immediate predecessor.

It follows that the ring A has at least one limit prime ideal if there exists an infinite subsequence $P_1, P_2, \cdots, P_i, \cdots$, in $\{P_\gamma\}$ such that P_{i+1} is a successor of P_i, $i = 1, 2, \cdots$.

We now prove that nonlimit prime ideals always exist. The concept of the McCoy radical of an ideal (see [10, §21]) will be utilized.

LEMMA 2.8. *Let a be any nonzero element of the ring A. Then the McCoy radical S of (a) is the smallest prime ideal containing (a), and has an immediate* *successor P. The ideal P is characterized as the unique prime ideal which is maximal with respect to disjointness from the multiplicative system*

$$\{a, a^2, \cdots, a^n, \cdots\}.$$

Proof. S coincides with the intersection of all the prime ideals containing (a) (see [10, Theorem 24]). Hence it is a prime ideal; and it is evidently the smallest prime ideal containing (a).

It is well known that there is a prime ideal P which is maximal with respect to disjointness from the multiplicative system $\{a, a^2, \cdots, a^n, \cdots\}$ (see [10, Lemma 2, p. 105]). Since the prime ideals of A form a chain, P is unique; thus P is the *largest* ideal (under set inclusion) disjoint from this system. Obviously P is a successor of S. Now any predecessor Q of P, being a prime ideal, must contain a power of a, and hence a itself. Thus $Q \supseteqq (a)$, so $Q \supseteqq S$, that is, S is a successor of Q.

It follows that if P_1 and P_2 are any distinct prime ideals of the ring A with $P_1 \supseteqq P_2$, then there exist prime ideals P_1', P_2' such that $P_1 \supseteqq P_1' \supseteqq P_2' \supseteqq P_2$, and where P_2' is the immediate successor of P_1'. For, we may choose any $a \in P_1 \sim P_2$, and let P_1' be the McCoy radical of (a).

We shall now see what special properties are possessed by the prime ideals in an integral domain C/P. Henceforth, P will always be assumed to be a nonmaximal prime ideal of C containing N^p, so that C/P is a totally ordered integral domain whose unique maximal ideal M^p/P is nonzero. In this case, the prime ideals of C/P are totally ordered under $\supseteqq$.

Let f be any nonnegative function in C, and let n be a positive integer. The function g defined by $g(x) = (f(x))^{1/n}$ is in C; it will be denoted by $f^{1/n}$. Given a positive element $a \in C/P$, we define $a^{1/n} \in C/P$ to be the image of $f^{1/n}$, where $f \in C$ is any nonnegative preimage of a. This definition does not depend upon the choice of f: If $h \geqq 0$ is the preimage of $b \in C/P$, and $a^{1/n} \neq b^{1/n}$, then we have, say, $a^{1/n} > b^{1/n} > 0$, whence $a = (a^{1/n})^n > (b^{1/n})^n = b$; that is, contrapositively, $f \equiv h \pmod{P}$ implies $f^{1/n} \equiv h^{1/n} \pmod{P}$.

DEFINITION 2.9. Let P be a nonmaximal prime ideal of C. For each positive $a \in M^p/P$, we call

$$Q^a = \{b \in C/P : |b| \leqq a^{1/n} \text{ for some positive integer } n\}$$

the *upper ideal* associated with a, and

$$Q_a = \{b \in C/P : |b| < a^n \text{ for all positive integers } n\}$$

the *lower ideal* associated with a. It is routine to verify that the sets Q^a and Q_a are actually ideals. Evidently $Q_a \subseteqq Q^a$, and $a \in Q^a \sim Q_a$.

THEOREM 2.10. *Let P be a nonmaximal prime ideal of C.*

(1) *For any positive $a \in M^p/P$, the upper ideal Q^a is a prime ideal. In fact, Q^a is the McCoy radical of the ideal (a).*

(2) *For any positive $a \in M^p/P$, the lower ideal Q_a is a prime ideal. In fact,*

Q_a is the unique ideal that is maximal with respect to disjointness from the multiplicative system $\{a, a^2, \cdots, a^n, \cdots\}$.

(3) *A prime ideal I of C/P has an immediate predecessor J if and only if, for some positive $a \in M^p/P$, we have $I = Q_a$ and $J = Q^a$.*

(4) *For any $b \in Q^a \sim Q_a$, $b > 0$, we have $Q^a = Q^b$ and $Q_a = Q_b$.*

(5) *Let Q be the prime ideal of C such that $Q/P = Q_a$. Then the subset of positive elements of the totally ordered integral domain C/Q has a countable coinitial subset, namely, the image of $\{a, a^2, \cdots, a^k, \cdots\}$ under the natural homomorphism $\phi: C/P \to C/Q$.*

Conversely, if Q is a prime ideal of C containing P such that the positive cone of the totally ordered integral domain C/Q has a countable coinitial subset, consisting of powers of an element α, then for any $a \in \nu^{-1}(\alpha)$, where ν is the natural homomorphism of C/P onto C/Q, we have $Q/P = Q_a$.

(6) *Every nonmaximal prime ideal Q of C/P is an intersection of lower ideals; and every nonzero prime ideal is a union of upper ideals.*

(7) *Let $a, b \in M^p/P$ be positive elements such that $Q^a \subseteqq Q_b$. Then there exists $c \in C/P$ such that $Q^a \subseteqq Q_c \subseteqq Q^c \subseteqq Q_b$, and all the inclusions are proper (in particular, $Q^a \neq Q_b$). Consequently, the set of lower (respectively upper) ideals is a dense totally ordered set (i.e., if Q_d is properly contained in Q_e, then there exists Q_c such that $Q_d \subseteqq Q_c \subseteqq Q_e$, with the inclusions proper); and the set of lower ideals is disjoint from the set of upper ideals.*

Proof. (1) Suppose $b \in Q^a$; then $|b| \leqq a^{1/n}$ for some positive integer n, so $b^2 \leqq a^{2/n}$. Hence $b^2 = ca^{1/n}$ for suitable $c \in C/P$ (see 2.7). Thus $b^{2n} = c^n a \in (a)$, so b is in the radical of (a). Conversely, let b be in the radical of (a), that is, $b^n = ca$ for some positive integer n and some $c \in C/P$ (we may assume that $c > 0$). Since $b^n \in (a)$, and a belongs to the convex ideal M^p/P, the same is true of bc; so we have $bc \leqq 1$. Thus $b^{n+1} = (bc)a \leqq a$, so $b \leqq a^{1/(n+1)}$. Hence $b \in Q^a$.

(2) Let Q be the ideal of C/P that is maximal with respect to disjointness from $\{a, a^2, \cdots, a^k, \cdots\}$. It is clear from the definition of Q_a that $Q_a \subseteqq Q$. Now if $b \in Q$, then, since Q is convex, $|b| < a^k$ for every positive integer k; that is, $b \in Q_a$. Hence $Q_a = Q$.

(3) The sufficiency follows directly from (1), (2), and Lemma 2.8. For the necessity, let $a \in J \sim I$. Then, since J is the smallest prime ideal containing (a), it is the McCoy radical of (a), that is, by (1), it is Q^a. And since I is the largest prime ideal disjoint from $\{a, a^2, \cdots, a^k, \cdots\}$, it is Q_a, by (2).

(4) This follows like the necessity of (3).

(5) By the remarks following 2.4, ϕ preserves order, so it suffices to show that the $\phi(a^k)$ are distinct. Now let n, k be any positive integers with $n > k$. Since $1 - a^{n-k}$ is not infinitely small, we have $1 - a^{n-k} \notin Q_a$. Thus, from the fact that $a^k \notin Q_a$ it follows that $a^k - a^n = a^k(1 - a^{n-k}) \notin Q_a$, that is, $a^k \not\equiv a^n \pmod{Q_a}$.

To prove the converse, let $a \in \nu^{-1}(\alpha)$; then $a^n \in \nu^{-1}(\alpha^n)$ $(n = 2, 3, \cdots)$. It is evident that Q/P is disjoint from the multiplicative system

$\{a, a^2, \cdots, a^k, \cdots\}$. To prove maximality, let $b \notin Q/P$, $b > 0$, be given, and consider the ideal $\nu((Q/P, b))$ in C/Q. For some positive integer n, we have $\alpha^n < \nu(b)$. Then $\alpha^{2n} < (\nu(b))^2$, so α^{2n} is a multiple of $\nu(b)$ (2.7). Hence $\alpha^{2n} \in \nu((Q/P, b))$, and $a^{2n} \in (Q/P, b)$.

(6) For each $a \notin Q$ with $a > 0$ and a nonunit, the multiplicative system $\{a, a^2, \cdots, a^k, \cdots\}$ is disjoint from Q. By (2), $Q \subseteqq Q_a$. Since $a \notin Q_a$, it follows that $Q = \bigcap_{a \notin Q} Q_a$. The proof of the second statement is the dual of the proof of the first.

(7) Since $\{a, a^{1/2}, \cdots, a^{1/k}, \cdots\}$ and $\{b, b^2, \cdots, b^k, \cdots\}$ are two countably infinite subsets of C/P of order type ω, ω^* respectively, with the first preceding the second, there is an element c between them, by Theorem 2.6. Now, if for some positive integers n, k, the relation $c^k \leqq a^{1/n}$ held, we should have $c \leqq a^{1/nk}$, which is impossible. Therefore $c^k > a^{1/n}$ for all n and k; similarly, $c^k < b^n$ for all n and k. It is then clear that $Q^a \subseteqq Q_c \subseteqq Q^c \subseteqq Q_b$. Reapplying the argument to the pairs Q^a, Q_c and Q^c, Q_b, we conclude that all of the inclusions are proper.

Now if $Q_d \subset Q_e$, we have, by (3), $Q^d \subseteqq Q_e$. Hence there exists c such that $Q_d \subseteqq Q^d \subseteqq Q_c \subseteqq Q^c \subseteqq Q_e$, and all of the inclusions are proper. In particular, Q_c lies strictly between Q_d and Q_e. The statement about upper ideals follows similarly, and the final statement is obvious from the first conclusion.

COROLLARY 2.11. *No upper ideal has an immediate predecessor, and no lower ideal has an immediate successor. In particular, the set of prime ideals of C/P is neither well-ordered nor inversely well-ordered.*

Proof. This follows from (7) and (3).

It is natural to enquire whether *every* prime ideal of C/P is either an upper ideal or a lower ideal. In view of the requirement that the element a in terms of which Q^a and Q_a are defined be a positive nonunit, it is evident that (0) cannot be an upper ideal and M^p/P cannot be a lower ideal, by default. We note however that (0) is *like* an upper ideal, and M^p/P is like a lower ideal when $p \in \upsilon X$: consider the sequences $0^{1/n}$ and $1/n$ ($n = 1, 2, \cdots$). In the next theorem we show that any ideal of C/P that comes from a $\mathfrak{Z}$-ideal of C is neither an upper ideal nor a lower ideal. Thus, the nontrivial possibility that M^p/P be an upper ideal is ruled out, and the nontrivial possibility tath (0) be a lower ideal is ruled out when P is a $\mathfrak{Z}$-ideal of C. As will be seen in §4, sometimes (0) and M^p/P are the only ideals that come from $\mathfrak{Z}$-ideals of C. But the answer to our question is still "no" in this case.

THEOREM 2.12. *If I is a $\mathfrak{Z}$-ideal of C containing P, then I/P is neither an upper nor a lower ideal.*

Proof. We show first that I/P is not an upper ideal. In view of the remarks preceding the theorem, we may assume that $I \neq P$. Let a be any positive element of I/P, and let $f \in C$ be a preimage of a such that $0 \leqq f \leqq \frac{1}{2}$

on X. Then $f \in I \sim P$. Define g as follows: $g(x) = 0$ for $x \in Z(f)$, $g(x) = -1/\log f(x)$ for $x \notin Z(f)$. Then $g \geqq 0$, and $Z(g) = Z(f)$. It is easily shown that $g \in C(X)$. Since $f \in I$, and I is a $\mathfrak{Z}$-ideal, we have $g \in I$. Let b denote the image of g in C/P.

Now if n is any fixed positive integer, then $\lim_{t \to 0^+} (-1/\log t)/t^{1/n} = +\infty$. Hence there exists $s \in R$ such that $0 < t \leqq s$ implies $-1/\log t \geqq t^{1/n}$. Let $Z = \{x \in X : f(x) \leqq s\}$. Note that $Z \in \mathfrak{Z}(N^p)$. Since $-1/\log f \geqq f^{1/n}$ on $Z \cap (X \sim Z(f))$, we have $g \geqq f^{1/n}$ on Z. It follows that $g - f^{1/n} - | g - f^{1/n} | \in N^p \subseteqq P$. Therefore $b \geqq a^{1/n}$. Since this is true for each positive integer n, we have $b \notin Q^a$. But $b \in I/P$; so $I/P \neq Q^a$.

We now show that I/P is not a lower ideal. This time we may assume that $I \neq M^p$. Let a be a positive element of $M^p/P \sim I/P$, and let f be a preimage of a such that $0 \leqq f \leqq 1$ on X. Define g as follows: $g(x) = 0$ for $x \in Z(f)$, $g(x) = e^{-1/f(x)}$ for $x \notin Z(f)$. Then $g \geqq 0$, and $Z(f) = Z(g)$. It is easily shown that $g \in C(X)$. Since $f \notin I$, and I is a $\mathfrak{Z}$-ideal, we have $g \notin I$. Let b denote the image of g in C/P. Using the fact that $\lim_{t \to 0^+} e^{-1/t}/t^n = 0$ for all positive integers n, we obtain $b \in Q_a$ by reasoning similar to that in the first part of the proof. Since $b \notin I/P$, we have $I/P \neq Q_a$.

COROLLARY 2.13. *Let $I \neq M^p$ be a $\mathfrak{Z}$-ideal of C containing P. Then there are infinitely many lower ideals and infinitely many upper ideals containing I/P.*

Proof. By Theorem 2.12, I/P is not a lower ideal. Now Theorem 2.20 below implies that I/P is a prime ideal; so by Theorem 2.10, (6), I/P is an intersection of lower ideals. There cannot be only a finite number of ideals in this intersection, for then the intersection would coincide with the last one of them. The statement about upper ideals now follows from the fact that every lower ideal is contained in the upper ideal associated with it.

In fact, no limit prime ideal is a lower ideal, for, by Theorem 2.10, (3), a lower ideal has an immediate predecessor.

On the other hand, the intersection of an ω-sequence of lower (upper) ideals need not be the image of a $\mathfrak{Z}$-ideal. For example, in the space E, the only $\mathfrak{Z}$-ideals containing N^e are N^e and M^e (see remarks following 4.1). Now it is easy to find two distinct lower ideals in $C(E)/N^e$, e. g., the lower ideals associated with the images of the functions in $C(E)$ determined by the sequences $\{1/n\}$ and $\{1/2^n\}$. By applying Theorem 2.10, (7), one can construct an ω-sequence of lower ideals lying between these two ideals. The intersection is clearly different from (0).

The next corollary is similar to Corollary 2.11.

COROLLARY 2.14. *If I is a $\mathfrak{Z}$-ideal of C containing P, then I/P has no immediate successor or predecessor.*

Proof. This follows from Theorems 2.12 and 2.10, (3), except for the fact that M^p/P has no immediate predecessor and (0) has no immediate successor. But this is trivial, since M^p/P is the first element and (0) is the last element of the set of prime ideals.

Since a prime ideal Q of C/P is an interval symmetric about zero, and $b < a^2$ implies that b is a multiple of a (2.7), it is clear that any cofinal subset of Q generates Q. Thus $\{a, a^{1/2}, \cdots, a^{1/n}, \cdots\}$ is a countable set of generators for Q^a. Moreover, combining this remark with Theorem 2.10, (4), we see that for any positive $b \in Q^a \sim Q_a$, the set $\{b, b^{1/2}, \cdots, b^{1/n}, \cdots\}$ is a countable set of generators for Q^a. This is contrasted with the situation for lower ideals in the following theorem.

THEOREM 2.15. *Let I be an ideal of C containing P such that I/P is a lower ideal Q_a. Then neither I nor Q_a is a countably generated ideal.*

Proof. Suppose $I = (f_1, f_2, \cdots)$. Then $Q_a = (b_1, b_2, \cdots)$, where b_i is the image of f_i in C/P. Since Q_a is a prime ideal, it must be infinitely generated (see 2.7); so it is no restriction to assume that $b_{i+1}^2 > b_i > 0$ for all i. By Theorem 2.6, there exists $c \in C/P$ such that $b_i < c < a^i$ for all i. Evidently, $c \in Q_a$. For some positive integer n and $d_i \in C/P$, we have $c = \sum_{i=1}^n d_i b_i$. Hence $c \leqq \sum_{i=1}^n |d_i| b_i \leqq (\sum_{i=1}^n |d_i|) b_n$. But $b_n < c$; so $1 \leqq \sum_{i=1}^n |d_i|$, which implies that $\sum_{i=1}^n |d_i|$ is a unit u. So we have $u^{-1}c \leqq b_n$. Now the hypothesis on the b_i implies that for any positive unit v of C/P, the inequality $b_n < vb_{n+1}$ holds. For, the contrary implies that $b_n^2 \geqq v^2 b_{n+1}^2 > v^2 b_n$, and hence $b_n > v^2$, so that b_n is a unit, which is absurd.

It follows that $b_n < u^{-1}b_{n+1} < u^{-1}c$, a contradiction. Thus, Q_a cannot be a countably generated ideal, and I cannot be either.

When $p \in vX$, the same proof, with a replaced by $\frac{1}{2}$, shows that neither M^p nor M^p/P is a countably generated ideal. Thus, when p is not a P-point with respect to X, M^p is not countably generated. It turns out that the conclusion is also valid when p is a P-point with respect to X, except when p is isolated. In fact, we are able to prove this without assuming that $p \in vX$.

The result is obtained as a corollary to the following theorem about the ideal N^p.

THEOREM 2.16. *If N^p is a countably generated ideal, then $\{p\}$ is a G_δ-set of βX.*

Proof. In this proof, the interior of a set will be denoted by "int".

Let $N^p = (f_1, f_2, \cdots, f_n, \cdots)$. We may assume that the f_i are bounded, since, for each f_i, there exists an $f_i^* \in C^*(X)$ that belongs to the same ideals of $C(X)$ (see [6, Lemma 1.5]). For any $x \in \beta X$ distinct from p, there exists $g \in C^*(X)$ such that $g \in N^p$, $g^\beta(x) = 1$. Now for some positive integer n and functions $h_i \in C(X)$, we have $g = \sum_{i=1}^n h_i f_i$. Suppose $x \in \bigcap_{i=1}^n \operatorname{int}(Z(f_i^\beta))$. Let V denote $\{y \in \beta X : g^\beta(y) > \frac{1}{2}\}$, which is a neighborhood of x in βX. Since X is dense in βX, the set $\bigcap_{i=1}^n \operatorname{int}(Z(f_i^\beta)) \cap V \cap X$, being the intersection with X of a nonempty open set of βX, is nonempty. But this contradicts the hypothesis $g = \sum_{i=1}^{n} h_i f_i$. Hence $x \notin \bigcap_{i=1}^{n} \text{int}(Z(f_i^\beta)) \supseteq \bigcap_{i=1}^{\infty} \text{int}(Z(f_i^\beta))$. It follows that $\{p\}$ is a G_δ-set of βX.

COROLLARY 2.17. *If $p \in \beta X \sim X$, then N^p is not a countably generated ideal.*

Proof. By 2.16, if the conclusion did not hold, then $\{p\}$ would be a closed G_δ-set of βX, contained in $\beta X \sim X$. But this is impossible, since the cardinal number of every nonempty subset of $\beta X \sim X$ that is a closed G_δ-set of βX is at least $2^{\mathfrak{c}}$ (see [8, Theorem 49]).

COROLLARY 2.18. *If p is a P-point with respect to X, and M^p is a countably generated ideal, then $p \in X$, and M^p is a principal ideal.*

Proof. Since $M^p = N^p$, 2.17 implies that $p \in X$. It follows from Theorem 2.16 that $\{p\}$ is a G_δ-set of X; so by [5, Corollary 4.3], p is isolated. As in [5, Theorem 5.9], M^p is a principal ideal; specifically, M^p is generated by the characteristic function of $X \sim \{p\}$.

In [6, Theorem 6.3(a)], it is shown that when X is a P-space, every finitely generated ideal of $C(X)$ is a principal ideal. It follows from 2.18 that every countably generated *maximal* ideal of $C(X)$ is a principal ideal when X is a P-space. It is not true that arbitrary countably generated ideals must be principal ideals, however. For instance, if X is the countable discrete space (hence a P-space), then the set of functions of $C(X)$ that vanish outside finite sets is a countably generated ideal of $C(X)$ which is not a principal ideal.

We now utilize the concept of upper ideal to conclude that if there are at least two prime ideals of C/P, then there are uncountably many.

THEOREM 2.19. *Let P be a nonmaximal prime ideal of C. Then there is an ω_1^*-sequence of upper ideals in C/P.*

Proof. We show first that C/P contains a nonzero lower ideal. Since P is not maximal, the zero ideal of C/P is an intersection of lower ideals, by 2.10, (6). It follows that there exists at least one nonzero upper ideal of C/P, say Q^b. Now, by 2.12, Q^b is not maximal, so it is also an intersection of lower ideals. Hence there exists a nonzero lower ideal $Q_a \supseteq Q^b$.

By 2.10, (6), Q_a is a union of upper ideals. Since each upper ideal is countably generated, if this collection had a countable coinitial subset, then Q_a would be countably generated, contradicting Theorem 2.15. It follows that the collection of upper ideals contains an ω_1^*-sequence.

We conclude this section with a theorem on $\mathfrak{Z}$-ideals connecting several of the ideas discussed earlier. Among the results it contains is the rather remarkable one that if C/I (I a $\mathfrak{Z}$-ideal) is a totally ordered ring, then it is necessarily an integral domain. The hypothesis that I contains some N^p is not actually a restriction; it is implied by each of the statements in the theorem (since this is the case for the first statement, and the proof of the equivalence of the statements makes no use of the hypothesis).

THEOREM 2.20. *Let I be a $\mathfrak{Z}$-ideal of C containing some N^p. The following statements are equivalent*:

(1) *I is a prime ideal.*
(2) *The prime ideals of C containing I form a chain.*
(3) *Every $\mathfrak{Z}$-ideal of C containing I is a prime ideal.*
(4) *C/I is a totally ordered integral domain.*
(5) *C/I is a totally ordered ring.*
(6) *Zero-sets are comparable on zero-sets of I.*
(7) *For $Z_1, Z_2 \in \mathcal{Z}(C)$, if $Z_1 \cup Z_2 \in \mathcal{Z}(I)$, then $Z_1 \in \mathcal{Z}(I)$ or $Z_2 \in \mathcal{Z}(I)$.*

Proof. (1) *implies* (2). This is a special case of 2.4.

(2) *implies* (3). Every $\mathfrak{Z}$-ideal of C is an intersection of prime ideals containing it [9, Lemma 2.2]. Hence, a $\mathfrak{Z}$-ideal containing I is an intersection of a chain of prime ideals; therefore it is a prime ideal (see, e. g., [4, Theorem 3.9]).

(3) *implies* (4). Since I is a $\mathfrak{Z}$-ideal, this is a special case of 2.1.

(4) *implies* (5), trivially.

(5) *implies* (6). Let $f, g \in C$, and let a, b denote the images of f, g respectively, in C/I. Since $Z(f) = Z(|f|)$ and $Z(g) = Z(|g|)$, we may suppose that $a, b \geqq 0$. If $a \geqq b$, then for some $Z \in \mathcal{Z}(I)$, we have $f \geqq g \geqq 0$ on Z, so $(Z(f) \cap Z) \subseteqq (Z(g) \cap Z)$; and similarly if $a \leqq b$.

(6) *implies* (7). Let $Z_1, Z_2 \in \mathcal{Z}(C)$ satisfy $Z_1 \cup Z_2 \in \mathcal{Z}(I)$. We may assume that $(Z_1 \cap Z) \supseteqq (Z_2 \cap Z)$ for some $Z \in \mathcal{Z}(I)$. Then

$$Z_1 \supseteqq Z_1 \cap Z = (Z_1 \cup Z_2) \cap Z \in \mathcal{Z}(I),$$

whence $Z_1 \in \mathcal{Z}(I)$.

(7) *implies* (1). Let $f, g \in C$ satisfy $fg \in I$. Since $Z(f) \cup Z(g) = Z(fg) \in \mathcal{Z}(I)$, either $Z(f) \in \mathcal{Z}(I)$ or $Z(g) \in \mathcal{Z}(I)$. Therefore either $f \in I$ or $g \in I$.

3. Totally ordered valuation rings

Let A be a commutative ring with identity. The ring A is said to be a *valuation ring* if for any pair of elements $a, b \in A$, either a is a multiple of b or b is a multiple of a.[4]

Trivially, every field is a valuation ring. If a valuation ring A is an integral domain, it can be embedded in its field of quotients F. The set of nonzero elements F^* of F is a multiplicative group, and the set U of units of A is a subgroup. The quotient group F^*/U is called the *value group* for the valuation ring A, and will be denoted by Γ. Obviously it is abelian; it is generally written as an additive group. The group Γ can be given a partial order as follows: $\gamma \geqq \delta$ if a representative of γ is a multiple by an element of A of a representative of δ. Antisymmetry ((1) (ii) of §1) follows from the fact that

[4] Proofs and more complete statements of many of the results summarized below may be found in [11, pp. 5–17 and p. 44].

F has no zero-divisors; and the hypothesis that A is a valuation ring implies directly that the order is *total*. It is evident that this order is invariant under translation, so that Γ is a totally ordered group.

From a slightly different viewpoint, we have defined a homomorphism V from the multiplicative group F^* onto the additive group Γ, with kernel U. Since each nonzero element of A is a multiple by an element of A of any element of U, and each element of U is a multiple by an element of A of any element of $F \sim A$, the set $V(A \sim \{0\})$ is precisely the subsemigroup Γ^+ of nonnegative elements of Γ. But the zero element of A is a multiple by an element of A of every element of A; so it is natural to extend V to zero by stipulating that $V(0) = \infty$, where ∞ denotes an element exceeding all of Γ^+ (and hence all of Γ), and satisfying the formal laws $\gamma + \infty = \infty + \gamma = \infty + \infty = \infty$ for $\gamma \in \Gamma$. We shall henceforth let V designate the restriction of this mapping to A, and let Γ^+ denote the extended semigroup.

Since V is a homomorphism, it satisfies $V(ab) = V(a) + V(b)$ for all $a, b \in A$. (The above convention on ∞ is set up so that this will be true even if a or b is 0.) It also satisfies the inequality $V(a + b) \geqq \min\{V(a), V(b)\}$ for all $a, b \in A$. For, assume (without loss of generality) that there exists $c \in A$ such that $a = cb$. Then $V(a) \geqq V(b)$; and $a + b = (c + 1)b$ implies that $V(a + b) \geqq V(b)$. The function V is called the valuation associated with the valuation ring A.

Let G be a totally ordered group. Recall that the convex subgroups of G form a chain (§1). The order type of the set of all proper convex subgroups (including $\{0\}$), ordered by set inclusion, is called the *rank* of G. If H is a convex subgroup of G, then the induced order on the quotient group G/H makes it a totally ordered group. If r, r_1 are the ranks of G and H, respectively, then the rank r_2 of G/H satisfies $r = r_1 + r_2$. A group has rank one if and only if it is archimedean. Each group of rank one is isomorphic, under an order-preserving isomorphism, to a subgroup of the additive group of all real numbers.

Now let Γ be the value group for a valuation ring A with valuation V. A subset of positive elements Ω of Γ is called an upper class of Γ if $\alpha \in \Omega$ and $\beta > \alpha$ imply $\beta \in \Omega$. The ideals of A are in one-to-one correspondence with the upper classes of Γ. In fact, if I is an ideal of A, then $V(I)$ is an upper class of Γ; and if Ω is an upper class of Γ, then $V^{-1}(\Omega)$ is an ideal of A. The principal ideals correspond to the upper classes possessing a least element. Furthermore, the *prime* ideals of A are in one-to-one correspondence with the convex subgroups of Γ. Specifically, if Q is a prime ideal of A, then the set $V(A \sim Q) \cup (-V(A \sim Q))$ is a convex subgroup of Γ; and every convex subgroup of Γ has this form for some prime ideal of A.

We remark that a valuation ring need not be totally ordered, as in the case of the field of complex numbers; and a totally ordered ring (in fact, integral domain) need not be a valuation ring, as in the case of the domain of integers. However, we shall be concerned exclusively with valuation rings which are also totally ordered rings, with the valuation and order related in a simple way. Since A and Γ^+ are now both ordered, it is reasonable to consider whether V can be a monotone function from the positive cone of A to Γ^+. It turns out that in the cases we shall discuss, V is a monotone decreasing function on this set. Due to the symmetry of V ($V(-a) = V(-1) + V(a) = V(a)$ for all $a \in A$), it will usually be convenient to think of the domain of V as being only the positive cone of A. In terms of elements of A, this monotonicity condition may be stated as follows:

(1) If $a, b \in A$ satisfy $0 \leqq a \leqq b$, then a is a multiple of b.

It is easily seen that (1) is equivalent to

(2) Every ideal of A is convex.

If A is a valuation ring which is also a totally ordered ring satisfying (1) (and hence (2)), we shall call A a *totally ordered valuation ring.*

Let A be an *arbitrary* commutative ring. Given any prime ideal P of A, we can form a ring A_P containing A as a subring, and such that every element in $A \sim P$ is a unit (see [10, Theorem 23]). When A is a totally ordered ring, the ring A_P can be made a totally ordered ring, whose order is an extension of the order on A.[5] Indeed, if $[a, s] \in A_P$ is the equivalence class with representative (a, s), we define $[a, s]$ to be positive if and only if $as > 0$.

In case A is a totally ordered valuation ring, every positive element $[a, s] \in A_P \sim A$ exceeds A. For, $[a, s] \notin A$ implies that $a \notin (s)$, and since (s) is convex, we must have $a > bs$ for every $b \in A$, if a and s are taken to be positive. But then $[a, s] > [bs, s]$, that is, $[a, s]$ exceeds b. Since A is a valuation ring, if $[a, s] \notin A$, then $[s, a] \in A$, so $[a, s]$ is a unit of A_P. Thus, A_P is also a valuation ring. Combining this with the previous statement, we have that A_P is a totally ordered valuation ring. The same is true of any homomorphic image of A_P.

Now suppose that P_1, P_2 are prime ideals of a valuation ring B such that P_2 is the immediate successor of P_1. Let Δ_i be the convex subgroup of Γ associated with P_i ($i = 1, 2$). Then the valuation ring B_{P_1}/P_2 has for its value group the quotient group Δ_2/Δ_1.[6] Furthermore, since there is no convex subgroup of Γ containing Δ_1 and contained in Δ_2, the rank of Δ_2/Δ_1 must be one, so Δ_2/Δ_1 is archimedean, and hence a subgroup of the additive group of all real numbers.

From now on, Γ denotes the value group of C/P, when P is prime and C/P is a totally ordered valuation ring. In order to avoid trivial special cases, we assume that C/P is not a field (i. e., that P is not maximal). It was pointed out in 2.7 that the only proper finitely generated ideal of C/P that is prime is (0). It might be conjectured that every nonprincipal ideal (hence every infinitely generated ideal) of C/P is prime. We show easily that this is false, by consideration of Γ.

[5] For the case $P = (0)$, cf. [1, p. 33, Proposition 2].

[6] This follows from [11, Chapter I, Lemma 12 and Theorem 5].

THEOREM 3.1. *Suppose that C/P is a totally ordered valuation ring, but not a field.*

(1) Γ^+ *is a dense totally ordered set, that is, if* $\gamma, \delta \in \Gamma^+$, $\gamma < \delta$, *then there exists* $\alpha \in \Gamma^+$ *such that* $\gamma < \alpha < \delta$.

(2) *Let Q be a nonmaximal prime ideal of C/P distinct from* (0). *Then* $\Gamma^+ \sim V(Q)$ *and* $V(Q)$ *define a gap in* Γ^+.

(3) *Let* $\gamma \in \Gamma$, $\gamma > 0$, *be given. Then* $I = V^{-1}(\{\delta \in \Gamma^+ : \delta > \gamma\})$ *is an infinitely generated ideal of C/P which is not prime.*

Proof. (1) Let $\gamma, \delta \in \Gamma^+$, with $0 \leqq \gamma < \delta$, be given; let $a, b \in C/P$ be any preimages of γ, δ respectively. We may assume that $0 \leqq b < a$ in C/P, and hence that there exists $h \in C/P$, $h \geqq 0$, such that $b = ha$, where $V(h) \neq 0$. If $h = 0$, then $b = 0$, and $\delta = \infty$; and we have $\gamma = V(a) < 2V(a) = V(a^2) < \infty$. If $h \neq 0$, we consider $ah^{1/2} \in C/P$. The inequality

$$V(b) = V(h) + V(a) > \tfrac{1}{2} V(h) + V(a) = V(ah^{1/2}) > V(a)$$

says that $\gamma < V(ah^{1/2}) < \delta$.

(2) As already remarked, Q is not a principal ideal; so the upper class $V(Q)$ has no first element. Now $\Gamma^+ \sim V(Q)$ is precisely the intersection of Γ^+ with the convex subgroup associated with Q. Since Q is not maximal, there is an element $\gamma > 0$ in $\Gamma^+ \sim V(Q)$. Hence $2\gamma \in \Gamma^+ \sim V(Q)$, and $2\gamma > \gamma$; so $\Gamma^+ \sim V(Q)$ has no last element.

(3) Since $V(I)$ is an upper class of Γ, I is an ideal of C/P. By (1), $V(I)$ has no first element, so I is not a principal ideal; hence it is infinitely generated. And if I were a prime ideal, then $\{\delta \in \Gamma : |\delta| \leqq \gamma\}$ would be a subgroup of Γ containing γ but not 2γ, which is absurd.

We now consider the elements which belong to an upper ideal Q^a but not to the associated lower ideal Q_a, when P is a $\mathfrak{Z}$-ideal. In this case, we may think of Q^a and Q_a as defined in terms of elements "a^r", where r is an arbitrary positive real number. For any such r, and any nonnegative $f \in C$, the function g, defined by $g(x) = (f(x))^r$, is in C; it is reasonable to denote g by f^r. Given a positive element $a \in C/P$, we define a^r to be the image of f^r, where $f \in C$ is any nonnegative preimage of a. Now let h be any other nonnegative function mapping into a. Then f and h coincide on a zero-set in $\mathcal{Z}(P)$, so f^r and h^r coincide on the same zero-set. Hence $f^r \equiv h^r \pmod{P}$, that is, the image of h^r is also a^r. Thus the expression "a^r" is well-defined. It is easy to see that

$$Q^a = \{b \in C/P : |b| \leqq a^r \text{ for some real } r > 0\},$$

and

$$Q_a = \{b \in C/P : |b| < a^r \text{ for all real } r > 0\}.$$

Now it turns out that every element in $Q^a \sim Q_a$ is "close" to a definite "real power" of the element a, in the sense that the element which appears as the multiplier will not be an element of small absolute value, but will be outside the upper ideal Q^a.

THEOREM 3.2. *Suppose that C/P is a totally ordered valuation ring, but not a field, and that P is a $\mathfrak{Z}$-ideal. Let a be a fixed positive nonunit element of C/P. Then for each $b \in Q^a \sim Q_a$, there exist a unique positive real number r and an element $h \in C/P \sim Q^a$, such that either $b = ha^r$ or $a^r = hb$. The element h appearing in either equation is unique.*

Proof. We assume for simplicity that $b > 0$. Define

$$t = \sup\{s \in R : s > 0, \quad b \leqq a^s\},$$

$$u = \inf\{s \in R : s > 0, \quad b \geqq a^s\}.$$

Then, since $b \in Q^a \sim Q_a$, we have $0 < u$ and $t < \infty$; and since C/P is totally ordered, it follows immediately that $u \leqq t$.

Suppose that $u < t$. Choose $s_1, s_2 \in R$ such that $u < s_1 < s_2 < t$. Then $s_2 < t$ implies $b \leqq a^{s_2}$; and $u < s_1$ implies $b \geqq a^{s_1}$. But $a^{s_2} < a^{s_1}$ (since $0 < a < 1$), so this is a contradiction. Hence $u = t$. We define r to be this number.

We shall show that $b = ha^r$ or $a^r = hb$, for some $h \in C/P \sim Q^a$. There are three cases:

(1) Suppose $b = a^r$. Then $b = 1 \cdot a^r$ (and $a^r = 1 \cdot b$).

(2) Suppose $b < a^r$. Then $b = ha^r$ for some $h \in C/P$. If $h \in Q^a$, we have $h \leqq a^s$ for some $s > 0$. Then $b = ha^r \leqq a^{s+r} = a^{s+t}$, contradicting the definition of t. Thus $h \notin Q^a$.

(3) Suppose $b > a^r$. Then $a^r = hb$ for some $h \in C/P$. Again, if $h \in Q^a$, we have $h \leqq a^s$ for some $s > 0$; since $b < 1$, we have $a^r < h$, so $s < r$. This implies that $h = ka^s$, where $k \in C/P$; evidently, $0 < k < 1$. Then $a^r = ka^s b$; and since $a \neq 0$, we have $a^{u-s} = a^{r-s} = kb \leqq b$, contradicting the definition of u. Thus $h \notin Q^a$.

The uniqueness of r is clear from the definition; and the uniqueness of h then follows from the fact that C/P is an integral domain.

We now apply 3.2 to obtain some information about archimedean quotient groups of subgroups of Γ. Although the result is not surprising, it is helpful in describing the value group Γ.

THEOREM 3.3. *Suppose that C/P is a totally ordered valuation ring, but not a field, and that P is a $\mathfrak{Z}$-ideal. Let a be any fixed positive nonunit element of C/P, and let Δ_1, Δ_2 be the convex subgroups of Γ associated with the prime ideals Q^a, Q_a respectively. Then the quotient group Δ_2/Δ_1 is isomorphic to the additive group of all real numbers.*

Proof. Set $C/P = A$. We define a function $V' : A_{Q^a}/Q_a \to R$ as follows: Given a positive $b' \in Q^a/Q_a$, let $b \in Q^a$ be a positive preimage of b'. By 3.2, there exists a unique positive real number r such that $b = ha^r$ or $a^r = hb$, with $h \in C/P \sim Q^a$; we set $V'(b') = r$.

We shall now show that this definition is independent of the representative chosen. Let c be any other (positive) representative of b', and suppose $c = ka^s$ or $a^s = kc$, with $s \neq r$ and $k \notin Q^a$. There are four possible cases:

$$(1) \quad b = ha^r, \quad c = ka^s, \qquad (2) \quad b = ha^r, \quad a^s = kc,$$

$$(3) \quad a^r = hb, \quad c = ka^s, \qquad (4) \quad a^r = hb, \quad a^s = kc.$$

In case (1), we may suppose that $r > s$. Since $ha^{r-s} \in Q^a$ and $k \notin Q^a$, we have $ha^{r-s} - k \notin Q^a$; so $ha^{r-s} - k > a$. Then $b - c = a^s(ha^{r-s} - k) > a^{1+s}$, contradicting $b - c \in Q_a$.

In case (2), we write $k(b - c) = a^s(kha^{r-s} - 1)$ if $r > s$, and $k(b - c) = a^r(kh - a^{s-r})$ if $r < s$. Then, in a manner similar to case (1), we obtain $k(b - c) > a^{1+s}$ if $r > s$, $k(b - c) > a^{1+r}$ if $r < s$, contradicting $k(b - c) \in Q_a$.

Case (3) may be treated just like case (2).

In case (4), we may suppose that $r > s$. Then $h(b - c) = a^s(a^{r-s}) - hc = c(ka^{r-s} - h)$. Since $ka^{r-s} \in Q^a$ and $h \notin Q^a$, we have $ka^{r-s} - h \notin Q^a$, whence $ka^{r-s} - h \notin Q_a$. But $c \notin Q_a$, so $h(b - c) \notin Q_a$, a contradiction.

We now extend V' to all nonzero elements of A_{Qa}/Q_a by setting $V'(b') = 0$ if $b' \notin Q^a/Q_a$, and $V'(c') = V'(-c')$ if $c' < 0$. The range of V' is evidently the nonnegative reals.

Now Theorem 3.2 shows that every element of the prime ideal Q^a/Q_a of the ring A_{Qa}/Q_a is the image of an element which is a multiple of some a^r by a unit of A_{Qa}. Moreover, if $r \neq s$, then a^r is not a unit multiple of a^s. Thus, it is clear that V' is actually the valuation associated with the valuation ring A_{Qa}/Q_a. But Δ_2/Δ_1 is the value group for the totally ordered valuation ring A_{Qa}/Q_a. Hence Δ_2/Δ_1 must be the additive group of all real numbers.

We now examine the situation in which C/I is a totally ordered valuation ring, where I is a $\mathfrak{Z}$-ideal of C containing some N^p (not assumed at the outset to be prime). From Theorem 2.20 we found that an order condition on C/I implies an algebraic condition—that it be an integral domain. Here we have an implication in the other direction. The algebraic condition that C/I be a valuation ring implies that it is totally ordered.

THEOREM 3.4. *Let I be a $\mathfrak{Z}$-ideal of C containing some N^p. The following statements are equivalent:*

(1) *C/I is a totally ordered valuation ring.*
(2) *The ideals of C containing I form a chain.*
(3) *C/I is a valuation ring.*
(4) *Every finitely generated ideal of C/I is a principal ideal.*

Proof. (1) *implies* (2). Since every ideal of C/I is convex, (2) follows from the remarks in §1.

(2) *implies* (3). It follows immediately from (2) that the principal ideals of C/I form a chain, whence C/I is a valuation ring.

(3) *implies* (4). In any valuation ring, every finitely generated ideal is

principal: the element with smallest image in the value group is a generator of the whole ideal (see [11, Corollary 2, p. 10]).

(4) *implies* (1). We shall show first that C/I is a totally ordered ring. Let $f \in C$ be arbitrary. Denote the images of f, $|f|$ in C/I by q_1, q_2 respectively. By hypothesis, there exists $r \in C/I$ such that $(q_1, q_2) = (r)$. Thus, we may write $f = gd$, $|f| = hd$, $d = sf + t|f|$ on Z, where $d \in C$ is a preimage of r, where $g, h, s, t \in C$, and where Z is a suitable zero-set in $\mathfrak{Z}(I)$. We now follow the proof of [6, Theorem 2.3, (e) implies (c)], except that certain equations are understood to be valid only on some zero-set in $\mathfrak{Z}(I)$. In this way, we find $k \in C$ and $Z' \in \mathfrak{Z}(I)$ such that $k = 1$ wherever f is positive in Z' and $k = -1$ wherever f is negative in Z'. Hence, there exists $Z'' \in \mathfrak{Z}(I)$ such that $Z'' \subseteqq Z'$, and either $f \geqq 0$ or $f \leqq 0$ on Z''.

This shows that C/I is a totally ordered ring, with $a \geqq 0$ if and only if for some preimage $f \in C$, we have $f \equiv |f| \pmod{I}$. Hence, by 2.20, C/I is a totally ordered integral domain. We must now show that if $a, b \in C/I$ satisfy $0 < a < b$, then a is a multiple of b. By hypothesis, there exists $c \in C/I$ such that $c > 0$ and $(a, b) = (c)$. So, for suitable $d, e, s, t \in C/I$, we have $a = dc$, $b = ec$, $c = sa + tb$. On substituting, $c = (sd + te)c$; and since $c \neq 0$, we obtain $sd + te = 1$, which implies $(d, e) = (1)$. Since the set of nonunits of C/I forms the unique maximal ideal, it follows that either d or e is a unit. But $e > d > 0$, and the maximal ideal is convex, so that e is a unit in any case. Hence $a = de^{-1}b$.

We remark that the proof of the equivalence of statements (1) and (4) is actually very explicit about generators. If (1) holds, and $0 < a < b$, then $(a, b) = (b)$. If (4) holds, whence, as shown, C/I is totally ordered, and $0 < a < b$, then the c whose existence is guaranteed by (4) may be chosen to be b: the proof shows that $(c) = (b)$. However, we prefer to state (4) without any mention of order.

4. βF-points

We now consider some special prime ideals. In [9], the concept of βF-point was introduced. A point $p \in \beta X$ is called a βF-point (with respect to X), if for each $f \in C(X)$ such that $\hat{f}(p) = 0$, there is an X-neighborhood of p on which one of the relations $f \geqq 0, f \leqq 0$ holds. A point p is a βF-point if and only if the ideal N^p is prime [9, Theorem 5.2]. Thus, since N^p is a $\mathfrak{Z}$-ideal of C, the statement that p is a βF-point is equivalent to each of the seven statements appearing in 2.20, with $I = N^p$. These results subsume 5.3, 5.6(a), 5.8, 5.9, and 5.11 of [9]. We might also note that the hypothesis in 5.13 of [9] can be weakened to "p is a βF-point".

Let p be a βF-point. Since $N^{*p} = N^p \cap C^*$, the final result of Theorem 2.5 may be specialized to the statement that C/N^p is isomorphic to C^*/N^{*p} if and only if $p \in \upsilon X$ (with the order preserved when the isomorphism exists). Thus all of the results obtained for C/N^p apply immediately to C^*/N^{*p} when $p \in \upsilon X$. However, in case $p \notin \upsilon X$, we may view $C^*(X)$ as $C(\beta X)$, and $C^*(X)/N^{*p}$

as the corresponding quotient ring of $C(\beta X)$. Since p is also a βF-point of βX (as follows from the remarks following [9, Definition 5.1]), this case is covered too. But it must be kept in mind that the rings $C^*(X)/N^{*p}$ and $C(X)/N^p$ are different.

We recall that every point of βX is a βF-point if and only if X is an F-space, that is, if and only if every finitely generated ideal of $C(X)$ is a principal ideal. Thus, one might conjecture that p is a βF-point if and only if every proper finitely generated ideal of C containing N^p is a principal ideal. In general, however, the second statement holds vacuously, that is, no proper ideal of C containing N^p can be finitely generated.

Indeed, we shall now show that there exists a proper finitely generated ideal of C containing N^p if and only if $\{p\}$ is a G_δ-set in X. Suppose that $\{p\}$ is not a G_δ-set in X, and that $I = (f_1, \cdots, f_n) \subseteqq M^p$. If $p \in X$, we have $Z(f_1^2 + \cdots + f_n^2) \neq \{p\}$; so there exists $q \in X$ distinct from p such that $q \in \bigcap_{i=1}^n Z(f_i)$. If $p \notin X$, then the same conclusion follows immediately from the hypothesis that $f_1, \cdots, f_n \in M^p$. By complete regularity, there is a $g \in N^p$ such that $g(q) = 1$. Evidently, $g \notin I$. Hence I does not contain N^p. Conversely, if $\{p\}$ is a G_δ-set in X, then it is a zero-set. Let $f \in C$ satisfy $Z(f) = \{p\}$. If $g \in N^p$, then $Z(g)$ contains a neighborhood U of $Z(f)$, so $g \in (f)$. (Define $h \in C$ by $h(x) = g(x)/f(x)$ for $x \notin U$, $h(x) = 0$ for $x \in \bar{U}$.) Thus $N^p \subseteqq (f)$.

Since a nonisolated βF-point in X cannot have a countable base of neighborhoods (the proof of this statement is the same as that of [6, Corollary 2.4]), the class of βF-points p such that $\{p\}$ is a G_δ-set but not open is rather restricted. It is not empty, however; the point e in the space E (§1) is such a point. Now there is a connection between βF-points and finitely generated ideals—of the ring C/N^p. The observation was made in the proof of Theorem 3.4 that statement (2) of 3.4 implies statement (2) of 2.20. It follows that p is a βF-point if every finitely generated ideal of C/N^p is a principal ideal.

It would be pleasant to be able to prove that the hypothesis that p be a βF-point is also sufficient for every finitely generated ideal of C/N^p to be a principal ideal—equivalently, for C/N^p to be a totally ordered valuation ring—thus obtaining the equivalence of all the statements (for N^p) appearing in Theorems 2.20 and 3.4. The validity of this implication remains in doubt. We have, however, obtained some partial results on this question. It has been shown that if p is a $\beta P'$-point or if X is an F-space (i. e., if *every* ideal N^q of $C(X)$ is prime), then C/N^p is a valuation ring [9, Theorem 5.7]. The remarks below show that the proof actually yields the result that C/N^p is a totally ordered valuation ring in these cases; alternatively, we could appeal to Theorem 3.4. (Whenever X is locally compact and σ-compact, the space $\beta X \sim X$ is an F-space [6, Theorem 2.7]. This provides examples of totally ordered valuation rings of the form C/N^p.) We now present two sufficient conditions, one of which is simply a natural weakening of the requirement that X be an F-space.

Suppose that C/N^p is a totally ordered ring. It is easily seen that every nonzero element of C/N^p that is neither an infinitely small element nor its negative must be a unit. Thus, if at least one of a pair of elements is not infinitely small, then one of them is a multiple of the other. And since the maximal ideal is convex, the monotonicity condition is satisfied. Clearly, then, to show that C/N^p is a totally ordered valuation ring, it suffices to consider the case of an arbitrary pair of distinct infinitely small elements.

THEOREM 4.1. *Let $p \in \beta X$ be a βF-point. Either of the following conditions suffices to ensure that C/N^p be a totally ordered valuation ring:*

(1) *There exists an X-neighborhood U of p that is an F-space, that is, every finitely generated ideal of $C(U)$ is a principal ideal.*

(2) *For any $f, g \in M^p \sim N^p$, the sets $Z(f)$ and $Z(g)$ coincide in some X-neighborhood of p.*

Proof. (1) Given a pair of distinct infinitely small positive elements of C/N^p, let f, g be functions in C which map into these elements. It is easily seen that we may assume that $0 \leqq f \leqq g \leqq 1$. Let V be an X-neighborhood of p with $\bar{V} \subseteqq U$. Then we can manage to make $Z(f)$ and $Z(g)$ subsets of V. Note that $Z(f) \supseteqq Z(g)$. We define $h' \in C^*(X \sim Z(g))$ by $h'(x) = f(x)/g(x)$. Then the restriction of h' to $U \sim Z(g)$ has a continuous extension k to all of U, by [6, Theorem 2.6]. Define a function h by $h(x) = h'(x)$ for $x \notin \bar{V}$, $h(x) = k(x)$ for $x \in U$. Since $Z(g) \subseteqq \bar{V}$, h is well-defined. And since $h \mid U$, $h \mid (X \sim \bar{V})$ are continuous, $h \in C(X)$. Obviously, $f = hg$, so $f \equiv hg \pmod{N^p}$. As remarked above, it follows from this that C/N^p is a totally ordered valuation ring.

(2) With the same assumptions and notation as in (1), it follows as in the proof of [9, Theorem 5.7] that the function $h \in C^*(X \sim Z(g))$ defined by $h(x) = f(x)/g(x)$ can be extended in a continuous manner to the point p. For simplicity we assume that the value of the extended function at p is zero. We extend h to all of X by setting $h(x) = 0$ on $Z(g)$.

Now the concepts of limit superior, limit inferior, and oscillation of a function at a point can be extended to functions on general topological spaces: Let f be a real-valued function on X, $p \in X$, and $\mathfrak{U}$ a base of neighborhoods of p. Then $\limsup_{x \to p} f(x)$ may be defined as $\inf_{U \in \mathfrak{U}} (\sup_{x \in U \sim \{p\}} f(x))$, $\liminf_{x \to p} f(x)$ as $\sup_{U \in \mathfrak{U}}(\inf_{x \in U \sim \{p\}} f(x))$, and the oscillation of f at p as $\limsup_{x \to p} f(x) - \liminf_{x \to p} f(x)$.

Since the extension of h is continuous *at* p and has the value zero at this point, there is a sequence U_n of X-neighborhoods of p such that $0 \leqq h(y) \leqq 1/n$ for all $y \in U_n$. In particular, the oscillation of h at each point of U_n is no greater than $1/n$. Now a sequence k_n can be chosen such that $k_n \in C(X)$, $0 \leqq k_n \leqq 1$, $\hat{k}_n(p) = 0$, and $Z(k_n) \subseteqq U_n$. Then $\bigcap_{n=1}^{\infty} U_n \supseteqq \bigcap_{n=1}^{\infty} Z(k_n) = Z(s)$ for some $s \in C(X)$ such that $\hat{s}(p) = 0$. (Specifically, we may set $s(x) = \sum_{n=1}^{\infty} 2^{-n} k_n(x)$.) By hypothesis, there exists an X-neighborhood U of p such that $(Z(s) \cap U) = (Z(g) \cap U)$. Then $((\bigcap_{n=1}^{\infty} U_n) \cap U) \supseteqq (Z(g) \cap U)$. At each point of $Z(g) \cap U$ the oscillation of h is zero. And for each point of $U \sim Z(g)$, the oscillation of h is also zero. Hence h is continuous on U.

By complete regularity, there is a function $k \in C^*(X)$ such that $k = 1$ on $X \sim U$, and $\hat{k}(p) = -1$. Then $k' = \max\{k, 0\} = 1$ on $X \sim U$ and $k' = 0$ on some X-neighborhood of p. Hence $f + k' \equiv f \pmod{N^p}$ and $g + k' \equiv g \pmod{N^p}$. Moreover, $Z(f + k') \subseteq Z(f) \cap U$ and $Z(g + k') \subseteq Z(g) \cap U$. Set $h'(x) = (f(x) + k'(x))/(g(x) + k'(x))$ for $x \notin Z(g + k')$, $h'(x) = 0$ for $x \in Z(g + k')$. Then the fact that h is continuous on U and h' is continuous on $X \sim Z(g + k')$ shows that $h' \in C(X)$. Evidently, $f \equiv f + k' \equiv h'(g + k') \equiv h'g \pmod{N^p}$. Thus, $C(X)/N^p$ is a totally ordered valuation ring.

It is difficult to obtain interesting examples of a βF-point (not a P-point with respect to X) which does not satisfy condition (1) of Theorem 4.1. It is always possible to obtain a somewhat artificial example by taking a P-point, every neighborhood of which contains a non-βF-point, and "gluing" it to a βF-point which is not a P-point—precisely, by identifying the two points, and taking neighborhoods of the new point as unions of neighborhoods of the original points. A specific case of this type of example is the space $X = W(\omega_1 + 1) \cup E$, with the points ω_1 and e identified (denote this point by p). Now it is evident that the presence of such non-βF-points can cause no harm. In fact, $C(X)/N^p$ is isomorphic to $C(E)/N^e$. A corresponding statement can be made in the general case.

A more interesting example is provided by the space $W(\omega_1 + 1) \times E$. The point (ω_1, e) is a βF-point, every neighborhood of which contains a non-βF-point. Here it is possible to construct continuous functions which, in every neighborhood of (ω_1, e), are zero at some non-βF-points and distinct from zero at others. However, a totally ordered valuation ring is still obtained, since condition (2) of 4.1 holds. Alternatively, a simple argument shows that the quotient ring in question is isomorphic to $C(E)/N^e$.

The author has been unable to construct any example which is *essentially* different from the two just described. It is possible that one of the two sufficient conditions of the theorem must always hold.

Condition (2) of Theorem 4.1 is equivalent to the more elegant statement:

(3) *The only $\mathfrak{Z}$-ideals containing N^p are N^p and M^p.*

For, suppose (2) holds, and I is a $\mathfrak{Z}$-ideal containing N^p properly. Let $f \in M^p \sim N^p$, and let $g \in I \sim N^p$. Then there is an X-neighborhood U of p such that $Z(f^2) \cap U = Z(g^2) \cap U$. If h is any function in N^p with $Z(h) \subseteq U$, we have $g^2 + h^2 \in I$, and hence $f \in I$. Conversely, suppose (2) does not hold, and $f, g \in M^p \sim N^p$ are such that $Z(f) \cap U \neq Z(g) \cap U$ for every X-neighborhood U of p. Then [9, Lemma 5.12] yields distinct $\mathfrak{Z}$-ideals containing N^p properly.

If p is a $\beta P'$-point (e. g., the point e in the space E), then condition (2) (and hence (3)) is evidently fulfilled. An example of a point satisfying (2) which is not a $\beta P'$-point is the point p in the space $X = W(\omega_1 + 1) \cup E$ discussed above.

We are now in a position to give a large class of examples of nonprime ideals of C which yield totally ordered quotient rings; the existence of such ideals was indicated in §2. By the discussion in §1, in order to produce an example, it suffices to find a nonprime convex ideal which contains a prime ideal. Let C/N^p be a totally ordered valuation ring but not a field; for example, let p be a point of an F-space which is not a P-point. Since every ideal of C/N^p is convex, the preimage of any nonprime ideal of C/N^p will be a nonprime convex ideal of C containing N^p. The set of ideals of C/N^p abounds with nonprime ideals. For instance, by 2.7, no nontrivial finitely generated ideal in this class is prime, which, by 3.4, is equivalent to the statement that no nontrivial principal ideal in this class is prime.

We conclude with some results related to Theorem 2.6.

Example 4.2. We now give an example of a βF-point p such that the positive cone of $C(X)/N^p$ has a countable coinitial subset. In particular, then, $C(X)/N^p$ is not an η_1-set.

As in [9, Example 5.5], we attach to each isolated point of the space E another copy of the space E. We now attach to each of the isolated points in the adjoined spaces still another copy of the space E, and repeat this process a countably infinite number of times. The spaces added at the $(n - 1)^{st}$ stage will be denoted generically by E_n.

The β-point of E_1 (the original copy of E) will be denoted by e. A neighborhood of e is the union of a neighborhood of e in E_1 with neighborhoods of the β-points in the spaces E_2 which lie in this neighborhood, and so on, through all the spaces E_n, for all n. The basic neighborhoods of any other point e' are defined in a similar manner (starting with the unique space E_n of which e' is the β-point). The resulting space will be designated by $E^{(\omega)}$. We remark that for any positive integer n, the union of all spaces E_j, $j \geqq n$, attached to one of the spaces E_n, is open in $E^{(\omega)}$. Since the complement of any basic neighborhood is a union of sets of this form, the basic neighborhoods are both open and closed. This shows that $E^{(\omega)}$ is zero-dimensional. (Every countable metric space is zero-dimensional, as is easily seen; but $E^{(\omega)}$ is not metrizable, since it is not first countable.)

We shall show first that e is a βF-point. Let f be any function in $C(E^{(\omega)})$ such that $f(e) = 0$.

If there is a neighborhood of e on which $f = 0$, we are finished. Otherwise, let n be the smallest integer such that on the intersection of every neighborhood of e with the spaces E_n, f is not identically zero. To present the general argument, we assume that $n > 2$. It is easy to see how to simplify the proof for application to the cases $n = 1$ and $n = 2$. In each of the spaces E_n, there exists a deleted neighborhood of its β-point on which $f > 0$, $f < 0$, or $f = 0$. Now a fixed space E_{n-1}, together with the spaces E_n which have been attached to it, form a space $E^{(2)}$ which is homeomorphic to the space described in [9, Example 5.5]. As in that example, we may select a neighborhood in the whole space $E^{(2)}$ of the β-point of the space E_{n-1}, on which $f \geqq 0$ or $f \leqq 0$.

Suppose this has been done for every space E_{n-1}. We then consider each fixed space E_{n-2}. Together with the spaces E_{n-1} which have been attached to it, it forms another copy of $E^{(2)}$. Repetition of the above argument yields a neighborhood in each such space $E^{(2)}$ of the β-point of the corresponding space E_{n-1} on which $f \geqq 0$ or $f \leqq 0$. Taking the union of all neighborhoods chosen thus far, we obtain a set on which $f \geqq 0$ or $f \leqq 0$. This set is contained in the union of the spaces E_n, E_{n-1}, and E_{n-2}.

Continuing in this manner, we obtain, in a finite number of steps, a set U containing e on which $f \geqq 0$ or $f \leqq 0$. This set is contained in the union of the spaces E_i, $i \leqq n$, and can be extended to a neighborhood of e in $E^{(\omega)}$ by adding appropriate sets A in the spaces E_i, $i > n$. Now, by hypothesis, we cannot have $f = 0$ on the intersection of U with the spaces E_n. And from the way in which U was constructed, it is clear that f has no zeros on the intersection of U with any one of these spaces, except perhaps at the β-point. Hence the choice of subsets A in the spaces E_i, $i > n$, used to extend U can be made (in an evident manner) so that f is actually different from zero on each of them. Thus, there is a neighborhood of e on which $f \geqq 0$ or $f \leqq 0$.

We now define a monotone decreasing sequence of nonnegative functions $\{g_j\}$ in $C(E^{(\omega)})$ such that $g_j(e) = 0$. Let $h \in C(E)$ be the function defined by $h(e_n) = 1/n$, $h(e) = 0$. Then set $g_1 = 0$ on the space E_1 and $g_1 = h$ on each of the spaces E_2; and on each space E_i, $i > 2$, put $g_1 = h + r$, where r is the constant function whose value coincides with the value already assigned to g_1 at the β-point of the space E_i.

In order to define the remaining g_j's we first note that, for any n, the union of all the spaces E_i, $i \geqq n$, attached to a single space E_n, is a space homeomorphic to $E^{(\omega)}$, in a natural way. We now define g_j to be identically zero on all the spaces E_i, $i \leqq j$, and like g_1 on each of the spaces obtained by forming the union of all spaces E_i, $i \geqq j$ that are attached to a single space E_j.

Then the set $\{\bar{g}_1, \bar{g}_2, \cdots\}$ is a countable coinitial subset of the positive cone of $C(E^{(\omega)})/N^e$. For, suppose $\bar{0} \leqq \bar{k} \leqq \bar{g}_j$, $j = 1, 2, \cdots$. Since $0 \leqq k \leqq g_j$ on some neighborhood of the β-point in each of the spaces E_j, and $g_j = 0$ on this space, we have that $k = 0$ on the neighborhood. The union of these neighborhoods is a neighborhood of e in $E^{(\omega)}$ on which $k = 0$. Hence $\bar{k} = \bar{0}$.

This space also provides an example for the case $p \notin vX$. Let $Y = E^{(\omega)} \sim \{e\}$. We show below that $E^{(\omega)}$ is homeomorphic to a subset of βY, under a mapping τ keeping Y pointwise fixed, so that e corresponds to a point z in $\beta Y \sim Y$. Since Y is a countable space, $vY = Y$; so $z \in \beta Y \sim vY$. The arguments about e may then be carried over, with Y in place of $E^{(\omega)}$, and Y-neighborhoods of z in place of neighborhoods of e.

Let τ coincide on Y with the identity mapping. To complete the definition of τ, it remains only to specify the image of e. Evidently, every bounded continuous function on the subspace $E_1 \sim \{e\}$ of Y can be extended to a bounded continuous function on all of Y. Thus, the closure of $E_1 \sim \{e\}$ in βY is homeomorphic to $\beta(E_1 \sim \{e\})$.[7] But $E_1 \sim \{e\}$ is homeomorphic to the countable discrete space N; so βY contains a subset homeomorphic to βN. Recall that E consists of N together with one point of $\beta N \sim N$. Define $\tau(e)$ to be the corresponding point z in $\beta Y \sim Y$ contained in the copy of βN found above. To show that τ is the desired homeomorphism, we must only show that the basic neighborhoods of z and e map into open sets.

Let F_n denote the union of all the spaces E_i, $i \geqq 2$, attached to the n^{th} space E_2 (i. e., the space E_2 that was attached to the n^{th} point in E_1). Then $Y = \bigcup_{n=1}^{\infty} F_n$, with each F_n open in Y. If U is a neighborhood of z in βY, then $U \cap F_n$ is open in F_n, and $U \cap (E_1 \sim \{e\})$ corresponds to a deleted neighborhood of e in E; thus $\tau^{-1}(U \cap (Y \cup \{z\}))$ is a neighborhood of e in $E^{(\omega)}$.

On the other hand, if V is a basic neighborhood of e in $E^{(\omega)}$, then $\tau(V \sim \{e\})$ is a union of basic neighborhoods of points of E_1 in the subspaces F_n. It was pointed out earlier that every basic neighborhood in $E^{(\omega)}$ is closed. Since each F_n is homeomorphic to $E^{(\omega)}$, the set $Y \sim \tau(V \sim \{e\})$ is a union of open sets; so $\tau(V \sim \{e\})$ is closed in Y. Hence the characteristic function χ of $\tau(V \sim \{e\})$ is continuous; and its extension to βY is 1 at z. Thus $\chi^{-1}(1)$ is a neighborhood of z in βY whose intersection with Y is exactly $\tau(V \sim \{e\})$.

Let $p \in \beta X \sim \upsilon X$. It was mentioned before Theorem 2.6 that the hyperreal field C/M^p, which by definition is the same as C/M'^p, is an η_1-set. (See [9, §5] for the definition of the ideal M'^p.) Evidently, every proper open subinterval of C/M'^p with a greatest lower bound and a least upper bound is also an η_1-set. We now consider the case where p is a βF-point in υX. Of course, the whole set C/M'^p will not be an η_1-set; but we find when the statement about subintervals is valid.

THEOREM 4.3. *Let $p \in \upsilon X$ be a βF-point. Then every proper open subinterval of C/M'^p with a greatest lower bound and a least upper bound is an η_1-set if and only if $M'^p \neq M^p$.*

Proof. The necessity of the condition is clear, since C/M^p is the real field. We turn to the sufficiency. From the definition of M'^p, it follows that $p \in X$.

First recall that M'^p is a prime ideal, so that C/M'^p is a totally ordered integral domain. Let ϕ be the natural order-preserving homomorphism of C/N^p onto C/M'^p. Given two subsets A, A' of C/M'^p of type ω, ω^* respectively, with $A < A'$, we can select representatives $D = \phi^{-1}(A)$, $D' = \phi^{-1}(A')$

[7] For, the closure of $E_1 \sim \{e\}$ in βY is a compact Hausdorff space in which $E_1 \sim \{e\}$ is dense; and, by the preceding sentence, every bounded continuous function on $E_1 \sim \{e\}$ can be extended to βY, and hence to this subspace. These are the characteristic properties of $\beta(E_1 \sim \{e\})$.

to obtain subsets of C/N^p having the same properties. Choosing $\delta \in C/N^p$ as in 2.6, we have $a \leqq \phi(\delta) \leqq a'$ for all $a \in A$, $a' \in A'$, whence $A < \phi(\delta) < A'$.

We now consider the case in which at least one of A, A' is finite. By reductions similar to those used in the proof of Theorem 2.6, it is easy to see that we need only show that the positive cone of C/M'^p has no countable coinitial subset. For the rest of this proof, the image of $f \in C$ in C/M'^p is denoted by $\bar{f}'$. Let $\{\bar{g}'_n\}$ be a decreasing sequence of positive elements. If no g_n vanishes at p, we take $\bar{h}'$ to be any infinitely small element, that is, the image of any $h \in M^p \sim M'^p$, with $h \geqq 0$. If some $g_n(p) = 0$ (so that we may assume all $g_n(p) = 0$), we follow Case I of the proof of 2.6. We must now show, independently, that $\bar{h}' > \bar{0}'$. We can adjust the g_n's, first, so that $Z(g_n) = \{p\}$ for all n (cf. the proof of [9, Theorem 5.7]), and second, so that they satisfy condition (2) in the proof of 2.6. Now for every $y \neq p$, we have $\phi(y) = g_1(y) \neq 0$. Hence, for some positive integer k, $1/(k+1) \leqq \phi(y) \leqq 1/k$; so $0 < g_{k+1}(y) \leqq h(y)$. Hence $Z(h) = \{p\}$. In particular, $\bar{h}' \neq \bar{0}'$.

As remarked after [9, §5.4], if $p \in X$ is a $\beta P'$-point, then $M'^p = N^p$. Thus, we obtain immediately:

COROLLARY 4.4. *If $p \in X$ is a $\beta P'$-point which is not a P-point, then every proper open subinterval of C/N^p with a greatest lower bound and a least upper bound is an η_1-set.*

We can show easily that the converse of 4.4 is false. If X is the space $W(\omega_1 + 1) \cup E$ (with e and ω_1 identified) described after 4.1, then, as mentioned there, $C(X)/N^p$ is isomorphic to $C(E)/N^e$. Now e is a $\beta P'$-point in the space E, whence every proper open subinterval of $C(E)/N^e$ with a greatest lower bound and a least upper bound is an η_1-set. But p is not a $\beta P'$-point in the space X.

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Purdue University
Lafayette, Indiana