TOROID TRANSFORMATION GROUPS ON EUCLIDEAN SPACE

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1. Introduction

We deal here with the operation of an r-dimensional toroid T^r on a *cohomology manifold* X^n which resembles euclidean n-space in a sense that is described below. Our main results give a detailed description of the action of T^r on X^n when $n \leqq 2r + 1$ and T^r operates effectively. We prove for example that the fixed point set F is either a point or a line, that all the isotropy subgroups are connected, that T^r has exactly 2^r isotropy subgroups, 2^r fixer subgroups, and r weights (see Section 3 for definitions).

In the case of a general r and n, we prove the inequality $0 \leqq \dim F \leqq \dim X - 2r$, provided that T^r operates almost effectively. Here F is a non-empty cohomology manifold resembling a euclidean space in the sense of our definition. Most of the proofs rely on recursion processes, which are based on the existence of *star circle subgroups*, that is, circle subgroups P of T^r whose fixed point sets are not the fixed point sets of any connected subgroup properly containing P (see Section 3).

In the case of an effective T^r on an HLC euclideanlike X^n with $n \leqq 2r + 1$ (in view of the aforementioned inequality, $n = 2r$ or $2r + 1$), we prove (Section 6) that the space of principal orbits U admits a global cross section and $U = B \times T^r$ where B is a euclideanlike cohomology manifold. The added hypothesis that X is an HLC space has been introduced, in order to allow us to employ Poincaré duality for open cohomology manifolds *over the integers* (see Section 4, Corollary 4.1). The proof of the existence of the global cross section involves a generalization of the fact that any map of euclidean space is homotopic to a constant (see Section 5).

Our results show that for the cases considered the action is closely related to the known linear action. The reader may find it helpful to keep the linear case in mind and interpret the definitions in this light. It can be shown that the action need not be equivalent to a linear action, however. This is not immediate but could be concluded, for example, from making use of some recent results of Bing (to appear in Annals of Mathematics) showing that E^4 is the product of a line and a space which is not a manifold.

2. Generalized manifolds

All of the spaces considered in this paper are locally compact, Hausdorff, and finite-dimensional. The cohomology groups (Alexander-Spanier) with coefficients in a group L are denoted by $H^i(X, L)$ and those with compact

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carriers by $H_c^i(X, L)$. The group $H^*(X, L)$ is the direct sum of the groups $H^i(X, L)$, and $H_c^*(X, L)$ is the direct sum of the groups $H_c^i(X, L)$. A space is called *clc* over L if for any x in X and compact neighborhood B of X, there is a compact neighborhood A of x such that $A \subset B$ and the natural map

$$H^i(B, L) \to H^i(A, L)$$

is trivial for all i. If A is any closed subset of X, there is the natural map $H_c^*(X, L) \to H_c^*(A, L)$. If U is an open subset of X there is also a natural map

$$H_c^*(U, L) \to H_c^*(X, L),$$

which will be denoted by I_{UX}. If $\bar{U}$ is compact, $H_c^*(U, L)$ can be identified with the relative group $H^*(X, X - U, L)$. We denote the group of integers by Z and integers modulo p by Z_p.

DEFINITION. The space X is said to have *local L-cohomology groups* at x if for each sufficiently small open neighborhood U of x, there exists a graded subgroup $H^*(x, U, L)$ of $H_c^*(U, L)$ satisfying

(a) For all open neighborhoods V satisfying $x \in V \subset U$, I_{VU} maps $H^*(x, V, L)$ isomorphically onto $H^*(x, U, L)$;

(b) Given a U for which $H^*(x, U, L)$ exists, there is an open V such that $V \subset U$, $x \in V$, and I_{VU} maps $H_c^*(V)$ into $H^*(x, U, L)$. If $H^*(x, U, L)$ exists, it is clearly isomorphic to $\lim_{\rightarrow} (\lim_{\leftarrow} (I_{VW}H_c^*(V); V \downarrow p); W \downarrow p)$.

The local L-cohomology group at x, $H^*(x, X, L)$, is taken to be any one of the groups $H^*(x, U, L)$. A space X which has local L-cohomology groups at each point is said to have *locally constant local L-cohomology groups* if for any x in X and sufficiently small open neighborhood U of x, there is an open V such that $x \in V$, $V \subset U$, and for any y in V, $H^*(y, U, L) = H^*(x, U, L)$.

DEFINITION. The space X is a cohomology manifold over L if

(a) X is locally compact finite-dimensional;

(b) X has locally constant local L-cohomology groups;

(c) For each $x \in X$, $H^i(x, X, L) \approx L$ for some i and is trivial for all other i.

When we deal with the group $H^*(X, L)$ (closed supports) we shall require that X be paracompact in addition.

A cohomology manifold over L is called a cohomology n-manifold over L if its local n-dimensional groups are isomorphic to L at *all* its points. It is readily seen that a connected cohomology manifold is a cohomology n-manifold for some n.

DEFINITION. A cohomology n-manifold X over L is called *orientable over L* if the sheaf (or the local system) of local n-dimensional cohomology groups $H^n(x, X, L)$ is constant.

Various definitions of manifolds in the sense of homology and cohomology have been introduced, notably by Wilder and Smith. More recently these definitions and their interrelations have been studied by Yang [12], Conner-Floyd [3], and Borel [1].

The definition that we adopt here coincides with Conner-Floyd's when $L = Z$, apart from the requirement of paracompactness, and with Borel's when L is a field. Moreover, it permits us to utilize the results of both as well as Smith's results on fixed point sets of transformations of homology manifolds.

In view of the various meanings attributed to "cohomology manifold" in the literature, we state explicitly below the properties that we use.

2.1. Translating a proof of Wilder for homology manifolds over a field into the language of cohomology over a general coefficient group L, Borel has proved in [1]:

A cohomology manifold over L is clc over L, provided that L is a principal ideal ring.

2.2. Conner-Floyd's definition of a cohomology manifold X is ours plus the requirement that X be clc over Z. In view of 2.1, a cohomology manifold over Z in our sense is also one in the sense of Conner-Floyd. Upon considering the exact cohomology sequences associated with $0 \to Z \to Z \to Z_p \to 0$ one proves just as in [3], that a cohomology n-manifold over Z is a cohomology n-manifold over Z_p for every p.[2]

2.3. Smith has given a definition of a homology manifold over Z_p ([9]). Its formulation in cohomology over an arbitrary principal ideal ring L is seen to coincide with our definition by a result of Yang (see [12] appendix, and [1] Section 1.4). Thus our cohomology manifolds over Z_p are Smith manifolds and we can apply the results of Smith on transformations. The relation of Smith manifolds to Wilder manifolds is described by Borel in [1].

DEFINITION. If X and Y are spaces, we mean by $X \sim_L Y$ that $H_c^k(X, L)$ and $H_c^k(Y, L)$ are isomorphic groups for each k. We say $X =_L Y$ if X and Y are each cohomology n-manifolds over L and in addition $X \sim_L Y$.

2.4. If X is a cohomology n-manifold over L, then n is the cohomological dimension of X over L in the sense of Cohen-Wallace [2] and is *at most* equal to the topological dimension, of course. We can assert the following "invariance of domain" result:

2.4.1. *If Y is a subset of a cohomology n-manifold X over L, and if in its relative topology Y is also a cohomology n-manifold over L, then Y is open in X.*

Proof. Since Y is locally compact, $U \cap Y$ is closed in U for sufficiently small open sets U of X. The exact cohomology sequence of a pair

$$\to H_c^n(U, L) \to H_c^n(U \cap Y, L) \to H_c^{n+1}(U, U \cap Y, L) \to$$

[2] For separable cohomology manifolds, one can say more; cf. Subsection 4.2.

yields that $H_c^n(U, L) \to H_c^n(U \cap Y, L)$ is surjective, since all $(n+1)$-dimensional cohomology over L vanishes (cf. [12]). Suppose now that $y \in Y$ and y is not an interior point. Select an open neighborhood U of y in X such that $U \cap Y$ is closed in U, $H^*(y, U, X)$ exists, and $H^*(y, U \cap Y, Y)$ exists. Since the local L-cohomology groups are locally constant, there exists an open neighborhood V in $U - Y$ with $I_{VU} H_c^n(V, L) = H^n(y, U, L)$, and an open neighborhood W of y in U with $I_{WU} H_c^n(W, L) = H^n(y, U, L)$, and $H_c^n(W \cap Y, L)$ mapping onto $H^n(y, U \cap Y, Y)$. From the commutativity of the diagram

$$\begin{array}{ccccc} H_c^n(V, L) & \to & H_c^n(U, L) & \to & H_c^n(U \cap Y, L) \\ & & \uparrow & & \uparrow \\ & & H_c^n(W, L) & \to & H_c^n(W \cap Y, L) \end{array}$$

it follows at once that the image of $H_c^n(V, L)$ in $H_c^n(U \cap Y, L)$ is isomorphic to L. On the other hand, since $V \cap U \cap Y$ is empty, the image of $H_c^n(V, L)$ in $H_c^n(U \cap Y, L)$ is zero, a contradiction. Hence each point of Y is an interior point, that is, Y is open in X.

We shall require the following result on orientable cohomology manifolds. In the special case that L is a field, a proof has been given by Borel in [1] based on Poincaré duality for cohomology manifolds. See also Yang [12] for a proof in his formulation. We sketch a proof.

2.4.2. *Let X be a connected cohomology n-manifold over L. Then $H_c^n(X, L) = L$ or $= 0$ according as X is orientable over L or not.*

Proof. Let C be the family of all connected open subsets Y with compact closure in X having the following property: for each open set $U \subset Y$ and point $x \in U$ for which $H^*(x, U, L)$ exists, I_{UY} maps $H^n(x, U, L)$ monomorphically into $H_c^n(Y, L)$.

Consider the Mayer-Vietoris exact sequence for open subsets of X with closure (see also the last section of [1])

$$\to H_c^q(U_1 \cap U_2, L) \xrightarrow{\sigma} H_c^q(U_1, L) + H_c^q(U_2, L)$$
$$\xrightarrow{\Delta} H_c^q(U_1 \cup U_2, L) \to H_c^{q+1}(U_1 \cap U_2, L) \to,$$

where $\sigma = (I_{U_1 \cap U_2, U_1}, I_{U_1 \cap U_2, U_2})$ and $\Delta = I_{U_1, U_1 \cup U_2} - I_{U_2, U_1 \cup U_2}$. (This corresponds to the relative Mayer-Vietoris sequence; cf. Eilenberg-Steenrod, *Foundations of Algebraic Topology*, p. 44.) Since X is n-dimensional over L, $0 = H^{n+1}(A, B, L) = H_c^{n+1}(A - B, L)$ for any compact pair by a result of H. Cohen [2]. Hence $H_c^n(U_1 \cup U_2, L)$ is generated by the images of $H_c^n(U_1, L)$ and $H_c^n(U_2, L)$ if $U_1, U_2 \in C$.

Since each set Y in C is a finite union of small subsets, we find that $H_c^n(Y, L)$ is generated by the images $I_{VY} H_c^n(V, L)$ with V small. The hypothesis that X is orientable means that $I_{VY} H_c^n(V, L) = I_{WY} H_c^n(W, L)$ for any two sufficiently small open sets V and W in a connected open set Y. Consequently $H_c^n(Y, L) = L$ for any Y in C.

Upon identifying $H_c^n(Y, L)$ with L, we can describe the image $\sigma H_c^n(U_1 \cap U_2, L)$ as the diagonal of $L + L$, provided $U_1 \cap U_2$ is not empty and $U_1, U_2 \in C$. Consequently, $H_c^n(U_1 \cup U_2, L) = L + L/L = L$ if U_1, $U_2 \in C$; that is, if U_1 and U_2 are in C, then $U_1 \cup U_2 \in C$. Since X is connected, we get readily from the definition of $H_c^*(X, L)$ that $H_c^*(X, L) = \lim_{\rightarrow} \{H^*(Y, L), I_{Y,Y'}\}$ over the directed set C. Consequently $H_c^*(X, L) = L$.

The converse follows at once from the fact that $I_{U,X}$ maps $H^n(x, U)$ onto $H_c^n(X)$ for any sufficiently small open neighborhood of a point $x \in X$.

2.5. We state here for future reference some fundamental theorems of P. A. Smith on the fixed point sets of p-groups (that is, groups whose orders are powers of p). If G is a group of transformations in a space X, we denote by $F(G, X)$ the subset of points of X that are fixed under all the transformations of G. We consistently denote by π a finite group of transformations of a space X.

THEOREM A. *If $X \sim_{Z_p} S^n$ with X compact and π is a p-group, then $F(\pi, X) \sim_{Z_p} S^m$ for some m.*

(Here S^0 denotes two points and S^{-1} the empty set.)

THEOREM B. *If X is a cohomology manifold over Z_p and π is a p-group, then $F(\pi, X)$ is a cohomology manifold over Z_p of lower dimension. If X is a compact orientable cohomology n-manifold over Z_p, then any connected component in $F(\pi, X)$ is orientable over Z_p.*

An orientable nonempty connected cohomology manifold has nontrivial global cohomology groups by 2.4.2. Thus one has as the immediate corollary

THEOREM C. *If $X =_{Z_p} S^n$ with X compact and if π is a p-group, then $F(\pi, X) =_{Z_p} S^m$ with $m \leqq n$, and $F(\pi, X)$ is connected if $m \neq 0$.*

If G is a compact connected r-dimensional abelian Lie group, then it is a toroid, that is, a direct product of r circle groups. If we denote by π_k the subgroup of G whose elements x satisfy $p^k x = 0$, then π_k is of order p^{kr} and the union of all the π_k is dense in G. If G operates on a space X, then $F(G, X) = \lim_{k\to\infty} F(\pi_k, X)$. It follows at once from Theorem C that if $X =_{Z_p} S^n$ with X compact and p prime, then $F(G, X) = F(\pi_k, X)$ for some k. Hence

COROLLARY C. *Let $X =_{Z_p} S^n$ with X compact and p prime. Let G be a toroid group operating on X. Then $F(G, X) =_{Z_p} S^m$ with $m < n$, and indeed $F(G, X) = F(\pi, X)$ for some p-group π in G.*

It is worth noting that if $X =_{Z_p} S^0$ or S^{-1} then $X = S^0$ or S^{-1}.

THEOREM D. *Let π be a commutative group whose elements all have prime order p. Let X be a compact space with $X =_{Z_p} S^n$. Assume $F(g, X) = F(\pi, X)$ for all $g \in \pi$. Then π is cyclic.*

2.6. Now we state some basic results of Floyd on fixed point sets of toroids. Further results in this area have been obtained by Conner and Floyd.

THEOREM E. *Let G be a compact abelian Lie group operating on a compact cohomology manifold over Z. Then G has only a finite number of distinct isotropy subgroups.*

THEOREM F. *Let X be clc over Z and compact. Then the Z-cohomology groups of X are finitely generated.*

THEOREM G. *Let G be a circle group, and let X be a cohomology manifold over Z. Assume G has only a finite number of distinct isotropy subgroups. Then $F(G, X)$ is a cohomology manifold over Z.*

If G is a toroid operating on X, a compact cohomology manifold over Z, then the point set union of all the isotropy subgroups other than G is a finite sum and cannot cover G. Hence this union fails to contain some one-parameter subgroup C. Clearly $F(C, X) = F(G, X)$, for if $z \in F(C, X)$, then C is contained in the isotropy subgroup of x and the latter must therefore be G. Thus one obtains

COROLLARY 1. *Let G be a toroid operating on X, a compact cohomology manifold over Z. Then $F(G, X)$ is a cohomology manifold over Z.*

COROLLARY 2. *Let G be a toroid operating on X. Assume X is compact and $X =_Z S^n$. Then $F(G, X) =_Z S^m$ with $m < n$.*

Proof. $X =_{Z_p} S^n$ for all p by the argument in 2.2. Hence $F(G, X) =_{Z_p} S^{m_p}$ with $m_p < n$ for each prime p, by Corollary C of 2.5. Suppose first that some m_{p_0} is positive; then $F(G, X)$ is connected, and therefore it is a cohomology m-manifold over Z for some m. Hence $m_p = m$ and $F(G, X) =_{Z_p} S^m$ for all primes p. Since the Z-cohomology groups of $F(G, X)$ are finitely generated, $F(G, X)$ has the Z-cohomology groups of S^m. Hence $F(G, X) =_Z S^m$. If on the other hand, $m_p \leqq 0$ for all p, then we see easily that all m_p are equal, and $F(G, X) =_Z S^0$ or $F(G, X) =_Z S^{-1}$. Here $F(G, X) = S^0$ or $F(G, X) = S^{-1}$.

We observe that the argument establishing Corollary 1 actually proves the following:

COROLLARY 3. *Let G be a toroid operating on a cohomology manifold X over Z. Assume that G has only a finite number of distinct isotropy subgroups. Then $F(G, X)$ is a cohomology manifold over Z. For each prime p, there is a p-subgroup π of G with $F(G, X) = F(\pi, X)$.*

The second assertion comes from the fact that no finite point-set union of proper closed subgroups can cover all the p-subgroups of G.

2.7. We shall be dealing with cohomology manifolds which resemble euclidean space in varying degrees. For that reason we list here some particulars which will be used repeatedly in the sequel.

Let X be a locally compact space, and suppose $X \sim_L E^n$, that is, $H_c^*(X, L)$ is isomorphic to the compact(-support) L-cohomology groups of n-dimensional euclidean space E^n. If $n > 0$, then X is not compact since $H_c^0(X, L) = 0$. We denote the *one-point compactification* of X by $X \cup \infty$. From the cohomology sequence of the pair $(X \cup \infty, \infty)$ it follows at once that $X \cup \infty \sim_L S^n$, $n > 0$. If $L = Z_p$ and π is a p-group operating on X, then π becomes a transformation group on $X \cup \infty$ if we define $\pi(\infty) = \infty$. By Theorem A of 2.5, $F(\pi, X \cup \infty) \sim_{Z_p} S^m$ if p is a prime. Since $F(\pi, X \cup \infty)$ contains ∞, we are certain that $m \geqq 0$. Thus $F(\pi, X \cup \infty)$ contains at least two points. That is, $F(\pi, X)$ is not empty. If $X \sim_Z E^n$, $n > 0$, then we get $X \sim_{Z_p} E^n$ by considering the cohomology sequence associated with $0 \to Z \to Z \to Z_p \to 0$. In this case too $F(\pi, X)$ is not empty for any p-group.

If $X =_Z E^n$, then $X =_{Z_2} E^n$. In this case, the sheaf of local Z_2-cohomology groups is constant on every connected component C of X and indeed $H_c^n(C, Z_2) = Z_2$. Hence $H_c^n(X, Z_2) = s \cdot Z_2$, where s is the number of connected components in X. Since $H_c^n(E^n, Z_2) = Z_2$, it follows that $s = 1$ and X is connected. If $X =_Z E^n$ and G is a toroid operating on X, then $F(G, X)$ is a cohomology manifold over Z [3] and hence over Z_2. By the preceding argument each connected component of $F(G, X)$ makes a nonzero contribution to $H_c^*(F(G, X), Z_2)$. Since $F(G, X) \sim_{Z_2} E^m$ for some m, we conclude that $F(G, X)$ is a connected m-cohomology manifold over Z. Indeed $F(G, X) \sim_{Z_p} E^{m_p}$ for all primes p. However we infer $F(G, X) =_Z E^m$ only after making some additional assumption such as $H_c(F(G, X), Z)$ is finitely generated. For in that case, $H_c^m(F(G, X), Z)$ contains Z as a factor; from this it follows that $m_p = m$ for all primes p.

Suppose that $X \cup \infty$ is a cohomology n-manifold over Z. Then $X \cup \infty =_Z S^n$. The results of 2.5 and 2.6 can then be applied to yield that $F(G, X) =_Z E^m$, $m < n$, if G is a toroid operating on X; also $F(\pi, X) =_{Z_p} E^m$, $m < n$, if π is a p-group with p prime and $F(\pi, X)$ is connected.

3. Star circle subgroups, fixers, and weights

Throughout this section, G denotes an r-dimensional toroid group operating on a locally compact finite-dimensional space X. If $Y \subset X$ and H is a subgroup of G, we call the intersection of all the isotropy subgroups H_y, y ranging over Y, the *fixer* of Y in H. If a subgroup H is a fixer in G, then H is the fixer of $F(H, X)$. A point $x \in X$ is called a *weight point* if its isotropy (or fixer (subgroup G_x is $(r - 1)$-dimensional. The homomorphism $G \to G/G_x$ is called the *weight* of G associated with x. If $H \subset G$, we denote by $H^+(G)$ the intersection of all the fixers in G containing H.

In order to have $H^+(G)$ contain H, we adopt the convention of regarding G as the fixer of the empty subset of X. We denote by $H^*(G)$ the connected component of the identity in $H^+(G)$. When there is no danger of ambiguity, we write H^+ and H^* for $H^+(G)$ and $H^*(G)$ respectively. A circle subgroup H of G is called a *star circle of* G if $H^*(G) = H$.

If $H \subset G$, then $H^+ = \bigcap G_x$, all $x \in F(H, X)$. Hence H^+ is the fixer of $F(H, X)$ and $F(H^+, X) = F(H, X)$. Also, $(H^+)^+ = H^+$. If H and K are fixers, then $H \cap K$ is the fixer of $F(H, X) \cup F(K, X)$.

This section is devoted to a discussion of fixer subgroups of toroids operating on euclideanlike spaces. It is convenient to introduce the following definition.

DEFINITION. Let G be a toroid operating almost effectively on a space X. We say that (G, X) satisfies Hypothesis F if

(1) $X =_Z E^n$,
(2) G has only a finite number of distinct isotropy subgroups,
(3) $H_c^*(F(H, X), Z) =_Z E^m$ for any toroid subgroup H,
(4) $F(\pi, X) =_Z E^m$ for any p-subgroup π and $F(\pi, X)$ is connected.

We shall mean by "dim X" the Cohen-Wallace dimension of X over Z.

LEMMA 3.1. *Let Y be an n-dimensional, locally compact (resp. and paracompact) space whose Z-cohomology groups with compact (resp. closed) supports vanish in all dimensions except n and possibly m, $0 \leqq m \leqq n$. Let π be a finite group operating freely on Y and trivially on $H_c^*(Y, Z)$. Then*
(1) $H^{n-m+1}(\pi, H_c^m(Y, Z)) = A/qA$, *where* $q =$ *order* π, $A = H_c^n(Y, Z)$.
(2) *If* $\pi \neq (1)$ *and* $H_c^*(Y, Z)$ *is free, then* $n - m$ *is odd* (*resp* $H^m(Y, Z)$ *in place of* $H_c^m(Y, Z)$).

Proof. We will carry out the proof for the case of cohomology with compact supports. The argument in the case of closed supports is exactly the same.

Let $\underline{Y}$ denote the complex of Alexander-Čech Z-cochains with compact supports. Then

$$H(\pi, \underline{Y}) = H(\underline{Y}^\pi) = H_c^*(Y/\pi, Z)$$

(see [8](c), p. 11–10). Moreover, there is a filtration for $H(\pi, \underline{Y})$ with the E_2 term of the associated spectral sequence given by

$$E_2^{p,q} = H^p(\pi, H_c^q(Y, Z)),$$

where p is the filtration degree and $p + q$ is the total degree. Since $H_c^q(Y, Z)$ vanishes for $q \neq m$ or n, we get by a simple familiar argument (ibid., p. 10–05) the exact sequence

$$\rightarrow {''E_2^k} \rightarrow H^k(\pi, \underline{Y}) \rightarrow {'E_2^k} \rightarrow {''E_2^{k+1}} \rightarrow$$

with ${'E_2^k} = E_2^{k-n,n}$ and ${''E_2^k} = E_2^{k-m,m}$. Setting $k = n$, we obtain

$$\rightarrow H_c^n(Y/\pi, Z) \rightarrow H^0(\pi, H_c^n(Y, Z)) \rightarrow H^{n-m+1}(\pi, H_c^m(Y, Z)) \rightarrow 0$$

using the fact that $H_c^{n+1}(Y/\pi, Z) = 0$ since Y/π is n-dimensional. Thus $H^{n-m+1}(\pi, H_c^m(Y))$ is isomorphic to the quotient of $H^0(\pi, H_c^n(Y, Z))$ by the image of $H_c^n(Y/\pi, Z)$. This latter image can be identified with the image of $H_c^n(Y/\pi, Z)$ in $H_c^n(Y, Z)$ (ibid., pp. 12–03, 04) and is therefore $qH_c^n(Y, Z)$ where $q =$ order π. Hence $H^{n-m+1}(\pi, H_c^m(Y, Z)) = A/qA$. (Note in particular that $H_c^m(Y, Z) \neq 0$ if $q \neq 1$ and A is not divisible by q.)

Suppose now $\pi \neq (1)$ and $H_c^*(Y, Z)$ is free. Then π contains a nontrivial cyclic subgroup π' of order p. Applying the foregoing to π', we have $H^{n-m+1}(\pi', H_c^m(Y, Z)) = A/pA \neq 0$. From the well-known formula for the cohomology groups of a cyclic group (see ibid., p. 3–07), we see that $H_c^k(\pi', H_c^m(Y, Z))$ vanishes for k odd. Hence $n - m + 1$ is even; that is $n - m$ is odd.

The next lemma is a direct application of Lemma 3.1. The second half of the conclusion can be inferred from P. A. Smith's theorem on commuting transformations (Theorem D of 2.5) under special circumstances. However, we shall require the full generality indicated below. See Cartan-Eilenberg, *Homological Algebra* for related results and further details on cohomology of groups and spectral sequences of a covering.

LEMMA 3.2. *Let (G, X) satisfy Hypothesis F, and let H be a connected subgroup of G. Then*

(1) $\dim X - \dim F(H, X)$ *is even.*

(2) *Any finite subgroup of G operating freely on $X - F(H, X)$ is cyclic.*

Proof. Since H and its topological closure $\bar{H}$ have the same fixed point set, and since $\bar{H}$ is a toroid, no generality is lost in assuming that H is a toroid. By hypothesis, $F(H, X)$ has finitely generated compact Z-cohomology groups. Since $X =_Z E^n$, we get $F(H, X) =_Z E^m$, $m < n$ (see 2.7). Set $Y = X - F(H, X)$. From the cohomology (with compact supports) sequence for a pair, we obtain at once that $H_c^*(Y, Z)$ is a free abelian group, vanishing in all dimensions except n and $m + 1$. By Lemma 3.1, $n - (m + 1)$ is odd. Consequently, $n - m$ is even. Also $H_c^n(Y, Z) = Z$ and $H_c^{m+1}(Y, Z) = Z$.

Suppose now that π is a finite subgroup of the toroid G and that π operates freely on Y. In order to prove that π is cyclic, it suffices to show that π does not contain $Z_p + Z_p$ for any prime p. Indeed, if π contained a subgroup isomorphic to $Z_p + Z_p$, then we would have $H^{n-m}(Z_p + Z_p, Z) = Z_{p_2}$, by Lemma 3.1 applied to $Z_p + Z_p$. But since Z_p is a direct factor of $Z_p + Z_p$, $H^{n-m}(Z_p, Z) = Z_p$ should be a direct factor of $H^{n-m}(Z_p + Z_p, Z) = Z_{p_2}$, which is not the case. It follows that π is cyclic.

LEMMA 3.3. *Let (G, X) satisfy Hypothesis F. Then*

(1) *G has a star circle subgroup if $G \neq (1)$.*

(2) $0 \leqq \dim F(G, X) \leqq \dim X - 2 \dim G$.

Proof. We can assume that $G \neq (1)$. By hypothesis, G has only a finite number of (distinct) isotropy subgroups. Inasmuch as any fixer subgroup is an intersection of isotropy subgroups, G has only a finite number of fixer subgroups. Let M be a fixer subgroup of minimal positive dimension; let U be the subgroup generated by all the fixer subgroups properly contained in M_1, the connected component of the identity in M. U is a finite group, and for any element g in $M_1 - U$, $(g)^+ \supset M_1$. Hence $F(g, X) = F(M_1, X)$. Let $s = \dim M$ and let p be a prime number not dividing the order of U. Let π be the subgroup of M_1 of elements of order p. Then π is isomorphic to $s \cdot Z_p$, the sum of s copies of Z_p. Moreover π operates freely in

$$Y = X - F(M_1, X).$$

Since $F(M_1, X) =_Z E^m$ with $m < n$ and $n - m$ even, we find from the cohomology sequence of the pair $(F(M_1, X), X)$ that $H_c^k(Y, Z) = Z$ for $k = n$, $m + 1$ and $= 0$ otherwise. Applying Lemma 3.2, we find that π is cyclic. Hence $s = 1$ and M_1 is a circle. Since $M_1^* = M_1$, M_1 is a star circle subgroup of G.

In order to prove $0 \leqq \dim F(G, X)$, we must prove that $F(G, X)$ is not empty. Inasmuch as G has only a finite number of isotropy subgroups, $F(G, X) = F(\pi, X)$ for some p-subgroups π with prime, by Corollary 3 of 2.6. By Smith's Theorem A of 2.5, $F(\pi, X \cup \infty) \sim_{Z_p} S^m$ for some m, since $X \cup \infty \sim_{Z_p} S^n$. Since π keeps ∞ fixed, $m \geqq 0$. Hence $F(\pi, X \cup \infty)$ contains more than one point and $F(\pi, X)$ is not empty. Consequently, $F(G, X)$ is not empty.

Let $r = \dim G$ and $n = \dim X$. We prove $\dim F(G, X) \leqq n - 2r$ by induction on r. We observe that for any subgroup P of G, G/P^+ operates effectively on $F(P, X)$. In particular if P is a star circle, G/P is almost effective on $F(P, X)$. It is obvious from what has been said above that if (G, X) satisfies Hypothesis F, then $(G/P, F(P, X))$ satisfies Hypothesis F. Hence by the induction hypothesis,

$$\dim F(G, X) = \dim F(G/P, F(P, X)) \leqq \dim F(P, X) - 2(r - 1).$$

We know from Lemma 3.2 that $\dim X - \dim F(P, X)$ is a nonzero even number. Hence $2 \leqq \dim X - \dim F(P, X)$, and

$$\dim F(G, X) \leqq \dim F(P, X) - 2r + \dim X - \dim F(P, X) \leqq n - 2r.$$

The proof of Lemma 3.3 is now complete.

THEOREM 3.1. *Let G be an r-dimensional toroid operating almost effectively on a locally compact (Hausdorff) space X. Assume that $X \cup \infty =_Z S^n$. Then*
(1) *G is spanned by its star circle subgroups.*
(2) $0 \leqq \dim F(G, X) \leqq n - 2r$.
(3) $F(G, X) \cup \infty =_Z S^m, \quad m \leqq n - 2r$.

Proof. By Floyd's Theorem E of 2.6, G has only a finite number of isotropy subgroups on $X \cup \infty$. By Corollary 1 and Theorem F of 2.6, $F(H, X \cup \infty)$ has finitely generated integral cohomology groups. It follows readily that (G, X) satisfies Hypothesis F. The assertion (2) now follows from Lemma 3.3. Assertion (3) follows from (2), Corollary 2 of 2.6, and 2.4. Also the existence of at least one star circle follows. We will establish the existence of additional ones by an induction argument. But first we take up separately the case $r = 2$.

Let P be a star circle subgroup of a two-dimensional toroid, and suppose that G contains no other star circle. Then for all $x \in X - F(P, X)$, the isotropy subgroup G_x is discrete. Let U be the subgroup generated by all the G_x, $x \in X - F(P, X)$. Since G has only a finite number of isotropy subgroups, U is finite. Let p be a prime number not dividing the order of U; let π be the subgroup of G of elements of order p. Then $\pi = Z_p + Z_p$ and $\pi \cap U = (1)$. Also, for any $g \in \pi$, $(g)^+ \supset P$. Hence $F(g, X) \subset F(P, X)$, and π operates freely on $X - F(P, X)$. On the other hand by Lemma 3.2, π is cyclic. This contradiction establishes the existence of a star circle subgroup distinct from P in the case $r = 2$.

Now we assume inductively that Theorem 3.1 is valid if the dimension of the toroid is less than r. Let P_r be a star circle subgroup of the r-dimensional toroid G. Then G/P_r^+ operates effectively on $F(P_r, X)$ and G/P_r operates almost effectively on $F(P_r, X)$. By Corollary 2 of 2.6, $F(P_r, X \cup \infty) =_Z S^m$. Hence $F(P_r, X) \cup \infty =_Z S^m$ and $(G/P_r, F(P, X))$ satisfies the hypotheses of Theorem 3.1. Applying the induction hypothesis, G/P_r is spanned by $r - 1$ of its star circle subgroups $Q_1, \cdots, Q_{r-1}$.

Let ρ denote the natural homomorphism of G onto G/P_r. Let $T_i = \rho^{-1}(Q_i)$, $i = 1, \cdots, r - 1$. Clearly

$$\rho(H^+(G)) \subset (\rho(H))^+ (\rho(G))$$

for any $H \subset G$. Hence $\rho(T_i^+(G))$ is a finite extension of Q_i, and $T_i^*(G) = T_i$ $(i = 1, \cdots, n)$. Now each T_i is a two-dimensional toroid; we can therefore assert the existence of a circle (distinct from P_r) subgroup $P_i \subset T_i$ with $P_i^*(T_i) = P_i$ $(i = 1, \cdots, n - 1)$. Inasmuch as $T_i^*(G) = T_i$, we have $P = P_i^*(T_i) = P_i^*(G)$, and P_i is a star circle subgroup of G, $i = 1, \cdots r - 1$. The circles $P_1, \cdots, P_{r-1}, P_r$ obviously span G. Proof of the theorem is now complete.

LEMMA 3.4. *Let π be a p-group operating on a connected locally compact space X with $X =_Z E^n$, p prime and $n > 0$. Assume that π operates trivially on $H_c^n(X - F(\pi, X), Z_p)$. Then $F(\pi, X) \sim_{Z_p} E^m$ with $m \leqq n - 2$.*

Proof. That $F(\pi, X) \sim_{Z_p} E^m$ with $m < n$ follows at once from Smith's Theorems A and B of 2.5. It remains only to prove that $m \neq n - 1$. We suppose $m = n - 1$, and we shall produce a contradiction.

From the cohomology sequence of the pair $(X, F(\pi, X))$ we find that $H_c^k(X - F(\pi, X), Z_p)$ is $Z_p + Z_p$ for $k = n$ and is zero for all other k. On the other hand, the sheaf of local n-dimensional Z_p-cohomology groups on X is constant since $H_c^n(X, Z_p) \neq 0$ (see 2.4.2). Hence each connected component of $X - F(\pi, X)$ has nonzero n-dimensional compact Z_p-cohomology.

Since the connected components of $X - F(\pi, X)$ are both open and closed, it follows that $X - F(\pi, X)$ has exactly two connected components, say Y_1 and Y_2. Moreover $Y_1 \sim_{Z_p} E^n$ $(i = 1, 2)$. Hence $Y_i =_{Z_p} E^n$ $(i = 1, 2)$. The hypothesis on π implies that π keeps the connected components invariant. Hence $F(\pi, Y_1)$ is not empty, by 2.7. This contradicts the fact that $Y_1 \cap F(\pi, X)$ is empty. Consequently $m \neq n - 1$.

Note. The above lemma asserts something that is not already contained in the well known results of Floyd [6], and Liao [7] only in the case $p = 2$. For $p \neq 2$, Floyd's result states that $n - m$ is even for any p-group; for $H_c^*(X, Z)$ finitely generated and π operating trivially on $H_c^n(X\ Z)$, Liao's result shows that $n - m$ is even when $p = 2$. Whether $n - m$ is even under the hypotheses of Lemma 3.4 remains open.

LEMMA 3.5. *Let (G, X) satisfy Hypothesis F, let G be effective, and assume moreover that the fixed point set of any star circle in G has codimension 2 in X. Let P be a star circle in G. Set $Y = X - F(P, X)$, $X_1 = Y/P$, and $G^1 = G/P$; let ϕ_p denote the orbit map of Y onto X_1, and let θ_p denote the natural homomorphism of G onto G^1. Then*

(1) *P operates freely on Y.*

(2) *(G^1, X_1) satisfies the hypothesis imposed on (G, X) in this lemma.*

(3) *Given a closed connected subgroup H^1 in G^1, there exists a unique subgroup H in G such that $\theta_p(H) = H^1$ and $\phi_p F(H, Y) = F(H^1, X_1)$; in addition, θ_p maps H isomorphically and $H = H^+ \cap \theta_p^{-1}(H^1)$.*

(4) *$P^+ = P$ (in G).*

(5) *θ_p maps the star circles of G other than P biuniquely onto the star circles of G^1.*

For notational convenience we denote θ_p and ϕ_p by θ and ϕ respectively in the course of the proof of this lemma.

Proof of (1). Suppose for some $x \in Y$, the isotropy subgroup $P_x \neq (1)$. Then P_x contains a cyclic subgroup π of prime order p, $p > 1$. By Hypothesis F, $F(\pi, X) =_{Z_p} E^m$. By Lemma 3.4, $m \leqq n - 2$. Since $F(\pi, X) \supset F(P, X)$ and $F(P, X) =_{Z_p} E^{n-2}$, we find that $m = n - 2$. Hence $F(P, X)$ is open as well as closed in the connected set $F(\pi, X)$. Therefore $F(P, X) = F(\pi, X)$ and $\pi^+ \supset P$, a contradiction. Thus $P_x = (1)$ for all $x \in X - F(P, X)$, that is, P operates freely on $X - F(P, X)$.

Proof of (2). The space $X - F(P)$ is fibered by the circle P. The cohomology of the fibre forms a constant system of coefficients over X_1. Thus there is Gysin's exact sequence

$$\rightarrow H_c^{k-2}(X_1) \rightarrow H_c^k(X_1) \rightarrow H_c^k(Y) \rightarrow H_c^{k-1}(X_1) \rightarrow,$$

where $H_c^s = 0$ for $s < 0$. For $k < n - 1$, we get $H_c^k(X_1) = H_c^{k-2}(X_1)$. Now $H_c^0(X_1) = 0$ since X_1 has no compact connected components. Thus $H_c^k(X_1) = 0$ for $k < n - 1$. For $k > n - 1$, $H_c^k(X_1) = 0$ since X_1 is $(n - 1)$-dimensional. From

$$H_c^n(X_1) \rightarrow H_c^n(Y) \rightarrow H_c^{n-1}(X_1) \rightarrow H_c^{n+1}(X_1)$$

we infer that $H_c^{n-1}(X_1) = H_c^n(Y)$. In short, X_1 has the compact integral cohomology of E^{n-1}. Inasmuch as small neighborhoods in Y are products of neighborhoods in X_1 by line segments, one can verify that X_1 is a cohomology manifold over Z just as Y is. Hence $X_1 =_Z E^{n-1}$.

Given any $x \in X_1$, we have $\phi^{-1}x = Py$ with $y \in Y$ and $\theta^{-1}(G_x^1)Py = Py$. Since P is transitive on Py, $\theta^{-1}(G_x^1) = \theta^{-1}(G_x^1)_y P = G_y P$. Hence $G_x^1 = \theta(G_y)$, where y can be taken as any element in $\phi^{-1}x$. Since G has only a finite number of isotropy subgroups on Y, the same is true of G^1 on X_1. Given now any toroid subgroup H^1 of G^1 it follows from 2.7 that $F(H^1, X_1)$ is connected. Furthermore for any $x \in F(H^1, X_1)$ and for any $y \in \phi^{-1}x$, we have $\theta^{-1}(H^1) = \theta^{-1}(H_x^1) = \theta^{-1}(H^1)_y P$. Since P operates freely on Y, $\theta^{-1}(H^1)_y \cap P = (1)$ and thus $\theta^{-1}(H^1)_y$ is isomorphic to H^1 for all $y \in \phi^{-1}F(H^1, X_1)$. Inasmuch as nearby closed isomorphic subgroups of a toroid coincide and $\phi^{-1}F(H^1, X_1)$ is connected, $\theta^{-1}(H^1)_y$ is constant as y ranges over $\phi^{-1}F(H^1, X_1)$; we denote it by H. Then $\phi^{-1}F(H^1, X_1) \subset F(H, Y)$. The converse inequality is obvious. Hence $F(H^1, X_1) = \phi F(H, Y)$. The subgroup H is unique for it is the connected component of the identity of the fixer in $\theta^{-1}(H^1)$ of $\phi^{-1}F(H^1, X_1)$. Thus (3) is proved.

Still remaining to establish in our proof of (2) is that

(a) $F(H^1, X_1) =_Z E^s$ for every toroid subgroup H^1 of G^1; and that

(b) $F(\pi^1, X_1) =_{Z_p} E^t$ and is connected for every p-subgroup π^1 of G, p prime.

By the results in 2.7, we can assert $F(H^1, X_1) =_Z E^s$ once we know that $H_c^*(F(H^1, X_1), Z)$ is finitely generated. In turn, this follows by Gysin's sequence from the fact that $F(H^1, X_1)$ is the finite-dimensional base space of a fibre map with circles as fibre and a total space Y having finitely generated compact integral cohomology groups. Thus (a) is proved.

Let π^1 be a p-subgroup of G^1 with p prime. For any $x \in F(\pi^1, X_1)$ let $\pi_x = \theta^{-1}(\pi^1) \cap G_y$ where y is any point of $\phi^{-1}x$. Since P is transitive on $\phi^{-1}x$, we see that $\pi_x P = \theta^{-1}(\pi^1)$. Thus $\theta(\pi_x) = \pi^1$ and $x \in \phi F(\pi_x, Y)$; also π_x is a p-group. Consequently, $F(\pi^1, X_1)$ is the union of all subsets of the form $\phi F(\pi, Y)$, with π a p-group in G such that $\theta(\pi) = \pi^1$. By definition of Hypothesis F, $F(\pi, X) =_{Z_p} E^m$ with some m for every p-subgroup π of G, and also $F(\pi, X)$ is connected. Hence $F(\pi, X)$ is an orientable cohomology m-manifold over Z_p. It follows that $F(\pi, Y)$ is orientable over Z_p. Since ϕ is a principal fibering, with circle as fibre on $F(\pi, Y)$, it follows that $\phi F(\pi, Y)$ is an orientable cohomology manifold over Z_p. Inasmuch as $F(\pi^1, X_1)$ is a cohomology manifold over Z_p, and simultaneously a finite union of orientable connected cohomology manifolds over Z_p, each of its connected components is an orientable cohomology manifold over Z_p and makes a nonzero contribution to $H_c^*(F(\pi^1, X_1), Z_p)$. But $F(\pi^1, X_1) \sim_{Z_p} E^t$ since $X_1 \sim_{Z_p} E^{n-1}$ (by Smith's result; see 2.7). It follows that X is connected and $F(\pi^1, X_1) =_{Z_p} E^t$. Proof of (2) is now complete.

Proof of (4). In (3) we take $H^1 = G^1/P$. Then we get $G = PH$ with $\phi F(H, Y) = F(H^1, X_1)$. Since $F(H^1, X_1)$ is not empty by 2.7, it follows that $F(H, X - F(P, X))$ is not empty. Now consider $P^+ \cap H$ where P^+ is the fixer of $F(P, X)$ in G. If $P^+ \cap H \neq (1)$, it has a subgroup π of prime order p. Then $F(\pi, X)$ keeps fixed the $(n - 2)$-dimensional subspace $F(P, X)$ together with some points in $X - F(P, X)$. It follows by Lemma 3.4 that π must operate trivially on X. If we assume therefore that G is effective on X, it follows that $P^+ \cap H = (1)$. Hence $P^+ = P(P^+ \cap H) = P$, and P^+ is connected.

Proof of (5). Let Q be a star circle subgroup of G distinct from P. By hypothesis, $F(Q, X)$ has codimension 2 in X. Since $F(P, X) \cap F(Q, X) = F(PQ, X)$, its codimension is at least 4 by Lemma 3.3. Hence $F(Q, Y)$ has codimension 2 in Y. It follows at once that $\phi F(Q, Y)$ has codimension at least 2 in X_1.

Since $F(\theta(Q), X_1) \supset F(Q, Y)$, $F(\theta_p(Q), X_1)$ has codimension at least 2. On the other hand $F(\theta(Q), X_1) =_Z E^m$, and it has nonzero even codimension in X_1. Hence $F(\theta_p(Q), X_1)$ has codimension 2. Let H^1 be the connected component of the identity in $\theta_p(Q)^*$. Then $F(H^1, X_1)$ has codimension 2 in X_1. From the fact that G is effective on X, it follows immediately that G^1 is effective on X_1. Then H^1 operates freely on $X_1 - F(H^1, X_1)$—otherwise the fixed point set of some of its elements would have to be of even codimension less than 2 and greater than 0. Let s be the dimension of H^1, and let p be a prime number. Then H^1 contains a subgroup isomorphic to sZ_p which operates freely on $Y_2 = X_1 - F(H^1, X_1)$. Since Y_2 has compact integral cohomology in only two dimensions, sZ_p must be cyclic by Lemma 3.2. Hence $s = 1$ and $\theta(Q)^* = \theta(Q)$. That is, $\theta(Q)$ is a star circle subgroup of G^1.

Conversely, suppose Q^1 is a star circle subgroup of G^1. By part (3) above, there is a unique circle subgroup Q in G with $\theta(Q) = Q^1$ and $\phi F(Q, Y) = F(Q^1, X_1)$. Since $\theta(Q^+) \subset (\theta(Q))^+$, $Q^+ \subset \theta^{-1}(\theta(Q)^+) = \theta^{-1}(Q^1)$. But $Q = Q^+ \cap \theta^{-1}(Q^1)$ by part (3). Hence $Q = Q^+$ and Q is a star circle subgroup in G. We can now assert that θ_p maps the star circles of G biuniquely onto the star circles of G^1.

THEOREM 3.2. *Let G be an r-dimensional toroid operating effectively on a space X. Assume that $X \cup \infty =_Z S^n$ and that $n = 2r$ or $2r + 1$. Then*

(1) *G has exactly r star circle subgroups and is their direct product.*

(2) *All the fixer subgroups are connected.*

(3) *For any two subgroups S and T, $(ST)^+ = S^+T^+$.*

(4) *G has exactly r weights, 2^r fixer subgroups, and 2^r isotropy subgroups.*

Proof. We follow the notation of Lemma 3.5.

Proof of (1). Let $P_1, \cdots, P_r$ be r star circle subgroups which span G, their existence being assured by Theorem 3.1. Let $\theta_1 = \theta_{p_1}$, and inductively define $\theta_2, \cdots, \theta_{r-1}$ by the formula $\theta_k = \theta_{Q_k} \cdot \theta_{k-1}$, where $Q_k = \theta_{k-1}(P_k)$. Similarly define spaces X_k and maps $\phi_k: X_k - F(\theta_k(P_k), X_k) \to X_{k+1}$ by the inductive definition:

$$X_0 = X,$$
$$X_1 = X - F(P_1, X)/P_1, \qquad \phi_1 = \phi_{P_1},$$
$$X_k = X_{k-1} - F(Q_k, X_{k-1})/Q_k, \qquad k = 1, 2, \cdots, r-1,$$

where the operation of $\theta_k(G)$ on X_k is induced from the operation of $\theta_{k-1}(G)$ on X_{k-1}. Clearly $\theta_k(G)$ is isomorphic to the toral group $G/P_1 P_2 \cdots P_k$, and by repeated applications of Lemma 3.5 $(\theta_k(G), X_k)$ satisfies the hypothesis of Lemma 3.5. Since $\theta_{r-1}(G)$ is a circle, it has only one star circle subgroup. By repeated applications of Lemma 3.5, part (5), we deduce that the group $\theta_k(G)$ has exactly $r - k$ star circles $(k = 1, \cdots, r-1)$ and that G has exactly r star circles.

In order to prove that $G = P_1 \cdot P_2 \cdots \cdot P_r$ as a direct product, it suffices to show that $(P_1 P_2 \cdots P_k) \cap P_{k+1} = (1)$. By repeated application of Lemma 3.5, parts (3) and (5), θ_k is an isomorphism of P_{k+1} onto $\theta_k(P_{k+1})$. Since $P_1 \cdots P_k$ is in the kernel of θ_k, we get $P_1 P_2 \cdots P_k \cap P_{k+1} = (1)$.

Proof of (2). Let H be a fixer subgroup of G. Let H_1 denote the connected component of its identity element. By Theorem 3.1, H_1 is spanned by its star circle subgroups. Since $H_1^* = H_1$ in G, any star circle in H_1 is a star circle in G. Thus $H_1 = P_1 P_2 \cdots P_s$, where each P_i is a star circle in G. Then

$$\theta_{s-1}(H_1^+) \subset (\theta_{s-1}(H_1))^+ = (\theta_{s-1}(P_s))^+ = \theta_{s-1}(P_s),$$

the last equality because star circle subgroups in $\theta_{s-1}(G)$ are fixers according to Lemma 3.5, part (4). Hence $(H_1)^+ \subset P_1 P_2 \cdots P_s$ and $H_1 = (H_1)^+$. Let $Y = F(H_1, X)$. Since H_1 is normal in G, $GY = Y$.

For any $g \in G$, let $\rho(g)$ denote the restriction of g to Y. Then $Y \cup \infty =_Z S^m$ by Theorem C of 2.5. From the definition of ρ it follows at once that the kernel of ρ is $(H_1)^+$ and that $\rho(K^+) = \rho(K)^+$ for any subgroup containing H_1. Thus

$$\rho(H)^+ = \rho(H^+) = \rho(H) = H/H_1.$$

That is, $\rho(H)$ is a discrete fixer subgroup of $\rho(G)$.

By Theorem 3.1, $0 \leqq \dim F(H_1) \leqq n - 2s$, where $n = \dim X$ and $s = \dim H_1$. Hence $\dim Y \leqq n - 2s \leqq 2 \dim \rho(G) + 1$. Since $\rho(G)$ is effective on Y, we have, again by Theorem 3.1, that $0 \leqq \dim Y - 2 \dim \rho(G)$. Therefore $\dim Y = 2 \dim \rho(G)$ or $2 \dim \rho(G) + 1$.

In short, $(\rho(G), Y)$ satisfies the hypotheses imposed on (G, X) in Theorem 3.2, and $\rho(H)$ is discrete. Thus assertion (2) of Theorem 3.2 has been reduced to:

(2′) *Any discrete fixer subgroup of G is trivial.*

This assertion will follow from the following lemmas.

LEMMA 3.6. *Let X be a locally compact space such that $X \cup \infty =_{Z_p} S^n$, p prime. Let G be an r-dimensional toroid operating almost effectively on X. Then*

(1) *G has a star circle subgroup.*

(2) $0 \leqq n - 2r$.

Proof. The proof of (1) is patterned after the proof of Lemma 3.4 with P. A. Smith's theorem on commuting transformations taking the place of Lemma 3.1.

We let M be a minimal fixer subgroup of positive dimension in G; the connected component of the identity in M is denoted by M_1. By Corollary C of 2.5, there is a maximal p-subgroup U of M_1 such that $F(U, X) \neq F(M_1, X)$. Let $H = G/U^+$, and let N denote the image of M_1 in H; set $Y = F(U, X)$. Then H operates effectively on Y and $F(n, Y) = F(L, Y)$ for any element n of order p in N. Let π denote the subgroup of all elements of order p in L. We have $\pi = s \cdot Z_p$ where $s = \dim L - \dim M$. By Smith's Theorem D of 2.5, π is cyclic. Hence $s = 1$ and M_1 is a star circle subgroup of G.

By Corollary C of 2.5 together with Lemma 3.5, we know that $F(P, X \cup \infty) =_{Z_p} S^m$ with $0 \leqq m \leqq n - 2$ for any circle subgroup P in G. By repeated application of assertion (1), we can find a sequence of r star circles $P_1, P_2, \cdots P_r$ in the r groups G, G/P, C, which operate almost effectively on the r compact Z_p-cohomology manifolds $X \cup \infty$, $F(P_1, X \cup \infty)$, $F(P_2, F(P_1, X \cup \infty))$, $\cdots$. Setting $S_0 = X \cup \infty$, $S_1 = F(P_1, X \cup \infty)$, $\cdots$, we have that $F(G, X \cup \infty) = F(P_r, S_{r-1})$. By Lemma 3.5, $F(P_k, S_{k-1}) =_{Z_p} S^{m_k}$ with $m_k \leqq m_{k-1} - 2$. Also $0 \leqq m_k$ since $\infty \in F(P_k, S_{k-1})$. It follows immediately that $0 \leqq m_r \leqq n - 2r$.

We now return to the proof of assertion (2′). Under the hypotheses of Theorem 3.2, any discrete fixer subgroup is trivial. To see this, let π be a nontrivial p-subgroup of a nontrivial discrete fixer subgroup of G. Then $F(\pi, X \cup \infty) =_{Z_p} S^m$ with $m \leqq 2r - 1$ by Lemma 3.4. On the other hand G/π^+ is an r-dimensional toroid that is effective on $F(\pi, X \cup \infty)$; therefore $m \geqq 2r$ by Lemma 3.6. This contradiction establishes assertion (2′). Proof of (2) is now complete.

Proof of (3). By applying assertions (5) and (4) of Lemma 3.5, it follows easily by induction that any product of star circles is a fixer.

Proof of (4). Since all the fixer subgroups are toroids, they can be described as m products of the r star circle subgroups $P_1, \cdots P_r$, $0 \leqq m \leqq r$. There are 2^r such subgroups, and among these exactly r are $(r - 1)$-dimensional. Thus there are exactly r weights.

Let H be a fixer subgroup of G. It remains only to prove that H is the isotropy subgroup of some point. Let $H_1, \cdots, H_s$ be the fixer subgroups containing H. Since G/H is effective on $F(H, X)$, $F(H_i, X)$ is a closed nowhere dense subset of $F(H, X)$ for each i.

Set $U = F(H, X) - (F(H_1, X) \cup \cdots \cup F(H_s, X))$. Clearly U is an open dense subset of $F(H, X)$. Also $G_u = H$ for $u \in U$. Hence H is an isotropy subgroup.

4. Poincaré duality for open Z-manifolds

4.1. The Poincaré duality theorem has been proved for an orientable paracompact n-dimensional cohomology manifold X over a *field* L by Wilder [11]; in Borel's formulation [1], it states that

$$H^{n-k}(X, L) = \mathrm{Hom}_L\,(H_c^k(X, L), L).$$

In case X is an orientable paracompact n-dimensional cohomology manifold over the integers Z, one means by Poincaré duality the assertion that the sequence

$$0 \to \mathrm{Ext}\,(H_c^{k+1}(X, Z), L) \to H^{n-k}(X, L) \to \mathrm{Hom}\,(H_c^k(X, Z), L) \to 0$$

is exact for any coefficient group L. The proof of Borel in the case L = a field would yield Poincaré duality in case $L = Z$ if only one knew that the group of Alexander-Spanier cochains defining $H_c^k(X, Z)$ were projective.

We prove here that *Poincaré duality holds for an orientable n-dimensional cohomology manifold over Z if X is separable and* HLC[3] (i.e., small singular cycles bound small singular chains). Inasmuch as singular cohomology coincides with Alexander-Spanier cohomology in an HLC space (see [8] (a), Ch. 16, p. 12), the usual sheaf argument (ibid., Ch. 17, p. 1) for $H_c^k(X, Z) = \tilde{H}_{n-k}(X, Z)$ combined with the Universal Coefficient Theorem for $\tilde{H}_{n-k}(X, L)$ will yield the Poincaré Duality Theorem for X as soon as we know that $\tilde{H}_k\,(X, X - x, Z) = \delta_{kn} Z$, where $x \in X$, $\tilde{H}_k\,(X, Y, L)$ (resp. $\tilde{H}_k\,(X, L)$) denotes the relative (resp. absolute) singular homology group of the pair (X, Y) (resp. space X) with coefficients in L, and δ_{kl} is the Kronecker symbol. Quite directly one sees that

$$\tilde{H}_k\,(X, X - x, Z) = \varinjlim\,(\tilde{H}_k\,(X, X - U, Z);\, U \downarrow x),$$

where U ranges over the open neighborhoods of x. We evaluate this direct limit in the following subsections.

4.2. Since X is locally compact and separable, a compact neighborhood in X is a compactum. If U is an open set in X with $\bar{U}$ compact, then by excision,

$$H^*(X, X - U, L) = H^*(F, F - U, L),$$

where F is a compact neighborhood of $\bar{U}$. For the compact pair, we have the Universal Coefficient Formula

$$0 \to H^k(F, F - U, Z) \otimes L \to H^k(F, F - U, L)$$
$$\to \mathrm{Tor}\,(H^{k+1}(F, F - U, Z), L) \to 0$$

for any coefficient group L.

Let U_0 be an open neighborhood of x with $\bar{U}_0$ compact and such that the subgroup $H^*(x, U_0, Z)$ of $H^*(X, X - U_0, Z)$ exists (see Section 2). Let U_1, U_2, U_3 be open neighborhoods of x with $U_3 \subset U_2 \subset U_1 \subset U_0$ and such that $I_{U_i U_{i-1}}$ maps $H^*(X, X - U_i, Z)$ onto $H^*(x, U_{i-1}, Z)$, $i = 1, 2, 3$. Since $I_{U_i U_{i-1}}$ is an isomorphism on $H^*(x, U_i, Z)$, we see that $H^*(X, X - U_i, Z) =$

[3] If X is a separable compact cohomology manifold, we can prove Poincaré duality over Z without the added hypothesis that X is HLC. We have learned that these facts about duality have been observed by F. Raymond and E. Dyer. By HLC we mean that any small singular cycle bounds a small singular chain.

$H^*(x, U_i, Z)$ + kernel $I_{U_iU_{i-1}}$ (direct) for $i = 1, 2, 3$. Let F be a compact neighborhood of $\bar{U}_0$. Upon identifying $H^k(X, X - U_i, L)$ with $H^k(F, F - U_i, L)$, we get the commutative diagram

$$\begin{array}{llll}
0 \to H^k(X, X - U_1, Z) \otimes L & \xrightarrow{i_1} H^k(X, X - U_1, L) & & \\
\quad\uparrow \alpha_2 & \quad \uparrow \beta_2 & \to \operatorname{Tor}(H^{k+1}(X, X - U_1, Z), L) \to 0 & \\
0 \to H^k(X, X - U_2, Z) \otimes L & \xrightarrow{i_2} H^k(X, X - U_2, L) & \quad\uparrow & \\
\quad\uparrow \alpha_3 & \quad \uparrow \beta_3 & \to \operatorname{Tor}(H^{k+1}(X, X - U_2, Z), L) \to 0 & \\
0 \to H^k(X, X - U_3, Z) \otimes L & \xrightarrow{i_3} H^k(X, X - U_3, L) & \quad\uparrow & \\
 & & \to \operatorname{Tor}(H^{k+1}(X, X - U_3, Z), L) \to 0 &
\end{array}$$

where the rows are exact, and the vertical maps are induced by $I_{U_iU_{i-1}}$, $i = 3, 2, 1$. Since $H^k(x, U_k, Z) = \delta_{kn} Z$, the induced maps of the Tor terms are zero; we find that

$$\begin{aligned}
\beta_2 \beta_3 H^k(X, X - U_3, L) &\subset \beta_2 i_2(H^k(X, X - U_2, Z) \otimes L) \\
&= i_1 \alpha_2(H^k(X, X - U_2, Z) \otimes L) \\
&= i_1(\delta_{kn} Z \otimes L) \\
&= \delta_{kn} L \qquad\qquad (k = 0, 1, \cdots)
\end{aligned}$$

for any coefficient group L. Thus $H^*(x, U_1, L)$ exists and X *is an n-cohomology manifold over any coefficient group L*. More precisely, $I_{U_3U_1} H_c^k(U_3, L) = \delta_{kn} L$.

4.3. Since U_i is an HLC space ($i = 1, 3$), we may identify $H_c^k(U_i, L)$ with the singular cohomology group $\tilde{H}_c^k(U_i, L)$. Taking for L, the group T of reals modulo one, we have

$$I_{U_3U_1} \tilde{H}_c^k(U_3, T) = \delta_{kn} T, \qquad k = 0, 1, \cdots .$$

We denote by $\tilde{H}^k(X, F, L)$, the derived group of the group of L-valued singular cochains which vanish on singular chains in F.

On the other hand, if $X - F$ is an open set with compact closure in X, the group $\tilde{H}_c(X - F, L)$ is canonically isomorphic with the derived group of the group of L-valued cochains which vanish on singular chains lying in some neighborhood of F. Thus if W, V, U are open sets with compact closure such that $\bar{W} \subset V \subset \bar{V} \subset U$, we have the following commutative diagram:

$$\begin{array}{ccccc}
\tilde{H}^k(X, X - W, L) & \xrightarrow{i^*_{WV}} & \tilde{H}^k(X, X - V, L) & \xrightarrow{i^*_{VU}} & \tilde{H}^k(X, X - U, L) \\
\searrow & \nearrow & \searrow & \nearrow & \\
 & \tilde{H}_c^k(V, L) & \xrightarrow{I_{VU}} & \tilde{H}_c^k(U, L) &
\end{array}$$

If U is a sufficiently small open neighborhood of x and if V is a sufficiently small open neighborhood in U, it follows that $i^*_{WU}\tilde{H}^k(X, X - W, T) = \delta_{kn} T$. Now the group of relative singular chains being projective, we can apply the Universal Coefficient Formula to obtain

$$0 \to \text{Ext}\,(\tilde{H}_{k-1}(X, X - U, Z), T) \to \tilde{H}^k(X, X - U, T) \to \text{Hom}\,(\tilde{H}_k(X, X - U, Z), T) \to 0.$$

Inasmuch as T is injective, we have

$$\tilde{H}^k(X, X - U, T) = \text{Hom}\,(\tilde{H}_k\,(X, X - U, Z), T).$$

The map i^*_{WU} being the transpose of the injection i_{WU} of $\tilde{H}_k(X, X - U, Z)$ into $\tilde{H}_k(X, X - W, Z)$, we can conclude that

$$i_{WU}\,\tilde{H}_k(X, X - U, Z) = \delta_{kn}\,Z, \qquad k = 0, 1, \cdots.$$

From this it follows at once that

$$\tilde{H}_k(X, X - x, Z) = \varinjlim\,(\tilde{H}_k\,(X, X - U, Z);\, U \downarrow x) = \delta_{kn}\,Z.$$

Poincaré duality for X now follows by the remarks in 4.1.

As an immediate consequence of Poincaré duality, we have:

COROLLARY 4.1. *Let $X =_Z E^n$. Assume that X is a separable HLC space. Then $H^k(X, L) = \delta_{k0}\,L$, $k = 0, 1, 2, \cdots$.*

5. Maps of euclideanlike Z-manifolds

5.1. Let X be a locally compact, paracompact space which is clc over a coefficient group L.

If α is a locally finite cover of X, we denote by $N(\alpha)$ the nerve of α. The support of a simplex of $N(\alpha)$ is defined to be the intersection of the sets corresponding to the vertices of the simplex.

Let α and β be locally finite covers of X by *closed* sets with β refining α. With Floyd [5], we say that "β strongly L-refines α up to n" if there exists a support enlarging projection $\pi_{\alpha\beta}$ of $N(\beta)$ into $N(\alpha)$ such that $\pi^*_{\alpha\beta}\,H^k(S, L) = 0$, $k = 1, \cdots, n$ and $\pi^*_{\alpha\beta}\,H^0(S, L) = L$ for the support S of every simplex in $N(\alpha)$. If α is any cover of a clc space over L and n is any integer, there exists a refinement β which strongly L-refines α up to n. If $L = Z$, β can be selected so as to strongly Z_q-refine α up to n, simultaneously for all integers q (see 2.1). If in additon X is separable, one can get simultaneous strong G-refinements for all coefficient groups G, by proving that X is clc over G with the method of 4.2. Two covers α and β are said to *determine* $H^*(X, L)$ up to n, if β refines α and if π^*_β maps $\pi^*_{\alpha\beta}\,H^k(N(\alpha), L)$ isomorphically onto $H^k(X, L)$, $k = 0, \cdots, n$, where $\pi_{\alpha\beta}$ is a support enlarging projection of $N(\beta)$ into $N(\alpha)$ and π^*_β is the canonical map of $H^*(N(\beta), L)$ into $H^*(X, L)$. In such a case, (α, β) is called a *determining L-pair up to n*. The basic facts about the relation between strong refinements and determining pairs were first proved by

Floyd [5] in the case of homology with compact coefficient groups on compact spaces. We call a cover open, closed, or compact according as its sets are open, closed, or compact. Results of this same kind have also been obtained by A. Grothendieck, Bull Soc. Math. France, vol. 84 (1956), pp. 1–7.

We require the following analogue for cohomology:

THEOREM 5.1. *Let X be a locally compact, paracompact space. Let α_0, $\alpha_1, \cdots, \alpha_{2n+1}$ be locally finite closed (resp. compact) covers of X with α_i strongly refining α_{i-1}, $i = 1, \cdots, 2n+1$. Then α_{n+1}, α_{2n+1} determine $H^*(X, L)$ (resp. $H_c^*(X, L)$) up to n.*

A short proof can be obtained by comparing the E_2 terms of the spectral sequence of a cover (see [8] (*c*), p. 20-11) associated with the covers $\alpha_0, \cdots, \alpha_{2n+1}$.

THEOREM 5.2. *Let X be a locally compact, paracompact, finite-dimensional space such that $H^k(X, Z) = 0$ for $k \neq 0$, $H^0(X, Z) = Z$, and X is clc over Z. Let Y be a k-simple, for all k, finite polyhedron, and let f be a continuous map of X into Y. Then f is homotopic to a constant.*

Proof. Let α_f be the cover of X whose sets are the nonempty $f^{-1}(\text{St } y)$, where St y denotes the star of a verex y in Y. Let K_f denote the nerve of α_f, and let σ denote the simplicial map of K_f into Y which sends each vertex $f^{-1}(\text{St } y)$ of K_f into the vertex y of Y. Then σ is a monomorphism of K_f into Y. Let ϕ be any continuous map of X into the geometric complex K_f which is subordinate to α_f; that is, for any set $A \in \alpha_f$ and for any $x \in A$, the image $\phi(x)$ is in St A. (Such a map can be constructed canonically from any partition of unity subordinate to α_f.) Then $\sigma \circ \phi \sim f$ ($\sigma \circ \phi$ is homotopic to f) since $\sigma(\phi(x))$ and $f(x)$ lie in the same simplex of Y for each $x \in X$.

Let n be the classical covering dimension of X. Since X is clc over Z, there exists a sequence of n-dimensional locally finite coverings $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that for all integers q, (α_i, α_{i+1}) is a Z_q-determining pair of covers $(i = 1, \cdots, n-1)$ and α_1 refines α_f. Let $\alpha_0 = \alpha_f$. Let π_i be a refinement projection of $K_i \to K_{i-1}$ $(i = 1, \cdots, n)$ where K_i denotes the nerve of α_i $(i = 0, \cdots, n)$. Set $\tau_i = \sigma \circ \pi_1 \circ \pi_2 \circ \cdots \circ \pi_i$, $\tau_0 = \sigma$, and let K_i^p denote the p-skeleton of K_i $(i = 0, \cdots, n)$.

We prove by induction that $\tau_i \mid K_i^i$, the restriction of τ_i to the i-skeleton of K_i, is homotopic to a constant. This is true when $i = 0$. For $H^0(X, Z) = Z$ implies that X is connected. Since $\sigma(K_f)$ can be described as the smallest closed subcomplex of Y containing $f(X)$, it follows that $\sigma(K_f)$ is connected. Hence $\sigma \mid K_f^0$ is homotopic to a constant map, and the induction is proved for $i = 0$.

Let $y_0 \in \sigma(K_f)$, and let C_i be the constant map of K_i^i into y_0. We prove that τ_i is homotopic to C_i $(i = 0, \cdots, n)$. This has been proved for $i = 0$. Suppose it is true for $i \leqq k$. Then $\tau_k \mid K_k^k$ is homotopic to C_k. As is well known, there exists an extension τ_k' of C_k to a map of K_k with $\tau_k' \sim \tau_k$. Since $\tau_k' \circ \pi_{k+1} \sim \tau_k \circ \pi_{k+1}$, it suffices to prove that $\tau_k' \circ \pi_{k+1} \mid K_{k+1}^{k+1} \sim C_{k+1}$.

Now $\tau_k' \circ \pi_{k+1}$ maps K_{k+1}^k into the point y_0, since $\tau_k'(K_k^k) = y_0$. Consider now the *deviation* cohomology class $\gamma^{k+1}(\tau_k' \circ \pi_{k+1}, C_{k+1})$, which we denote by h^{k+1}. It is an element of $H^{k+1}(K_{k+1}, f^*\lambda_{k+1}(Y))$, where $\lambda_{k+1}(Y)$ is the local system of coefficients defined by the $(k+1)$-dimensional homotopy group of Y, and $f^*\lambda_{k+1}(Y)$ is the induced system on K_{k+1}. By hypothesis, Y is $(k+1)$-simple. Hence $\lambda_{k+1}(Y)$ is a constant system of coefficients, and thus $f^*\lambda_{k+1}(Y)$ is constant also. Inasmuch as $C_{k+1} = C_k' \circ \pi_{k+1}$ where C_k' is the constant map of K_k^{k+1} into y_0, and inasmuch as

$$\gamma^{k+1}(\tau_k' \circ \pi_{k+1}, C_k' \circ \pi_{k+1}) = \pi_{k+1}^* \gamma^{k+1}(\tau_k', C_k'),$$

we obtain $h^{k+1} \in \pi_{k+1}^* H^{k+1}(K_k, f^*\lambda_{k+1}(Y))$; since (K_k, K_{k+1}) is a Z_q-determining pair, we know that

$$\pi_{k+1}^* H^{k+1}(K_k, Z_q) = H^{k+1}(X, Z_q) \qquad \text{for} \quad k \geqq 0, \quad q = 0, 1, 2, \cdots.$$

From the exact cohomology sequence associated with the exact sequence $0 \to Z \to Z \to Z_q \to 0$, we find that $H^{k+1}(X, Z_q) = 0$ for all nonnegative integers k and q. Since Y is a finite polyhedron, the homotopy groups of Y are finitely generated, and therefore $f^*\lambda_{k+1}(Y)$ is a direct sum of groups $Z_{q_1} + \cdots + Z_{q_s}$. Consequently

$$H^{k+1}(X, f^*\lambda_{k+1}(Y)) = H^{k+1}(X, Z_{q_1}) + \cdots + H^{k+1}(X, Z_{q_s}) = 0$$

for all integers $k \geqq 0$. It follows at once that the deviation $h^{k+1} = 0$ and $\tau_k' \circ \pi_{k+1} \sim C_{k+1}$. The induction proof is now complete.

Let ϕ be a map of X into K_n that is canonically associated with a partition of unity subordinate to the cover α_n. Then $\pi_1 \circ \pi_2 \circ \cdots \circ \pi_n \circ \phi$ is subordinate to the cover α_f. Hence

$$f \sim \sigma \circ (\pi_1 \circ \cdots \circ \pi_n \circ \phi) = (\sigma \circ \pi_1 \circ \cdots \circ \pi_n) \circ \phi = \tau_n \circ \phi \sim c_n \circ \phi.$$

Since $c_n \circ \phi$ is the constant map onto the point y_0, the theorem is proved.

COROLLARY 5.3. *Let X be a locally compact, paracompact finite-dimensional space such that $H^k(X, Z) = 0$ for $k \neq 0$, $H^0(X, Z) = 0$, and X is clc over Z. Then any principal fibering with a compact connected Lie group as fibre and X as base space is trivial.*

Proof. Let n be the classical covering dimension of X, and let G be the compact connected Lie group. Let Y be the base space of an n-universal bundle N. As is well known, Y can be taken to be a finite polyhedron. Since N is simply connected and G is connected, Y is simply connected. Any principal G-bundle with X as base is induced by a map f of X into Y, and it is a trivial bundle if and only if f is homotopic to a constant. The corollary is now seen to follow immediately from the theorem above.

6. A cross section of the principal orbits

Suppose that G is an r-dimensional toroid operating effectively on a locally compact space X such that $X \cup \infty =_Z S^n$ with $n \leqq 2r + 1$. Then of course

$n = 2r$ or $2r + 1$ by Theorem 3.1. If P is a star circle subgroup of G, then G/P is almost effective on $F(P, X)$. Consequently $F(P, X)$ has codimension at most 2, by Theorem 3.1. On the other hand, by 2.5, $F(P, X) = F(\pi, X)$ for some p-subgroup of P with p prime; hence $F(P, X)$ has codimension at least 2 (by Lemma 3.6 or [6] or [7]). Thus $F(P, X)$ has codimension 2 for any star circle subgroup P.

Let U denote the subset of points of X whose isotropy subgroup consists only of the identity. By repeated applications of Lemma 3.5 we can describe U as $X - \sum_i F(P_i, X)$ where $P_1, \cdots, P_r$ is the set of star circle subgroups of G. Thus U is a nonempty open dense subset of X which is fibered principally by the orbits of G; that is the fibres are r-dimensional toroids.

Let ϕ denote the projection, and let B denote the base space. Then again by r repeated applications of Lemma 3.5, we see that $B =_Z E^{n-r}$. This means by definition that B is a cohomology manifold over Z, and that $H_c^*(B, Z) = H_c^*(E^{n-r}, Z)$.

THEOREM. *Let X, G, U, B continue to have the same meaning as above. Assume moreover that X is HLC and separable. Then $U = B \times G$.*

Proof. Clearly B is locally compact and separable. Hence it is paracompact. Since G is a compact Lie group, the orbit fibering of U admits local sections. From this one sees readily that B is an HLC space. Knowing that $B =_Z E^{n-r}$, we conclude by Poincaré duality that

$$H^k(B, Z) = H^k(E^{n-r}, Z) = \delta_{k,n-r} Z, \qquad k = 0, 1, \cdots .$$

We can therefore apply Corollary 5.3 to conclude that $U = B \times G$.

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