ON THE EXISTENCE OF THE STIELTJES MEAN σ-INTEGRAL

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The object of this paper is to establish a necessary and sufficient condition (Theorem 1) for the existence of the Stieltjes mean σ-integral $\int_a^b fdg$, where f is a bounded function on $[a, b]$ and g is of bounded variation on $[a, b]$. We then use this condition to settle a question raised by T. H. Hildebrandt[2] (Corollary 1.1) and to show that the mean σ-integral is absolutely convergent[3] (Corollary 1.2) when f and g satisfy the above condition and $\int_a^b fdg$ exists. We base our proof of Theorem 1 on an earlier existence theorem due to T. H. Hildebrandt.

We recall that the Stieltjes mean σ-integral is one of several limits introduced by H. L. Smith (cf. [2]) and is defined as follows: if each of f and g is a function on $[a, b]$, then the statement that the number J (hereafter denoted by $\int_a^b fdg$) is the Stieltjes mean σ-integral of f with respect to g on $[a, b]$ means that for each $\varepsilon > 0$, there exists a subdivision D of $[a, b]$ such that if E is any refinement of D, then $| \int_a^b fdg - S_E(f, g; I) | < \varepsilon$, where

$$S_E(f, g; I) = \sum_{i=0}^{m} 2^{-1}[f(y_i + 1) + f(y_i)][g(y_{i+1}) - g(y_i)],$$

y_i, $i = 0, 1, \cdots, m + 1$, are the terms of E, and I denotes the interval $[a, b]$.

If for each subinterval I' of $[a, b]$, we set

$$w(f, g; I') = \text{LUB } [| S_E(f, g; I') - S_F(f, g; I') |; \text{all subdivisions } E \text{ and } F \text{ of } I'],$$

then, according to Hildebrandt (cf. [1], Theorem 2.13), $\int_a^b fdg$ exists if, and only if,

$$\text{GLB } [\textstyle\sum_D w(f, g; I_p); D] = 0,$$

where D is a subdivision of $[a, b]$ and I_p, $p = 0, 1, \cdots, n$, are the subintervals formed by D.

We shall need the following definitions and lemmas.

DEFINITION 1.1. If f is a bounded function on $[a, b]$ and k a positive number, then $M(f, k+)$ denotes a subset of $[a, b]$ such that x is in $M(f, k+)$ if, and only if, $x \neq b$ and for each $y > x$ there exist points t and t' in the segment (x, y) such that $t < b$, $t' < b$, and $| f(t) - f(t') | \geqq k$. $M(f, k-)$ denotes a subset of $[a, b]$ such that x is in $M(k, k-)$ if, and only if, $x \neq a$ and for each $y < x$

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[2] Cf. [1], p. 274.

[3] H. S. Wall, private communication.

there exist points t and t' in the segment (y, x) such that $a < t$, $a < t'$, and $| f(t) - f(t') | \geqq k$. $M(f, k)$ denotes the logical sum $M(f, k+) + M(f, k-)$.

DEFINITION 1.2. If f is a bounded function on $[a, b]$, g a nondecreasing function on $[a, b]$, and $k > 0$, then the statement that $M(f, k)$ has directed content $C_g M(f, k)$ means that $C_g M(f, k)$ is the largest nonnegative number S such that if $\{[a_p, b_p]\}_{p=1}^{n}$ is a finite collection of nonoverlapping subintervals of $[a, b]$ satisfying

(i) a_p is not in $M(f, k-)$ and b_p is not in $M(f, k+)$ for $p = 1, \cdots, n$, and

(ii) $\{[a_p, b_p]\}_{p=1}^{n}$ covers $M(f, k)$,

then $\sum_{p=1}^{n} [g(b_p) - g(a_p)] \geqq S$.

LEMMA 1. *If f is a bounded function on $[a, b]$, g of bounded variation on $[a, b]$, $k > 0$, and $[c, d]$ a subinterval of $[a, b]$ such that, either c is in $M(f, k+)$, d is in $M(f, k-)$, or there exists a point t common to the segment (c, d) and $M(f, k)$, then*

$$w(f, g; [c, d]) \geqq 2^{-1}k \, | g(d) - g(c) | .$$

Proof. Suppose $k > \varepsilon > 0$. There exist points t' and t'' in the segment (c, d) such that,

$$| f(t') - f(t'') | \geqq k - \varepsilon,$$

$$[f(t') - f(t'')]\cdot[g(d) - g(c)] = | f(t') - f(t'') | \cdot | g(d) - g(c) | ,$$

and

$$| g(t') - g(t'') | \leqq \varepsilon N^{-1},$$

where N is an upper bound of $| f |$ on $[a, b]$. If E and D are subdivisions of $[c, d]$ consisting of just the terms $c < t' < d$ and $c < t'' < d$ respectively, then

$$\begin{aligned} |S_E(f, g; [c, d] - S_D(f, g; [c, d] | &\geqq 2^{-1} | [f(t') - f(t'')][g(d) - g(c)] \\ &+ [f(c) - f(d)][g(t') - g(t'')] | \geqq 2^{-1}(k - \varepsilon) | g(d) - g(c) | - \varepsilon. \end{aligned}$$

Since this holds for all $\varepsilon > 0$, we get

$$w(f, g; [c, d]) \geqq 2^{-1}k \, | g(d) - g(c) |.$$

LEMMA 2. *If f is a bounded function on $[a, b]$, g of bounded variation on $[a, b]$, $k > 0$, and $[c, d]$ a subinterval of $[a, b]$ such that,*

(1′) *c is not in $M(f, k+)$,*

(2′) *d is not in $M(f, k-)$, and*

(3′) *for each x in the segment (c, d), x is not in $M(f, k)$,*

then there exists a finite collection $\{[y_{i+1}, y_i]\}_{i=1}^{m}$ of nonoverlapping subintervals of $[c, d]$ covering $[c, d]$ such that

$$\sum_{i=1}^{m} w(f, g; [y_{i+1}, y_i]) \leqq 6k[g^*(d) - g^*(c)],$$

where g^* is the variation function of g.

Proof. Suppose the lemma is false and c' denotes the midpoint of $[c, d]$. In view of (3′), we see that each one of the intervals $[c, c']$ and $[c', d]$ satisfies

(1′), (2′), and (3′). Consequently, let $\{I_n\}_{n=1}^{\infty}$, $I_1 = [c, d]$, be a sequence of nested intervals having just one point, say t, in common such that, for each n, I_n satisfies the hypotheses of the lemma, but for which the conclusion is false. Suppose $t < d$. There exists a point s' such that $t < s' < d$ and for any point s in the segment (t, s'), $|f(x) - f(y)| \leqq k$ for $t < x \leqq s$ and $t < y \leqq s$. Let N be an upper bound of $|f|$ on $[a, b]$ and γ a nonnegative number such that $\gamma \leqq k(2N)^{-1}[g^*(s) - g^*(t)]$. There exists a point z in the segment (t, s) such that $|g(u) - g(v)| < \gamma$ for $t < u \leqq z$ and $t < v \leqq z$.

If A and B are subdivisions of $[t, z]$, then

$$| S_A(f, g; [t, z]) - S_B(f, g; [t, z]) | \leqq 2k[g^*(z) - g^*(t)];$$

if D and E are subdivisions of $[z, s]$, then

$$| S_D(f, g; [z, s]) - S_E(f, g; [z, s]) | \leqq (k)\cdot[g^*(s) - g^*(z)].$$

Similarly, if $c < t$, then there exists a point r' in the segment (c, t) such that for any point r in the segment, there exists a point y in the segment (r, t) which has the same properties as the point z has in the segment (t, s).

Suppose, now, I_n is an interval in the nested sequence which is also a subinterval of $[r', s']$ such that t is an interior point of I_n, which is the worst case. If we pick the r and s so that $[r, s] = I_n$, then $[r, y]$, $[y, t]$, $[t, z]$, and $[z, s]$ is a collection of nonoverlapping subintervals of I_n satisfying the conclusion of the lemma. This contradiction completes the proof of Lemma 2.

THEOREM 1. *If f is a bounded function on $[a, b]$ and g is of bounded variation on $[a, b]$, then a necessary and sufficient condition that the mean σ-integral $\int_a^b f dg$ exist is that for each positive number k, $C_{g^*} M(f, k) = 0$, where g^* is the variation function of g on $[a, b]$.*

Proof. In order to prove the necessity, let us suppose $\int_a^b f dg$ exists, $k > 0$, $C_{g^*} M(f, k) > 0$, and $\varepsilon > 0$. Let F be a subdivision of $[a, b]$ and E a refinement of F such that, (1) $| \int_a^b f dg - S_E(f, g; [a, b]) | < \varepsilon$, and (2) if x_i, $x = 0, 1, \cdots, n + 1$, denote the terms of E, then $\sum_{i=0}^{n} | g(x_{i+1}) - g(x_i) | > \sum_{i=0}^{n}[g^*(x_{i+1}) - g^*(x_i)] - \varepsilon$. There exists a finite collection $\{[a_p, b_p]\}_{p=1}^{m}$ of nonoverlapping intervals such that,

(i) for each $p \leqq m$, there exists $i \leqq n$ such that $a_p = x_i$ and $b_p = x_{i+1}$,

(ii) $b_m \notin M(f, k+)$ and $a_1 \notin M(f, k-)$,

(iii) if, for some $p < m$, $b_p \in M(f, k+)$, then $b_p = a_{p+1}$ and there exists x such that $a_p \leqq x < b_p$ and $x \in M(f, k+)$, or $a_p < x < b_p$ and $x \in M(f, k)$,

(iv) if for some $p > 1$, $a_p \in M(f, k-)$, then $a_p = b_{p-1}$ and there exists a point x such that $a_p < x \leqq b_p$ and $x \in M(f, k-)$, or $a_p < x < b_p$ and $x \in M(f, k)$, and

(v) for each p, $[a_p, b_p]$ contains a point of $M(f, k)$.

Upon applying Lemma 1 to the intervals $\{[a_p, b_p]\}_{p=1}^{m}$, we get

$$\sum_E w(f, g; [x_{i+1}, x_i]) \geqq \sum_{p=1}^{m} w(f, g; [a_p, b_p] \\ \geqq 2^{-1}k \sum_{p=1}^{m} | g(b_p) - g(a_p) | \geqq 2^{-1}k[C_{g^*} M(f, k) - \varepsilon].$$

This implies that GLB $[\sum_D w(f, g; I); D] \geqq 2^{-1}k\, C_{g^*} M(f, k) > 0$, which is a contradiction. This completes the proof of the necessity.

In order to prove the sufficiency, suppose $k > 0$, $\varepsilon > 0$, and $\{[a_p, b_p]\}_{p=1}^{m}$ is a collection of pair-wise mutually exclusive subintervals of $[a, b]$ satisfying conditions (i) and (ii) of Definition 1.2 such that

$$\sum_{p=1}^{n}[g^*(b_p) - g^*(a_p)] < \varepsilon N^{-1},$$

where N is an upper bound of $|f|$ on $[a, b]$. Consequently,

$$\text{(A)} \qquad \sum_{p=1}^{m} w(f, g; [a_p, b_p]) < 2\varepsilon.$$

There exists a finite collection $\{[c_g ; d_g]\}_{g=1}^{m}$ of nonoverlapping subintervals of $[a, b]$ such that,

(i) except possibly for $c_1 = a$, each c_s is some b_p, and except possibly for $d_m = b$, each d_s is some a_p,

(ii) $(\{[a_p, b_p]\}_{p=1}^{n} + \{[c_s, d_s]\}_{s=1}^{m})$ forms a nonoverlapping covering of $[a, b]$, and

(iii) each $[c_s, d_s]$ satisfies the hypotheses of Lemma 2.

In view of this and (A) above, we see that the sufficiency part of Theorem 2.13 of [1] is satisfied, so that $\int_a^b fdg$ exists. This concludes the proof of Theorem 1.

As a consequence of Theorem 1, we have:

COROLLARY 1.1. *If f is a bounded function on $[a, b]$ and g is of bounded variation on $[a, b]$, then $\int_a^b fdg$ exists if, and only if, $\int_a^b fdg^*$ exists.*

In view of the fact that $C_{g^*} M(f, k) = 0$ implies $C_{g^*} M(|f|, k) = 0$, we have:

COROLLARY 1.2. *If f is a bounded function on $[a, b]$, g of bounded variation on $[a, b]$, and $\int_a^b fdg$ exists, then $\int_a^b |f|\, dg$ exists.*

COROLLARY 1.3. *If f is a bounded function on $[a, b]$ and g is of bounded variation on $[a, b]$ such that no discontinuity of g is in $M(f, k)$ for every $k > 0$, then $\int_a^b fdg$ exists if, and only if, $l_{g^*} M(f) = 0$, where $l_{g^*} M(f)$ denotes the outer g^*-length of $M(f)$ and $M(f)$ denotes the logical sum $\sum_{n=1}^{\infty} M(f, n^{-1})$.*

Proof. Since no point in $M(f)$ is a discontinuity of g and $M(f, k)$ is a closed point set for $k > 0$, $l_{g^*} M(f) = 0$ is equivalent to $C_{g^*} M(f, k) = 0$.

COROLLARY 1.4. *If f is a bounded function on $[a, b]$ and g is of bounded variation on $[a, b]$ such that f and g have no common discontinuity and $\int_a^b fdg$ exists, then the Riemann-Stieltjes integral* RS$\int_a^b fdg$ *exists.*

Proof. Suppose D denotes the set of discontinuities of f. If D_p, $p = 1, 2, \cdots$, is a subset of D such that x is in D_p if, and only if, x is in D, x is not in $M(f)$, and the ordinary oscillation of f at x is greater than or equal to p^{-1}, then $D = M(f) + \sum_{p=1}^{\infty} D_p$. For each p, D_p is at most a countable set and $l_{g^*} D_p = 0$, since f and g have no common discontinuity. It follows from this and Corollary 1.3 that $l_{g^*} D = 0$; i.e. RS$\int_a^b fdg$ exists.

References

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