CLOSE-PACKING AND FROTH

In commemoration of G. A. Miller

BY

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Cannon-balls may aid the truth,
But thought's a weapon stronger;
We'll win our battles by its aid;—
Wait a little longer.

CHARLES MACKAY (1814–1889)
("The Good Time Coming")

1. Algebraic introduction

The abstract groups (2, p, q), defined by

$$R^p = S^q = (RS)^2 = 1,$$

or

$$R^p = S^q = T^2 = RST = 1,$$

or

$$S^q = T^2 = (ST)^p = 1,$$

have been studied intensively ever since Hamilton [11] expressed (2, 3, 5) in the form

$$\iota^2 = \kappa^3 = \lambda^5 = 1, \qquad \lambda = \iota\kappa$$

and wrote, "I am disposed to give the name 'Icosian Calculus' to this system of symbols." Dyck ([8, p. 35]; see also [4, p. 407]) expressed the symmetric and alternating groups

$$\mathfrak{S}_3, \quad \mathfrak{A}_4, \quad \mathfrak{S}_4, \quad \mathfrak{A}_5$$

in the form (2, 3, q) with $q = 2, 3, 4, 5$, respectively. Miller [19, p. 117] remarked that the case when $q = 6$ is entirely different. In fact [20], the group (2, p, q) is finite if and only if

1.1 $$(p - 2)(q - 2) < 4.$$

Thus the finite groups in the family are

the dihedral group (2, 2, q) of order $2q$,
the tetrahedral group (2, 3, 3) of order 12,
the octahedral group (2, 3, 4) of order 24,
the icosahedral group (2, 3, 5) of order 60.

The inequality 1.1 is a necessary and sufficient condition for the finiteness of the number c, which we define to be the period of any one of the elements

$$R^2S^2, \quad R^{-1}S^{-1}RS, \quad TSR, \quad S^{-1}TST.$$

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In the infinite case, Brahana [3] obtained interesting factor groups by assigning a finite period to these elements.

When $(2, 3, q)$ is expressed in the form

$$S^q = T^2 = (ST)^3 = 1,$$

the commutator $TS^{-1}TS = STS \cdot S$ is conjugate to S^3T; thus c is the period of S^3T. If $q = 3$, we have $S^3T = T$, so that $c = 2$. If $q = 4$, $S^3T = (TS)^{-1}$, so that $c = 3$. If $q = 5$, S^3T is conjugate to $S^{-1}TS^{-1} = TST$, so that $c = 5$. In §5 we shall find a natural way to combine these three results in the single formula

$$c = \frac{12}{7 - q} - 1 \qquad (2 < q < 6)$$

and to express the order of $(2, 3, q)$ in the form $2c(c + 1)$.

In §6 we shall obtain the criterion

$$p - 4/p + 2q + r - 4/r < 12$$

for the finiteness of the group [29, p. 217]

1.2 $$R^p = S^q = T^r = (RS)^2 = (RST)^2 = (ST)^2 = 1 \quad (p, q, r > 2).$$

2. Geometric introduction

Hamilton's "Icosian Calculus" and Klein's "Lectures on the Icosahedron" remind us that one of the most significant properties of the polyhedral groups $(2, p, q)$ is their occurrence as finite groups of rotations in three-dimensional Euclidean (or non-Euclidean) space, i.e., as the rotation groups of the Platonic solids $\{p, q\}$. In the words of the late Hermann Weyl [30, pp. 78, 79], "These groups . . . are an immensely attractive subject for geometric investigation."

In the Schläfli symbol $\{p, q\}$, p is the number of vertices (or sides) of a face, and q is the number of faces (or edges) at a vertex; e.g., the cube is $\{4, 3\}$. If such a polyhedron has V vertices, E edges, and F faces, we easily verify that

$$qV = 2E = pF$$

[5, p. 11]. With the aid of Euler's formula $V - E + F = 2$, we can deduce expressions for V, E, F as functions of p and q. For instance, if $q = 3$,

2.1 $$V = \frac{4p}{6 - p}, \qquad E = \frac{6p}{6 - p}, \qquad F = \frac{12}{6 - p}.$$

In 5.1 we shall see why E is always the product of two consecutive integers.

A *Petrie polygon* of $\{p, q\}$ is a skew $2c$-gon whose sides are $2c$ edges of the solid, so chosen that any two consecutive sides, but no three, belong to a face; e.g., the Petrie polygon of a cube is a skew hexagon. Every edge belongs to two faces, and therefore also to two Petrie polygons. Since there are E edges, there are E/c Petrie polygons.

Since a Petrie polygon is symmetrical by a half-turn about the line joining the midpoints of two opposite sides, the midpoints of its $2c$ sides lie in a plane and are the vertices of a plane $2c$-gon, $\{2c\}$.

Reciprocating $\{p, q\}$ with respect to the sphere that touches its edges, we obtain the *reciprocal* polyhedron $\{q, p\}$, which has F vertices, E edges, and V faces. Its edges cross those of $\{p, q\}$ at right angles. Thus the Petrie polygon of $\{q, p\}$ is of the same type as that of $\{p, q\}$, and the plane polygon formed by the midpoints of its sides is the same $\{2c\}$.

In §5 we shall follow the procedure of Steinberg [25] to obtain an explicit formula for c.

A *vertex figure* of $\{p, q\}$ is the plane q-gon, $\{q\}$, whose vertices are the midpoints of the q edges meeting at one vertex, i.e., the section of the solid by the plane through these midpoints. Thus $\{p, q\}$ may be described as having face $\{p\}$ and vertex figure $\{q\}$.

The four-dimensional analogues of the five Platonic solids are the six regular hypersolids, which include the regular simplex $\{3, 3, 3\}$ and the hypercube $\{4, 3, 3\}$. Such a four-dimensional polytope $\{p, q, r\}$ is a configuration of equal polyhedra $\{p, q\}$, called *cells*, fitting together in such a way that each face $\{p\}$ belongs to two cells, and each edge to r cells. It follows that the arrangement of the cells at a vertex corresponds to the arrangement of the faces of a $\{q, r\}$, in the sense that each face of the $\{q, r\}$ is a vertex figure of the corresponding cell. This $\{q, r\}$, whose vertices are the midpoints of the edges at one vertex of $\{p, q, r\}$, is naturally called the *vertex figure* of the polytope [5, p. 129]. Thus $\{p, q, r\}$ may be described as having cell $\{p, q\}$ and vertex figure $\{q, r\}$.

In §6 we shall obtain a new version (6.1) of one of the principal results in *Regular Polytopes* [5, p. 232], and use it to obtain a criterion for the available values of p, q, r.

The notation $\{p, q, r\}$ extends naturally from finite polytopes to infinite honeycombs so as to suggest the symbol $\{4, 3, 4\}$ for the three-dimensional honeycomb of cubes, whose vertices may be taken to be all the points (x, y, z) for which x, y, z are integers. Its cell is the cube $\{4, 3\}$, and its vertex figure is the octahedron $\{3, 4\}$ whose 8 faces are the vertex figures of the 8 cubes that surround a vertex.

In §7 we shall attempt to describe a hypothetical regular honeycomb $\{p, 3, 3\}$, which has a value of p lying between 5 and 6 but still may be regarded as existing in a statistical sense. It provides a theoretical explanation for some experimental results obtained by Matzke and his colleagues.

3. Close-packing

In old war memorials one often sees a pyramidal heap of cannon balls: one at the top resting on four others which, in turn, rest on nine, and so on, the n^{th} horizontal layer containing n^2. Each interior ball touches twelve others: four in its own layer, four above, and four below. The centers of these twelve spheres are the vertices of a cuboctahedron, i.e., they are the mid-edge points of a cube. The shape of the whole square pyramid is just the "top" half of a regular octahedron, since each sloping face is an equilateral triangle formed by $1 + 2 + 3 + \cdots$ cannon balls.

This triangular arrangement suggests the simpler problem of packing equal circles in a plane [9, p. 58], or stacking circular cylinders. Each circle touches six others whose centers are the vertices of a regular hexagon. In other words, the circles are the incircles of the cells of the regular tessellation $\{6, 3\}$, which has three hexagons at each vertex.

One way to pack equal spheres is to begin with a horizontal layer whose "equators" form such a packing of circles. The same arrangement in the next layer above can be so placed that each sphere rests on three, and this can be done in two equivalent ways. When we come to the third layer, the two ways are no longer equivalent [12, p. 46]; [27, p. 170]: in *hexagonal* close-packing each sphere in the third layer is exactly above one in the first layer, but in *cubic* close-packing the repetition is delayed till the fourth layer. It was Barlow [1] who first pointed out that the latter is the same as the normal piling of cannon balls (after the application of a suitable rotation). Hexagonal close-packing and cubic close-packing are equally dense, but the latter is more nearly "isotropic" since the spheres occur in straight rows in six different directions: the centers of the spheres form a *lattice* (in the crystallographic sense). Cubic close-packing is actually the densest lattice-packing. Gauss's original proof has been simplified by Mordell [21] and Dempster [7].

This lattice is called the "face-centered cubic lattice", because it can be derived from the simple cubic lattice $\{4, 3, 4\}$ by taking not only the vertices but also the centers of the square faces. In other words, the spheres are the inspheres of the cells of the honeycomb of rhombic dodecahedra [27, p. 153].

The densest lattice-packing is not necessarily the densest packing. A first suspicion in this direction arises from the existence of equally dense non-lattice packings: the hexagonal close-packing and also several hybrids [18]. This suspicion is increased by carefully examining the twelve neighbors of any one sphere. In the lattice-packing their centers are the vertices of a cuboctahedron (the reciprocal of the rhombic dodecahedron), whose faces consist of 8 triangles and 6 squares. Another familiar polyhedron having 12 vertices is the regular icosahedron, whose faces are 20 triangles. If a sphere is surrounded by 12 equal spheres located in this manner, the 12 spheres, while all touching the first, will not touch one another at all. Accordingly, if we let them roll on the first sphere until they are concentrated in one direction, it seems plausible that they might somehow make room for one more, so that the first sphere would touch thirteen others. According to H. W. Turnbull, who has studied the unpublished notebooks of David Gregory, this idea originated in a conversation of Gregory with Newton about 1694, apropos of the distribution of stars of various magnitudes! It remained an open question till 1953, when its impossibility was established by Schütte and van der Waerden [24]; see also [15]. Although the thirteenth sphere cannot quite touch the one in the middle, it can be pushed in far enough to make a hopeful beginning for a dense packing that might conceivably be continued. Boerdijk [2] describes such a packing in an infinite tubular region, but there is apparently no satisfactory way to fill space by stacking such regions.

Stephen Hales stated, in his *Vegetable Staticks* [10a, pp. 95, 206]: "I compressed several fresh parcels of Pease in the same Pot, with a force equal to 1600, 800, and 400 pounds; in which Experiments, tho' the Pease dilated, yet they did not raise the lever, because what they increased in bulk was, by the great incumbent weight, pressed into the interstices of the Pease, which they adequately filled up, being thereby formed into pretty regular Dodecahedrons."

Marvin [16] and Matzke [17] repeated Hales's experiment, replacing his peas by lead shot, "carefully selected under a microscope for uniformity of size and shape," in a steel cylinder, compressed with a steel plunger at a sufficient pressure (about 40,000 pounds) to eliminate all interstices. They found that, if the shot were stacked in cannon-ball fashion and compressed, nearly perfect rhombic dodecahedra were formed. But "if the shot were just poured into the cylinder the way Hales presumably put his peas into the iron pot, irregular 14-faced bodies were formed. . . . They were never rhombic dodecahedra." Nearly all the faces were either quadrangles, pentagons, or hexagons, with pentagons predominating.

Hulbary [13] examined cells in undifferentiated vegetable tissues, and concluded that the internal cells have an average of approximately 14 faces. Among 650 such cells, chosen without special selection, he found a remarkable variety of shapes. The most prevalent (32 of the 650) had only 13 faces: 3 quadrangles, 6 pentagons, 4 hexagons; 33 edges, and 22 vertices.

4. Froth

Lord Kelvin [14] believed that, of the various polyhedra which can be repeated to fill Euclidean space without interstices, the shape with the smallest surface for its volume is the truncated octahedron, whose faces consist of 8 hexagons and 6 squares. (For this solid he coined the outrageous name "tetrakaidecahedron," as if it were the chief or only polyhedron having fourteen faces! Actually it is one of the thirteen Archimedean solids, and it appears as one of the perspective drawings of models made by Leonardo da Vinci in Fra Luca Paccioli's *Divina proportione* [22, p. 240]. The name *truncum octaëdron* is due to Kepler.) Kelvin's conjecture is supported by the fact that, if S is the surface and C the volume, the value of S^3/C^2 is $150.1 \cdots$ for the truncated octahedron, and $152.8 \cdots$ for the rhombic dodecahedron [9, p. 174].

In the space-filling of truncated octahedra (whose centers form the "bodycentered cubic lattice," consisting of the vertices and cell-centers of the simple cubic lattice), there are three cells at each edge, four at each vertex. To this extent it agrees with the theoretical specification for a froth of approximately uniform bubbles. But the balancing of surface tensions requires equal angles of 120° between the three faces that come together at an edge. Seeing that all the dihedral angles of the truncated octahedron are different from 120° (some greater, some less), Kelvin proposed a modification in which the flat hexagons are replaced by monkey-saddle-shaped minimal surfaces (cf. [12, p. 192], where however, Figure 200 is obviously incorrect).

This "remarkable conformation" [28, p. 552] remained unchallenged till 1940, when Matzke made a microscopic examination of an actual froth of 1900 measured bubbles, each one-tenth of a cc. [17, p. 225]. "For 600 central bubbles examined, the average number of contacts was 13.70 $\cdots$, however, not a single bubble ... had the configuration which Kelvin had described and which Thompson had accepted. The commonest combination was 1-10-2 (118 of 600 bubbles). There were no rhombic dodecahedra". ("1-10-2" means "1 quadrangle, 10 pentagons, 2 hexagons".)

5. The Platonic solids and their characteristic triangles

The planes of symmetry of the regular polyhedron $\{p, q\}$ meet a concentric sphere in great circles which we shall call *circles of symmetry*. They decompose the sphere into $4E$ congruent right-angled spherical triangles $P_0 P_1 P_2$, where P_0 is on the radius through a vertex, P_1 is on the radius through the midpoint of an edge, and P_2 is on the radius through the center of a face [5, p. 24].

We are assuming here that p and q are integers, greater than 2, satisfying 1.1, so that the polyhedron $\{p, q\}$ is nondegenerate. Clearly, the reciprocal polyhedron $\{q, p\}$ yields the same network of spherical triangles with the symbols P_0 and P_2 interchanged.

Figure 1 shows six such *characteristic triangles* in the neighborhood of an edge $P_0 P_0$. The three points marked P_1, being the midpoints of three consecutive sides of a Petrie polygon, are three consecutive vertices of a $\{2c\}$ (see §2). The great circle containing these $2c$ points P_1 is called an *equator* ([5, p. 67], where $2c$ is denoted by h). There are E/c equators: one for each

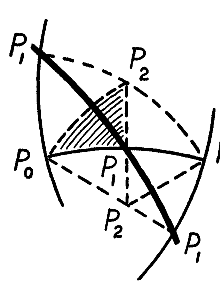

FIGURE 1

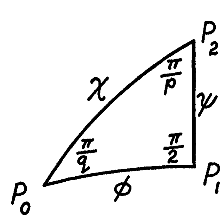

FIGURE 2

Petrie polygon. Since the

$$\frac{E}{c}\left(\frac{E}{c} - 1\right)$$

points of intersection of pairs of equators are just the E points P_1 (which occur in $\frac{1}{2}E$ pairs of antipodes), we have

$$\frac{E}{c}\left(\frac{E}{c} - 1\right) = E,$$

whence

5.1 $$E = c(c + 1).$$

Any equator is crossed by each circle of symmetry in a pair of antipodal points. It is crossed twice at each point P_1 (by two perpendicular circles of symmetry) and once at the midpoint of each arc $P_1 P_1$ (by the hypotenuse $P_0 P_2$ of a triangle $P_0 P_1 P_2$). Hence the number of circles of symmetry (or of planes of symmetry) is $3c$; cf. [5, p. 68].

Figure 3 shows the 9 circles of symmetry (light) and the 4 equators (dark) for the cube $\{4, 3\}$ or the octahedron $\{3, 4\}$, drawn in stereographic projection; cf. [4, frontispiece].

The $3c(3c - 1)$ points of intersection of pairs of circles of symmetry consist of $\frac{1}{2}p(p - 1)$ at each of the F points P_2, $\frac{1}{2}q(q - 1)$ at each of the V points P_0,

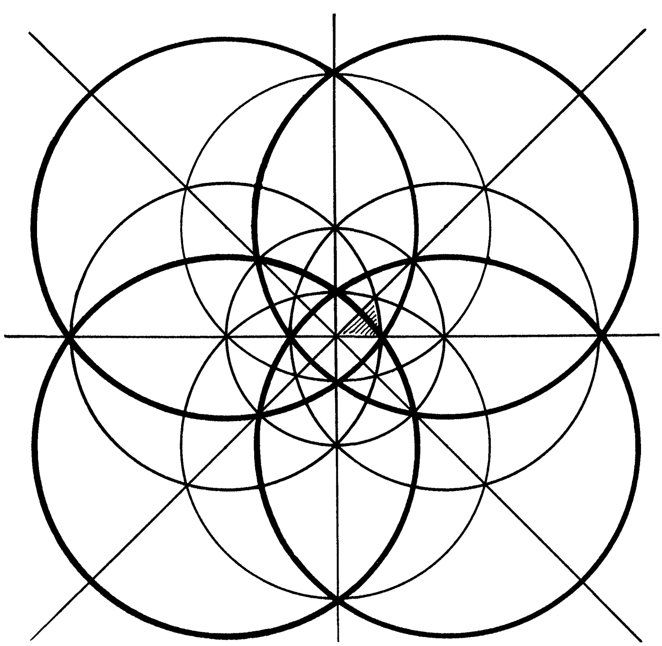

FIGURE 3

and one at each of the E points P_1. Hence

$$3c(3c-1) = \tfrac{1}{2}p(p-1)F + \tfrac{1}{2}q(q-1)V + E$$
$$= (p-1)E + (q-1)E + E$$
$$= (p+q-1)\,c(c+1),$$

and therefore

5.2 $$c + 1 = \frac{12}{10 - p - q}.$$

The connection with the c of §1 is seen by writing

$$R = R_1 R_2, \qquad S = R_2 R_3, \qquad T = R_3 R_1,$$

where R_1, R_2, R_3 are the reflections in the sides $P_1 P_2$, $P_2 P_0$, $P_0 P_1$ of the characteristic triangle. Since $2c$ is the period of the product $R_1 R_2 R_3$ [5, p. 91], c itself is the period of

$$(R_1 R_2 R_3)^2 = RTS = R^2 S^2.$$

The spherical triangle $P_0 P_1 P_2$ has angles π/p at P_2, $\pi/2$ at P_1, π/q at P_0. Let the respectively opposite sides be denoted by ϕ, χ, ψ, as in Figure 2 [5, p. 24]. Since the sides of all the $4E$ triangles are arcs of the $3c$ circles of symmetry, each described twice, we have

$$4E(\phi + \chi + \psi) = 6c \cdot 2\pi,$$

whence

5.3 $$\phi + \chi + \psi = \frac{3c\pi}{E} = \frac{3\pi}{c+1} = \frac{(10-p-q)\pi}{4}$$

[5, p. 74].

6. The regular hypersolids and their characteristic tetrahedra

The hyperplanes of symmetry of a finite polytope $\{p, q, r\}$ (that is, the hyperplanes which act as "mirrors" reflecting the polytope into itself) meet a concentric hypersphere in "great spheres" which we naturally call *spheres of symmetry*. They decompose the hypersphere into (say) g congruent quadrirectangular spherical tetrahedra $P_0 P_1 P_2 P_3$, whose four vertices are on the radii through a vertex, the midpoint of an edge, the center of a face, and the center of a cell [5, p. 139]. This *characteristic tetrahedron* is said to be "quadrirectangular" because all four faces are right-angled triangles. It can most easily be visualized by comparing it with the Euclidean tetrahedron that arises from the analogous simplicial subdivision of the cubic honeycomb $\{4, 3, 4\}$ in ordinary space [5, p. 71], where the edges $P_0 P_1$, $P_1 P_2$, $P_2 P_3$ are not only mutually orthogonal (as they always must be) but straight, and equal in length (since in this case $P_0 P_1$ is half an edge of a cube, $P_1 P_2$ is the inradius of a square face, and $P_2 P_3$ is the inradius of the whole cube). Figure 4 shows

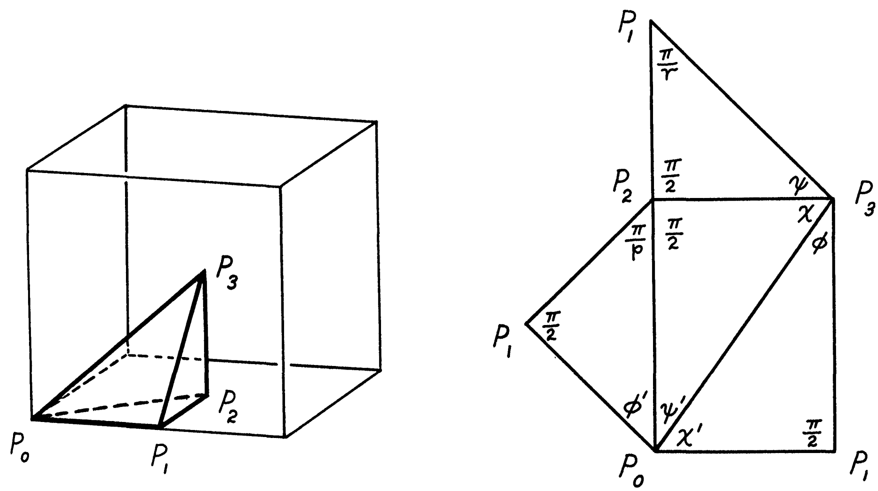

FIGURE 4

an unfolded "net" which the reader may like to copy on thick paper, cut out, and fold up to make a solid model.

The angles between the edges $P_3 P_0$, $P_3 P_1$, $P_3 P_2$ are equal to the sides of the characteristic triangle for the cell $\{p, q\}$, namely

$$\angle P_0 P_3 P_1 = \phi, \qquad \angle P_0 P_3 P_2 = \chi, \qquad \angle P_1 P_3 P_2 = \psi.$$

Since P_2 is the center of a spherical p-gon of which $P_0 P_1$ is half a side, $\angle P_0 P_2 P_1 = \pi/p$. Since the face $P_0 P_1 P_2$ is perpendicular to the edge $P_3 P_2$, $\angle P_0 P_2 P_1$ can alternatively be obtained as the dihedral angle at this edge, which is the angle π/p of the characteristic triangle for $\{p, q\}$.

Similarly, the angles between the edges $P_0 P_1$, $P_0 P_2$, $P_0 P_3$ are equal to the sides (say ϕ', χ', ψ') of the characteristic triangle for the vertex figure $\{q, r\}$, namely

$$\angle P_1 P_0 P_2 = \phi', \qquad \angle P_1 P_0 P_3 = \chi', \qquad \angle P_2 P_0 P_3 = \psi'.$$

Since the face $P_1 P_2 P_3$ is perpendicular to the edge $P_0 P_1$, $\angle P_2 P_1 P_3$ is equal to the dihedral angle at this edge, which is the angle π/r of the characteristic triangle for $\{q, r\}$.

Each sphere of symmetry is tessellated with triangular faces of characteristic tetrahedra. Let a denote the average number of such triangles covering a sphere of symmetry. (This is actually the precise number of triangles, since every sphere of symmetry contains the same number; see [26, Corollary 5.2], where a is denoted by g/h, as in [5, p. 231].) Since each of the g tetrahedra has 4 faces, and each face belongs to 2 tetrahedra, the total number of triangles is $2g$, and the number of spheres of symmetry is $2g/a$.

In terms of the radius of the hypersphere as unit, the total area of the $2g/a$ great spheres is $8\pi g/a$. This must be equal to the sum of the angular excesses of the $2g$ spherical triangles. Since the sum of all the angles of all the triangles is $\frac{1}{2}g$ times the sum of the twelve face-angles of a single tetrahedron (Figure 4), we can use 5.3 to obtain

$$\begin{aligned} 8\pi g/a &= \tfrac{1}{2}g(\phi + \chi + \psi + \phi' + \chi' + \psi' + \pi/p + \pi/r + 4\pi/2) - 2g\pi \\ &= \tfrac{1}{2}g\cdot\tfrac{1}{4}\pi(10 - p - q + 10 - q - r + 4/p + 4/r + 8 - 16) \\ &= \tfrac{1}{8}\pi g(12 - p - 2q - r + 4/p + 4/r), \end{aligned}$$

whence [25]

6.1 $$64/a = 12 - (p - 4/p) - 2q - (r - 4/r).$$

Since a is positive for a four-dimensional polytope and infinite for a three-dimensional honeycomb, we must have

$$p - 4/p + 2q + r - 4/r \leqq 12,$$

with equality only in the case of a honeycomb, namely when

$$p = r = 4 \quad \text{and} \quad q = 3.$$

It is interesting to observe how this algebraic criterion has the same effect as Schläfli's trigonometrical criterion [23, p. 215]

$$\sin(\pi/p)\sin(\pi/r) \geqq \cos(\pi/q).$$

Since the vertex figure $\{q, r\}$ must be one of the five Platonic solids

$$\{3, 3\}, \quad \{3, 4\}, \quad \{4, 3\}, \quad \{3, 5\}, \quad \{5, 3\},$$

the regular hypersolids can be enumerated by assigning these particular values to q, r, and using the inequality

$$p - 4/p \leqq 12 - 2q - r + 4/r$$

or

6.2 $$p^2 - (12 - 2q - r + 4/r)p - 4 \leqq 0$$

to determine the possible values for p.

When $q = 3$ and $r = 3$, we have $p \leqq (13 + \sqrt{313})/6 = 5.115 \cdots$.

When $q = 3$ and $r = 4$, we have $p \leqq 4$.

When $q = 4$ and $r = 3$, we have $p \leqq (7 + \sqrt{193})/6 = 3.48 \cdots$.

When $q = 3$ and $r = 5$, we have $p \leqq (9 + \sqrt{481})/10 = 3.09 \cdots$.

When $q = 5$ and $r = 3$, we have $p \leqq (1 + \sqrt{145})/6 = 2.17 \cdots$.

Thus the only finite polytopes $\{p, q, r\}$ are Schläfli's

$$\{3, 3, 3\}, \quad \{4, 3, 3\}, \quad \{5, 3, 3\}, \quad \{3, 3, 4\}, \quad \{3, 4, 3\}, \quad \{3, 3, 5\}.$$

A connection with group-theory is seen in Todd's observation that the direct symmetry-operations of the polytope $\{p, q, r\}$ constitute a group of order $\frac{1}{2}g$ for which the relations 1.2 provide an abstract definition. In fact,

$$R = R_1 R_2, \qquad S = R_2 R_3, \qquad T = R_3 R_4,$$

where R_1, R_2, R_3, R_4 are the reflections in the faces $P_1 P_2 P_3$, $P_2 P_3 P_0$, $P_3 P_0 P_1$, $P_0 P_1 P_2$ of the characteristic tetrahedron.

7. A statistical honeycomb

As we remarked in §3, the closest packing of equal circles in the Euclidean plane is provided by the incircles of the cells of the regular tessellation of hexagons $\{6, 3\}$. In fact, three equal circles are packed as closely as possible when they all touch one another, and the two-dimensional packing problem is easy because any number of further circles can be added in such a way as to continue the pattern systematically over the whole plane.

Analogously, four equal spheres are packed as closely as possible when they all touch one another, and some further spheres can be added so as to form the beginning of a pattern apparently consisting of the inspheres of the cells of a regular honeycomb $\{p, 3, 3\}$. This beginning can be continued for spheres of a suitable size in spherical (or elliptic) space with $p = 5$, and again in hyperbolic space with $p = 6$ [10, p. 159]; [6, p. 266]. The conclusion is inescapable that a compressed close-packing of equal lead shot, or a froth of equal bubbles, is trying to approximate to a Euclidean honeycomb $\{p, 3, 3\}$ in which p lies between 5 and 6. The fractional value of p means that this "honeycomb" exists only in a statistical sense, but the agreement with experiment is striking.

Setting $q = r = 3$ in the equation

7.1 $$p^2 - (12 - 2q - r + 4/r)p - 4 = 0$$

(cf. 6.2), we obtain

$$p^2 - (13/3)p - 4 = 0,$$

whence

$$p = (13 + \sqrt{313})/6 = 5.115 \cdots,$$

in agreement with Matzke's observation that pentagons are prevalent (especially in froth) while hexagons are more frequent than quadrangles.

The cell $\{p, 3\}$ has an average of F faces, E edges, and V vertices, where, by 2.1,

$$F = \frac{12}{6 - p} = \frac{23 + \sqrt{313}}{3} = 13.56 \cdots,$$

$$E = \frac{6p}{6 - p} = 17 + \sqrt{313} = 34.69 \cdots,$$

and $V = \frac{2}{3}E = 23.13 \cdots$.

In conclusion, I wish to thank Michael Goldberg and John Satterly for drawing my attention to the experimental work of Matzke, whose estimate

$$F = 13.70$$

(see §4) motivated my choice of the equation 7.1 in place of the equally plausible equation

$$\sin(\pi/p)\sin(\pi/r) = \cos(\pi/q),$$

from which the substitution $q = r = 3$ yields $p = \pi/\kappa$ in the notation of [5, p. 293], whence

$$\frac{12}{6-p} = 13.398\cdots, \qquad \frac{4p}{6-p} = 22.796\cdots.$$

That other approach has been extended to n dimensions by Rogers [22a].

Note added in proof. Professor Bernal [1a] has used two independent experiments to obtain for F the approximate values 13.6 and 13.3 He refers to Meijering [17a, p. 282], who applied statistical methods of an entirely different kind to the so-called "Johnson-Mehl model," obtaining

$$V = 22.56.$$

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