PROPAGATION OF CURVED SHOCKS IN PSEUDO-STATIONARY THREE-DIMENSIONAL GAS FLOWS[1]

BY

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1. Introduction

Recently [1, 2] we discussed the curved shocks in three-dimensional steady gas flows. In that discussion formulas were derived which make possible the determination of the derivatives of velocity, density, pressure, and entropy behind the shock surface when the flow in front is known. Furthermore the explicit determination of the vorticity components behind the shock was made. That led to the formulation of a general theorem regarding the characterization of surfaces behind which the flow will remain irrotational. It was found that a plane, a right circular cone, a cylinder, and a developable helicoid are the only such surfaces. The main purpose of this paper is to discuss the same problem in the case of unsteady flows. In the case of plane unsteady flows Taub [3] has solved the corresponding problem by introducing a dimensional argument which indicates that, when viscosity and heat conductivity are neglected, there is no intrinsic length in the problem and the problem may be stated in terms of the independent variables x^i/t alone instead of x^i and t. In this notation x^i are the cartesian orthogonal coordinates, and t is the time. The same argument has been introduced in the following analysis.

In this discussion it is found that at time t the flow behind a shock wave will be irrotational if the shock is plane, or if it propagates normal to itself relative to the fluid, or if it is developable and the direction of its propagation, relative to the fluid, at every point on it lies in the plane determined by the generator and the normal to the shock surface at that point. It is further found that, as in the case of steady flows, the component of the vorticity normal to the shock vanishes at every point on the shock.

2. Equations of motion

The motion of a gas when the effects of viscosity and heat conduction may be neglected is described by the following set of differential equations:

(1) $$\frac{d\rho}{dt} + \rho \frac{\partial u^i}{\partial x^i} = 0, \qquad \text{(equation of continuity)},$$

(2) $$\rho \frac{du^i}{dt} + \frac{\partial p}{\partial x^i} = 0, \qquad \text{(equations of motion)},$$

(3) $$\frac{dS}{dt} = 0, \qquad \text{(changes of state are adiabatic)},$$

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where summation over index i is implied by its repetition in a term. The symbols u^i, S, p, and ρ represent, respectively, the components of velocity, entropy, pressure, and density in the gas. The derivative d/dt is the material derivative which indicates the variation on following a particle of the fluid. Since there is no distinction between covariant and contravariant indices within a rectangular system, we may write any index as a superscript or subscript without modification of the value of the term in which the index occurs.

The shock configuration in a three-dimensional gas flow at time t may be represented by the equations:

$$x^i = t\, a^i(y_1, y_2), \tag{4}$$

where y_1 and y_2 are the Gaussian coordinates on the shock surface. This substitution leads to the following form of the differential operators in the equations (1) to (3):

$$\frac{dg}{dt} = \frac{1}{t}(u^i - a^i)g_{,i} = \frac{1}{t}U^i g_{,i}, \tag{5}$$

$$\frac{\partial g}{\partial x^i} = \frac{1}{t}g_{,i}, \tag{6}$$

where $g(x^1/t, x^2/t, x^3/t)$ is any function into which the coordinates and time enter in the manner indicated and the derivative $\partial g/\partial a^i$ has been written $g_{,i}$. The symbol U^i is defined in the equation (5).

When this substitution is introduced into the equations (1) to (3), they become

$$U_i\,\rho_{,i} + \rho U_{i,i} + 3\rho = 0, \tag{7}$$

$$\rho U_i + \rho U_j\, U_{i,j} + p_{,i} = 0, \tag{8}$$

$$U^i S_{,i} = 0. \tag{9}$$

A flow which meets these requirements is called pseudo-stationary [4]. From (4) we notice that the components of the velocity of the shock are given by

$$v_i = a_i\,. \tag{10}$$

Since the u_i are the components of the particle velocity, the quantities $U_i = u_i - a_i$, when evaluated on either side of the shock front, give the velocity of the flow relative to the shock at the corresponding points. The Rankine-Hugoniot equations may be written as

$$[U_i] = -\frac{\delta U_{1n}\xi_i}{1+\delta}, \tag{11}$$

$$[p] = \frac{\delta}{1+\delta}\rho_1 U_{1n}^2, \tag{12}$$

$$[\rho] = \delta\rho_1, \tag{13}$$

where

$$\delta = \frac{2(\rho_1 U_{1n}^2 - \gamma p_1)}{2\gamma p_1 + (\gamma - 1)\rho_1 U_{1n}^2}, \tag{14}$$

and the bracket [ ] denotes the difference of the values on the two sides of the shock surface of the quantity enclosed. ξ_i are the components of the unit normal vector to the shock. The normal is assumed to be directed from region 1 into region 2.

3. Derivation of the partial derivatives of the flow quantities behind the shock surface

As in the analysis of steady gas flows [1, 2] we take the lines of curvature as the Gaussian coordinates y_1, y_2. In the following we shall use Latin letters for the indices referring to the space variables and Greek letters for the indices referring to the surface variables. Thus the Latin indices will assume values 1, 2, 3; and Greek indices the values 1, 2.

The surface unit tangent vectors to the coordinate curves are $\delta_1^\alpha/\sqrt{g_{11}}$ and $\delta_2^\alpha/\sqrt{g_{22}}$ where $g_{\alpha\beta}$ are the components of the first fundamental form of the surface [5]. The corresponding space components of the unit tangent vectors are $x_1^i/\sqrt{g_{11}}$ and $x_2^i/\sqrt{g_{22}}$, respectively, where we have put

$$x_\alpha^i = \frac{\partial x^i}{\partial y^\alpha} = t\,\frac{\partial a^i}{\partial y^\alpha}. \tag{15}$$

As in the steady case we define the dimensionless variables

$$\tau_\alpha = \frac{V_\alpha}{\sqrt{g_{\alpha\alpha}}U_{1n}}, \qquad \chi_\alpha = \frac{V_\alpha}{\sqrt{g_{\alpha\alpha}}U_{2n}}, \qquad M_{1n} = \frac{U_{1n}}{c_1}, \qquad M_{2n} = \frac{U_{2n}}{c_2}, \tag{16}$$

where α is not summed, $V_\alpha = U_i x_\alpha^i$, and $c^2 = \gamma p/\rho$. It is clear that $V_\alpha/\sqrt{g_{\alpha\alpha}}$ (α not summed) are the physical components of the velocity field U_i along the lines of curvature on the shock surface.

Differentiating the relations (11) to (13) with respect to y^1 and y^2, we get relations along the shock surface of the form

$$U_{i,j}\, x_\alpha^j/t = U_{1i,j}\, x_\alpha^j/t + A_{i\alpha} = A_{i\alpha}^*, \tag{17}$$

$$p_{,j}\, x_\alpha^j/t = p_{1,j}\, x_\alpha^j/t + B_\alpha = B_\alpha^*, \tag{18}$$

$$\rho_{,j}\, x_\alpha^j/t = \rho_{1,j}\, x_\alpha^j/t + C_\alpha = C_\alpha^*, \tag{19}$$

where for simplicity we have omitted the subscript 2 on the quantities appearing in the left members of these equations. The explicit values of $A_{i\alpha}$, B_α and C_α are obtained by differentiating the right members of the equations (11) to (13).

We first eliminate the derivatives $p_{,i}$ from the six relations given by (7), (8), (9), (17), (18), and (19), and obtain the following equations, which we represent as two sets of equations for convenience of reference, namely,

$$\rho_{,j}\, x^j_\alpha = t\, C^*_\alpha\,, \tag{20}$$

$$\rho_{,j}\, U_j + \rho U_{k,k} + 3\rho = 0, \tag{21}$$

and

$$U_{i,j}\, x^j_\alpha = t\, A^*_{i\alpha}\,, \tag{22}$$

$$\rho U_{i,j}\, x^i_\alpha U_j = -\,(t\, B^*_\alpha + \rho V_\alpha), \tag{23}$$

$$U_i\, U_j\, U_{i,j} - c^2 U_{i,i} = -\,q^2 + 3c^2, \tag{24}$$

where $q^2 = U_i\, U_i$. Now from (22) to (24) we can obtain $U_{i,j}$, and then from (20) and (21) $\rho_{,i}$ can be found; after this the equation (8) yields the value $p_{,i}$. As in the steady case this determination can be effected by introducing the matrix:

$$\begin{Vmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{Vmatrix} = \begin{Vmatrix} \lambda^1_1 & \lambda^2_1 & \lambda^3_1 \\ \lambda^1_2 & \lambda^2_2 & \lambda^3_2 \\ \dfrac{U_1}{U_n} & \dfrac{U_2}{U_n} & \dfrac{U_3}{U_n} \end{Vmatrix}, \tag{25}$$

where λ^i_α denote the space components of the unit tangent vector to the y^α curve and are given by the relation

$$\lambda^i_\alpha = x^i_\alpha / \sqrt{g_{\alpha\alpha}}\,, \qquad \alpha \text{ not summed.} \tag{26}$$

The determinant of this matrix can be easily seen to be unity. Hence we can define the quantities D_{ik} by

$$D_{ik}\, C_{kj} = \delta_{ij}\,, \qquad \text{or} \qquad D_{ki}\, C_{jk} = \delta_{ij}\,. \tag{27}$$

The second relation (27) follows readily from the first relation, and conversely. The matrix $\| D_{ij} \|$ is given by

$$\begin{Vmatrix} D_{11} & D_{21} & D_{31} \\ D_{12} & D_{22} & D_{32} \\ D_{13} & D_{23} & D_{33} \end{Vmatrix} = \begin{Vmatrix} \dfrac{(\lambda^2_2 U_3 - \lambda^3_2 U_2)}{U_n} & \dfrac{(\lambda^3_1 U_2 - \lambda^2_1 U_3)}{U_n} & \xi_1 \\ \dfrac{(\lambda^3_2 U_1 - \lambda^1_2 U_3)}{U_n} & \dfrac{(\lambda^1_1 U_3 - \lambda^3_1 U_1)}{U_n} & \xi_2 \\ \dfrac{(\lambda^1_2 U_2 - \lambda^2_2 U_1)}{U_n} & \dfrac{(\lambda^2_1 U_1 - \lambda^1_1 U_2)}{U_n} & \xi_3 \end{Vmatrix}. \tag{28}$$

Now define the quantities B_{ij} by

$$B_{ij} = U_{l,m}\, C_{li}\, C_{mj}\,. \tag{29}$$

Then from (22), (23), (24), and (29) we have

$$\text{(30)} \quad \begin{Vmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{Vmatrix} = \begin{Vmatrix} \dfrac{A^*_{i1} x^i_1 t}{g_{11}} & \dfrac{A^*_{i2} x^i_1 t}{\sqrt{g_{11} g_{22}}} & \dfrac{-(B^*_1 t + \rho V_1)}{\rho \sqrt{g_{11}}\, U_n} \\ \dfrac{A^*_{i1} x^i_2 t}{\sqrt{g_{11} g_{22}}} & \dfrac{A^*_{i2} x^i_2 t}{g_{22}} & \dfrac{-(B^*_2 t + \rho V_2)}{\rho \sqrt{g_{22}}\, U_n} \\ \dfrac{A^*_{i1} U_i t}{\sqrt{g_{11}}\, U_n} & \dfrac{A^*_{i2} U_i t}{\sqrt{g_{22}}\, U_n} & \dfrac{(U_{k,k} + 3)}{M^2_n} - \dfrac{q^2}{U^2_n} \end{Vmatrix}.$$

If we multiply both sides of (29) by $D_{is} D_{jt}$ and use the relations (27), we readily obtain

$$\text{(31)} \qquad U_{l,m} = B_{ij} D_{il} D_{jm}.$$

The first eight elements of the matrix can be determined from the right hand sides of the equations (17) and (18). However, we must still express B_{33} in terms of similar quantities. Setting $l = m$ in (31) and using (28), we get

$$\text{(32)} \qquad \begin{aligned} U_{k,k} = B_{11}(1 + \chi_1^2) + B_{22}(1 + \chi_2^2) + 2 B_{(12)} \chi_1 \chi_2 \\ - 2B_{(13)} \chi_1 - 2B_{(23)} \chi_2 + B_{33}, \end{aligned}$$

where χ's are defined by the relations (16) and for brevity we have introduced

$$B_{(ij)} = \tfrac{1}{2}(B_{ij} + B_{ji}).$$

Thus we arrive at the value of B_{33}, namely

$$\text{(33)} \qquad B_{33} = \frac{\{B_{11}(1 + \chi_1^2) + B_{22}(1 + \chi_2^2) + 2B_{(12)} \chi_1 \chi_2 - 2B_{(13)} \chi_1 - 2B_{(23)} \chi_2\} - M^2}{M^2_n - c^2},$$

where $M = q/c$ and M_n is defined by the relation (16).

To effect the determination of the derivatives $\rho_{,j}$ behind the shock, we observe that the set of equations (20) and (21) can be written in the form

$$\text{(34)} \qquad \rho_{,j} C_{ji} = d_i,$$

where

$$\text{(35)} \quad d_1 = C^*_1 t/\sqrt{g_{11}}, \qquad d_2 = C^*_2 t/\sqrt{g_{22}}, \qquad d_3 = -\rho(U_{k,k} + 3)/U^2_n.$$

Hence we have

$$\text{(36)} \qquad \rho_{,j} = d_i D_{ij}.$$

Finally from (8) the derivatives $p_{,i}$ behind the shock can be determined by the equations

$$\text{(37)} \qquad p_{,i} = -\rho U_j U_{i,j} - \rho U_i = -\rho U_n B_{j3} D_{ji} - \rho U_i.$$

4. Calculation of the invariants d_i and B_{ij}

It is possible to give an explicit formulation of the invariants d_i and B_{ij} under the assumption that the flow in front of the shock surface is uniform, i.e. u_i, p_1, and ρ_1 are constants, in a manner analogous to the steady case. In addition to the expressions $g_{\alpha\beta}$, x^i_α, and λ^i_α introduced in the previous section, we need some more results of the differential geometry [4] which we write down for the sake of ready reference. The expressions for normal curvatures in the direction of the coordinate curves are

$$K_1 = b_{11}/g_{11}\,; \qquad K_2 = b_{22}/g_{22}\,, \tag{38}$$

where $b_{\alpha\beta}$ are the components of the second fundamental form of the shock surface. As we have taken the lines of curvature as the coordinate curves on the shock, we have

$$g_{12} = b_{12} = 0. \tag{39}$$

Moreover, if $g^{\alpha\beta}$ denotes the components of the tensor conjugate to $g_{\alpha\beta}$, i.e.,

$$g_{\alpha\beta}\, g^{\alpha\gamma} = \delta^\gamma_\beta\,, \tag{40}$$

we obtain

$$g^{11} = g_{22}/g\,, \qquad g^{12} = g^{21} = 0, \qquad g^{22} = g_{11}/g, \tag{41}$$

where $g = |\, g_{ij} \,| = g_{11}\, g_{22}$. In this notation the space components of the unit normal vector to the surface are given by the relations

$$\xi^i = \tfrac{1}{2}\varepsilon^{\alpha\beta} e_{ijk}\, x^j_\alpha\, x^k_\beta\,, \tag{42}$$

where $\varepsilon^{\alpha\beta}$ and e_{ijk} are, respectively, the components of the surface and space permutation tensors. The Weingarten's formulas give

$$\xi^i_{;\alpha} = -\, g^{\gamma\beta} b_{\alpha\gamma}\, x^i_\beta\,, \tag{43}$$

where the subscript ; denotes the surface covariant differentiation.

With the help of these results it can be readily found that

$$U_{1n;\alpha} = U_{1i}\,\xi^i_{;\alpha} = -\, V_\alpha K_\alpha\,, \qquad \alpha \text{ not summed}, \tag{44}$$

and

$$\xi^i_{;\alpha} = -\, K_\alpha\, x^i_\alpha\,, \qquad \alpha \text{ not summed}. \tag{45}$$

Now for uniform flow

$$A^*_{i\alpha} = -\, x^i_\alpha/t + A_{i\alpha}\,; \qquad B^*_\alpha = B_\alpha\,; \qquad C^*_\alpha = C_\alpha\,. \tag{46}$$

The components of the matrix (30) i.e. $\|\, B_{ij} \,\|$ can now be easily calculated and are given as

$$\begin{Vmatrix} (\delta K_1 U_n^{-1})t & 0 & \dfrac{4V_1 K_1 t}{\sqrt{g_{11}}(\gamma+1)} - \dfrac{V_1}{\sqrt{g_{11}}U_n} \\ 0 & (\delta K_2 U_n^{-1})t & \dfrac{4V_2 K_2 t}{\sqrt{g_{22}}(\gamma+1)} - \dfrac{V_2}{\sqrt{g_{22}}U_n} \\ Z\dfrac{V_1 K_1 t}{\sqrt{g_{11}}} - \dfrac{V_1}{\sqrt{g_{11}}U_n} & Z\dfrac{V_2 K_2 t}{\sqrt{g_{22}}} - \dfrac{V_2}{\sqrt{g_{22}}U_n} & B_{33} \end{Vmatrix}, \tag{47}$$

where

$$Z = \frac{4}{\gamma + 1} + \frac{\delta^2}{1 + \delta}$$

and B_{33} is obtained from the relation (33) by substituting the values of the quantities B_{ij} from the above matrix.

Correspondingly the d_i are given by the expressions

$$(48) \quad \begin{cases} d_1 = \dfrac{C_1 t}{\sqrt{g_{11}}} = \dfrac{2}{\gamma + 1} \dfrac{V_1 K_1 t}{\sqrt{g_{11}}\, U_n} \{(\gamma - 1)\rho - (\gamma + 1)\rho_1\}, \\ d_2 = \dfrac{C_2 t}{\sqrt{g_{22}}} = \dfrac{2}{\gamma + 1} \dfrac{V_2 K_2 t}{\sqrt{g_{22}}\, U_n} \{(\gamma - 1)\rho - (\gamma + 1)\rho_1\}, \\ d_3 = -\rho(U_{k,k} + 3), \end{cases}$$

where $U_{k,k}$ is known from the relations (32) and (47).

5. The expression for the vorticity vector behind the shock

The components of the vorticity vector w_i behind the shock are given by

$$(49) \quad \begin{aligned} w_i &= e_{ijk}\, \partial U_k/\partial x^j = (1/t)\, e_{ijk}\, U_{k,j} = (1/t)\, e_{ijk}\, B_{lm}\, D_{lk}\, D_{mj} \\ &= \left(-\frac{K_1 V_1}{\sqrt{g_{11}}} \lambda_2^i + \frac{K_2 V_2}{\sqrt{g_{22}}} \lambda_1^i \right) \frac{\delta^2}{1 + \delta}, \end{aligned}$$

wherein we have made use of the relations (6), (28), and (31). Now for w_i to be zero at time t, it is necessary as well as sufficient that either $\delta^2/(1 + \delta)$ or

$$\left(-\frac{K_1 V_1}{\sqrt{g_{11}}} \lambda_2^i + \frac{K_2 V_2}{\sqrt{g_{22}}} \lambda_1^i \right)$$

vanish. In the latter case coefficients of both λ_1^i and λ_2^i must vanish because λ_1^i and λ_2^i are perpendicular to each other. Thus in order that the flow be irrotational behind the shock surface at least one of the following conditions must be satisfied:

(i) $\delta = 0$, i.e. *no shock surface*;
(ii) $K_1 = 0 = K_2$, i.e. *the shock surface is plane*;
(iii) $V_1 = V_2 = 0$, *i.e. the shock propagates in the direction normal to itself relative to the fluid*;
(iv) $K_1 = 0 = V_2$;
(v) $K_2 = 0 = V_1$.

It is clear that (iv) and (v) lead to the same result. Let us take the case (v), i.e. $K_2 = 0 = V_1$. Now $K_2 = 0$ implies that the shock is a developable surface, and as such its generators and their orthogonal trajectories form its congruences of lines of curvature; while $V_1 = 0$ implies that every point on these orthogonal trajectories is propagated, relative to the fluid, in the plane determined by the generator and the normal to the surface through that point.

Thus *the flow behind the shock wave will be irrotational if one of the following conditions holds*:

1. *The shock surface is a plane.*

2. *The wave propagates normal to itself relative to the fluid.*

3. *The shock surface is developable, and at every point the direction of propagation of the shock, relative to the fluid, is in the plane determined by the generator and the normal to the surface at that point.*

Furthermore, $w_i \xi^i = 0$, i.e. the component of the vorticity normal to the shock vanishes at every point of the shock. This was also found to hold in the case of steady gas flows [1, 2].

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