CONDITIONS FOR A MINIMUM IN ABSTRACT SPACE

BY
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Introduction

In what follows we shall seek conditions that a point in a Banach space should furnish a minimum to a functional defined on a region of that space in the class of all points in the region satisfying an operator equation.

A partial solution to this general problem was given by the author [2]. The solution given there, however, is made possible by means of certain restrictive hypotheses that are in general not easy to verify in a given situation. More recently Graves has, in an as yet unpublished paper, succeeded in removing most of these restrictions.[1] However there remains one essential restriction of an artificial sort. In what follows this restriction is removed and a general solution is given.

More specifically, we give in Sections 1–3 an excursus into a theory of linear dependence of linear, continuous operators. This is then used in Section 4 to formulate and prove two necessary conditions: the multiplier rule, and another condition related to the so-called second variation. In Section 5 these conditions, suitably strengthened, are combined to provide sufficient conditions for a minimum. The proof is carried out by means of a Taylor's expansion in abstract space and does not require the assumption of "normality". In the last Section the meaning of "normality" is explored somewhat.

1. Formulation of the problem

In what follows we shall be concerned with a functional G defined on a region of a Banach space U, and an operator F defined on the same region but with values in another Banach space V. We assume that both F and G are of class C'' at each point of the region [3, p. 651]. A point u in this region will be called *admissible* in case it satisfies the equation

$$F(u) = 0.$$

Our problem can then be defined as that of finding in the class of admissible points one which furnishes G with a minimum. More precisely, it is to find conditions on an admissible point u that are necessary for G to be a minimum in the class of admissible points neighboring u. Also it is to find conditions on u that are sufficient to ensure that G is a minimum in some neighborhood of u.

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[1] This work of Graves was presented to the Society in April 1948 and was recently made available to me in draft form. It also contains a treatment of the Weierstrass condition.

2. Linear dependence of operators

In the next three sections we shall put aside temporarily the main problem just formulated and instead study certain linear relationships between linear, continuous operators. To this end let R, S, T be Banach spaces, and let f on R to S and g on R to T be linear, continuous operators. It will be convenient to consider the linear, closed manifolds R_f, R_g of all points in R for which

$$f(r) = 0, \qquad g(r) = 0,$$

respectively. We shall consider two cases, $R_f \subset R_g$ and $R_f \not\subset R_g$, and in each case shall enquire whether or not there is a linear relationship between f and g.

We consider first the case

$$R_f \subset R_g ; \tag{2.1}$$

this is equivalent to assuming that every zero of f is a zero of g. It is thus natural to form the quotient space $P = R/R_f$. Each element p in P is the class of all r in R for which $f(r)$ has a constant value, i.e. if r_1, r_2 are both in p, then $f(r_1) = f(r_2)$. Thus the functions H, J defined as

$$H(p) = f(r_p), \qquad J(p) = g(r_p) \tag{2.2}$$

are well-defined, linear operators on P, where r_p is a "representative" element of p, i.e. r_p is in p; it is understood that $p_1 + p_2$, ap are the points in P determined by $r_{p_1} + r_{p_2}$, ar_p, with r_{p_1}, r_{p_2}, r_p in p_1, p_2, p, respectively.

The linear operator H is defined on all of P and has

$$H(P) = S_0 = f(R). \tag{2.3}$$

Further, $H(p) = 0$ implies that $p = 0$, since it is equivalent to $f(r)$ vanishing for every r in p. Thus H establishes a one–one mapping of P onto S_0, and H has an inverse H^{-1} on S_0 which is linear.

LEMMA 2.1. *The operators H, J of relation* (2.2) *are linear, and the former one has a linear inverse on the linear manifold S_0 of* (2.3).

Consider now the operator L, defined on S_0 with values in T,

$$L \equiv JH^{-1}.$$

Clearly L is linear, and if s is in S_0,

$$Ls = Lf(r) = JH^{-1}H(p) = Jp = g(r),$$

where r is any one of the points in R which, according to (2.3), is such that $s = f(r)$ and p is the class in P containing r. Thus

$$g(r) = Lf(r). \tag{2.4}$$

THEOREM 2.1. *If $R_f \subset R_g$, then there exists a linear operator L on $S_0 = f(R)$ with values in T such that the relation* (2.4) *holds identically in R. This operator* *is unique. If the linear manifold S_0 is closed, then L is continuous, closed, and bounded.*

The uniqueness of L is clear. For, if there were two operators satisfying (2.4), then their difference would vanish identically, and they would then be identical on S_0. If S_0 is closed, it is a Banach space. Thus if s_n is a sequence of points in S_0 converging to an s_0 in S_0, then by a result of Banach [1, p. 40] there is a sequence r_n of points in R converging to an r_0 in R such that $s_n = f(r_n)$, $s_0 = f(r_0)$. Thus by (2.4)

$$Ls_0 = g(r_0) = \lim_{n=\infty} g(r_n) = \lim_{n=\infty} Ls_n = \lim_{n=\infty} Lf(r_n). \tag{2.5}$$

Thus (2.5) implies that L is both closed and continuous and hence, of course, bounded as well.

It is of course important to consider what obtains when S_0 is not closed. Let us examine some special cases to see the consequences of this assumption. First, choose the spaces R, S, and T to be the space of functions defined and continuous on the closed interval $-1, +1$ and vanishing at $t = -1$. Let

$$f(r) = \int_{-1}^{t} r(t')\, dt', \qquad g(r) = r. \tag{2.6}$$

Then both f and g are linear and continuous and have $R_f = R_g = [0(x)]$, the class containing only the zero function. Thus $R_f \subset R_g$. S_0 now consists of all continuous functions possessing a continuous first derivative. This set is clearly not closed. The transformation L is seen to be

$$Ls = ds/dt \tag{2.7}$$

and is defined on S_0. This operator L is certainly not continuous as can be seen at once; it is, however, closed.

Second, choose S and T to be the space $L^{(2)}$ and R to be C, the space of continuous functions. As before, define f and g with the help of the relations (2.6), and L with (2.7). Again $S_0 = f(R)$ is not closed. We show that L is not now a closed operator. Let

$$s_n(x) = \exp\,[-(x^2 + 1/n)^{1/2}], \qquad s(x) = \exp\,(-|\,x\,|), \quad -1 \leqq x \leqq 1.$$

Then $s_n \to s$, and $Ls_n = -x(x^2 + 1/n)^{-1/2} s_n(x) \to -\operatorname{sgn} x \cdot s(x)$, where $\operatorname{sgn} x = +1$ for $x > 0$, $= 0$ for $x = 0$, $= -1$ for $x < 0$. But

$$-\operatorname{sgn} x \cdot s(x) \neq ds(x)/dx$$

for x in $-1, 1$. Thus L is not closed [6, p. 75]. In fact we notice that s has no derivative at $x = 0$.

These examples serve to emphasize the somewhat unsatisfactory state of the result contained in Theorem 2.1. In the case when S_0 is not closed, so far no continuity criteria have been stated for L, although it is obvious that for L chosen arbitrarily as a linear operator, the function

$$g'(r) = Lf(r)$$

is not necessarily continuous.

Let R_1 be the unit sphere in R, and let

$$(2.8) \qquad Ng \equiv \text{lub}\, [\,|\, g(r) \,|\,|\, r \in R_1], \qquad B_L \equiv \text{lub}\, [\,|\, Ls \,|\,|\, s \in f(R_1)].$$

Then

$$B_L = Ng < +\infty.$$

Conversely, if B_L is finite, then

$$g'(r) = Lf(r)$$

is a linear, continuous operator on R with values in T. Another, but equivalent, way to view the situation is to introduce into S_0 a new metric in the following way: For each s in S_0 let R_s be the set of all r in R for which $f(r) = s$. Then

$$(2.9) \qquad |\, s \,|_f = \text{glb}\, [\,|\, r \,|\,|\, r \in R_s].$$

Note that for $|\, f \,| \neq 0$, we have

$$|\, s \,|_f \geqq |\, s \,| \,/\, |\, f \,|.$$

It is evident that $(S_0\,, |\ \ |_f)$ is a normed, linear space.

We go to show that L is continuous relative to the norm just introduced, i.e. if s_n is a Cauchy sequence in S_0, then Ls_n is such a sequence in T. Actually it is easier to show that L is a bounded operator. We have

$$|\, Ls \,| = |\, Lf(r) \,| = |\, g(r) \,| \leqq Ng \cdot |\, r \,| \qquad (r \in R_s).$$

Thus

$$|\, Ls \,| \leqq Ng \cdot |\, s \,|_f\,,$$

and we have shown that $NL \leqq Ng$. We summarize in

THEOREM 2.2. *The unique linear operator L whose existence is asserted in Theorem* 2.1 *is continuous and bounded when it is considered as being defined on the set S_0 with the norm defined in* (2.9). *If the bound of L is $N(L)$, then*

$$N(L) = B_L = Ng,$$

where B_L and Ng are defined in (2.8) *and*

$$NL = \text{lub}\, [\,|\, Ls \,|\,|\, s \in S_1],$$

S_1 being the unit sphere in S_0 relative to $|\ \ |_f$.

We show first that $N(L) \geqq N(g)$. To this end we note that

$$|\, g(r) \,| = |\, Ls \,| \leqq N(L) \cdot |\, s \,|_f \leqq N(L) \cdot |\, r \,|, \quad r \in R_s\,, \quad s \in S_0$$

implies the desired conclusion.

In what follows we shall always consider the space R with its original norm, but with regard to the subspace S_0 of S we shall have occasion to use not only the original norm but also the new one, $| \ |_f$.

THEOREM 2.3. *If L is an arbitrary operator which is defined on S_0, has values in T, and is linear and continuous relative to the norm $| \ |_f$ of relation* (2.9), *then the operator*

$$g'(r) \equiv Lf(r)$$

is defined on R and is linear and continuous.

To establish this result we notice that g' is linear since both f and L are. The continuity property of g' is equally easy. If $r_n \to 0$, then the $s_n = f(r_n)$ are such that

$$| s_n |_f \leqq | r_n |,$$

as one sees directly from (2.9). Thus $s_n \to 0$ in the sense of the norm $| \ |_f$.

THEOREM 2.4. *The operator L of Theorem 2.2 vanishes exactly on the set $f(R_g)$. If f and g vanish on exactly the same set, i.e. if $R_f = R_g$, then L vanishes only at the origin. In this case it has a linear inverse L^{-1} for which the relations*

$$g(r) = Lf(r), \qquad f(r) = L^{-1}g(r)$$

hold identically in R. If furthermore $S_0 = f(R)$ is closed, then L is continuous, and if $T_0 = g(R)$ is closed, L^{-1} is continuous. Also if S_0 is closed, L^{-1} is closed, and if T_0 is closed, L is closed.

To prove the first part we notice that, by definition, $Ls = 0$ implies

$$JH^{-1}s = 0.$$

Let $p = H^{-1}s$; then $J(p) = 0$. But if r_p is in p, this becomes

$$0 = J(p) = g(r_p),$$

i.e. r_p is in R_g, the set of zeros of g. However $p = H^{-1}s$ implies that

$$s = H(p) = f(r_p),$$

i.e. that s is in $f(R_g)$. Since s is an arbitrary zero of L, the conclusion follows.

If now $R_f = R_g$, then

$$f(R_g) = f(R_f) = [0],$$

and thus L vanishes exactly at the origin. In this case it establishes a one-one correspondence between S_0 and $T_0 = g(R)$. Thus L has a linear inverse, as stated. If S_0 is closed, then S_0 is a Banach space, and thus L is continuous. Similarly if T_0 is closed, L^{-1} is continuous.

Finally, suppose S_0 is closed, and consider a sequence t_n of points in T_0 converging to a point t in T such that $s_n = L^{-1}t_n$ converges to a value s in S. Then s must be in S_0, and L must be continuous. It follows at once that $t_n = Ls_n$ converges to Ls. But by hypothesis t_n also converges to t, and thus $Ls = t$. Similarly, T_0 closed implies L is closed.

3. Linear dependence. Continuation

Having treated the case where $R_f \subset R_g$, we go now to consider the contrary one, namely where $R_f \not\subset R_g$. To do this we need first the following result:

THEOREM 3.1. *Given a linear manifold* U_0 *in a Banach space* U, *another Banach space* V, *and a point* u^* *not in the closure of* U_0, *there is a linear and continuous operator* h *on* U *to* V *such that* h *is orthogonal to* U_0 *but not to* u^*.

It is well-known that there is a linear and continuous functional m on U which is orthogonal to U_0 but not to u^* [1, p. 57]. Let v^* be any point in V not the zero point, and let

$$h(u) = v^* \cdot m(u)$$

for every u in U. Then h is effective in the theorem.

Let us now consider the linear, closed manifold R_{fg} determined (i.e. spanned) by R_f and R_g. We suppose first that

$$R_{fg} \neq R. \tag{3.1}$$

Then by the previous theorem we can find a linear, continuous operator h on R to T such that h is orthogonal to R_{fg} and is not identically zero. Then we see that R_h, the manifold of zeros of h, contains R_{fg} and therefore also R_f and R_g. We now apply Theorem 2.1 to the operators h and f and find on $S_0 = f(R)$ a linear, continuous operator L relative to $|\ |_f$, with values in T such that identically in R

$$h(r) = Lf(r). \tag{3.2}$$

This operator L is not identically zero on S_0, since h is not identically null. Next we apply Theorem 2.1 to the operators h and g and find on $T_0 = g(R)$ a linear operator M, continuous relative to $|\ |_g$, with values in T such that the relation

$$h(r) = Mg(r) \tag{3.3}$$

holds identically in R. The operator M is not identically zero on T_0 since h is not identically zero. Combining the relations (3.2) and (3.3), we find

THEOREM 3.2. *If the linear closed manifold spanned by* R_f *and* R_g *is not the entire space* R, *there are linear operators* L *on* S_0 *and* M *on* T_0 *with values in* T *such that the relation*

$$Lf(r) = Mg(r) \tag{3.4}$$

holds identically on R. *Neither* L *nor* M *is identically zero. Both have the continuity, closure, and boundedness properties described in Theorem* 2.2 *for the operator* L *appearing there.*

If $R_{fg} = R$, then the operator h appearing in relations (3.2) and (3.3) is identically zero. If either

$$\bar{S}_0 = \overline{f(R)} \neq S \quad \text{or} \quad \bar{T}_0 = \overline{g(R)} \neq T, \tag{3.5}$$

then either L or M can be chosen not identically zero. Thus we have

COROLLARY 3.1. *If the linear closed manifold spanned by R_f and R_g is R, then the conclusion of Theorem (3.2) still holds provided (3.5) is valid and the statement that neither L nor M is identically zero is amended to read not both are identically zero.*

It is possible to improve somewhat the result given in this corollary. We do so by replacing the conditions (3.5) by

$$\overline{f(R_g)} \neq S \quad \text{or} \quad \overline{g(R_f)} \neq T. \tag{3.6}$$

Consider now linear, continuous operators L on S to T and M on T to T which are orthogonal to $f(R_g)$ and $g(R_f)$, respectively. If one or the other of conditions (3.6) is valid, not both L and M are identically zero. Let r be an arbitrary point in R. Then since $R_{fg} = R$, r is expressible in the form

$$r = \lim_n (r_{fn} + r_{gn}),$$

where r_{fn} is in R_f and r_{gn} is in R_g. Then by the continuity of $L, M, f,$ and g,

$$\begin{aligned} Lf(r) &= \lim_n Lf(r_{fn} + r_{gn}) = \lim_n Lf(r_{gn}) = 0, \\ Mg(r) &= \lim_n Mg(r_{fn} + r_{gn}) = \lim_n Mg(r_{fn}) = 0, \end{aligned} \tag{3.7}$$

since L, M are orthogonal to $\overline{f(R_g)}$, $\overline{g(R_f)}$, respectively. We clearly can combine the relations (3.7) to obtain (3.4).

If neither of the conditions (3.6) is valid, then (3.4) is possible only for $L \equiv 0, M \equiv 0$. For, if s is an arbitrary point in S and t an arbitrary point in T, then they are expressible in the form

$$s = \lim_n f(r_{gn}), \qquad t = \lim_n g(r_{fn}),$$

where r_{fn}, r_{gn} are points in R_f, R_g, respectively. Thus if L, M exist as linear, continuous operators, we must have

$$Ls = \lim_n Lf(r_{gn}) = \lim_n Mg(r_{gn}) = 0.$$

$$Mt = \lim_n Mg(r_{fn}) = \lim_n Lf(r_{fn}) = 0.$$

This implies $L \equiv 0$, $M \equiv 0$.

THEOREM 3.3. *If the linear closed manifold spanned by R_f and R_g is the entire space R, there are linear, continuous operators L on S and M on T with values in T such that the relation (3.4) is valid on R. The operators L, M*

may be chosen not both identically zero if and only if either of the conditions (3.6) *obtains.*

COROLLARY 3.2. *The operators L, M are orthogonal to the sets $\overline{f(R_g)}$ and $\overline{g(R_f)}$, respectively.*

4. The multiplier rule

We shall be concerned in this section with an application of some of our previously obtained results, notably Theorems 2.1, 3.2, 3.3, to the minimum problem formulated in Section 1. In what follows, the space T, the contradomain of g, is the set of real numbers. Then g is a linear, continuous functional, and the operators L of Theorems 2.1, 2.2, 2.4, 3.2, 3.3 are linear functionals; the operators M of Theorems 3.2, 3.3 are real numbers.

LEMMA 4.1. *The relation $R_f \subset R_g$ is valid if and only if $R_{fg} \neq R$ or g is identically zero.*

Suppose $R_f \subset R_g$. Then $R_{fg} = R_g$. Now if $R_g = R$, then g vanishes identically. Conversely, if $g \equiv 0$, then $R_g = R \supset R_f$. If $R_{fg} \neq R$ and $g \not\equiv 0$, then by Theorem 3.2 there are a linear functional L on S_0 and a constant l such that

$$lg(r) + Lf(r) = 0; \tag{4.1}$$

neither l nor L is identically zero. But then (4.1) implies that $R_f \subset R_g$.

COROLLARY 4.1. *If $R_f \not\subset R_g$, then $g(R_f)$ is the space of real numbers.*

This follows at once from the fact that if $f(r) = 0$ and $g(r) \neq 0$, then $ag(r)$ is in $g(R_f)$ for every real a.

LEMMA 4.2. *If $R_{fg} = R$, then $\bar{S}_0 = S$ is equivalent to $\overline{f(R_g)} = S$.*

If $\overline{f(R_g)}$ were not S, then by Corollary 3.2 the operator L would be orthogonal to $\overline{f(R_g)}$ and not identically zero. Moreover, with the help of the relations (3.7) derived in the proof of Theorem 3.3, we see that L would be orthogonal to $\bar{S}_0$ and $L \neq 0$; thus $\bar{S}_0$ would be properly contained in S.

Conversely, we have

$$S = \overline{f(Rg)} \subset \overline{f(R)},$$

and thus $\bar{S}_0 = S$.

In what follows it is convenient to introduce certain notations and definitions. They are these:

At each admissible point $\tilde{r}$ let

$$f(r) = dF(\tilde{r}; r), \qquad g(r) = dG(\tilde{r}; r). \tag{4.2}$$

An admissible point $\tilde{r}$ *satisfies the multiplier rule* in case there are a constant l and a linear functional L defined on some linear manifold in R such that the relation

$$0 = lg(r) + Lf(r) = ldG(\tilde{r}; r) + LdF(\tilde{r}; r) \tag{4.3}$$

holds identically on R with l, L not both identically zero. An admissible point $\bar{r}$ is *normal* in case $S_0 = f(R) = S$ and $R_f \subset R_g$. In the contrary case, $\bar{r}$ is *abnormal.*

THEOREM 4.1. *Every normal point satisfies the multiplier rule with a set l, L for which $l = 1$ and L is unique, defined, and continuous on S. Every abnormal point either satisfies the multiplier rule with a set l, L for which $l = 0$ or is such that $S_0 = S$, $R_f \not\subset R_g$.*

The case where the point in question is normal follows almost at once from the first part of Theorem 2.1. The continuity of L follows from the second part of that theorem since $S_0 = S$. We examine now the case where the point is abnormal. Then either $S_0 \neq S$, or $R_f \not\subset R_g$. We distinguish two cases: $S_0 \neq S$, and $S_0 = S$ together with $R_f \not\subset R_g$. In the former case choose s^* to be a point in $S - S_0$, and consider the set S_1 of all s of the form $s_0 + as^*$, where s_0 is in S_0 and a is an arbitrary real number. On S_1 we define an L as follows:

$$Ls = L(s_0 + as^*) = a, \qquad s \in S_1.$$

This L is defined on S_1, is linear, is orthogonal to S_0, but is not identically zero. Then the set $l = 0$, L is effective in the theorem.

We come now to the remaining case: $S_0 = S$, $R_f \not\subset R_g$. In this case the multiplier rule cannot be satisfied. If the multiplier rule were satisfied with $l \neq 0$, then clearly $R_f \subset R_g$. But in our case $R_f \not\subset R_g$, and therefore $l = 0$. Thus we would have

$$Lf(r) = 0, \qquad r \in R. \tag{4.4}$$

Since $S_0 = S$, L would be orthogonal to all of S and thus identically zero.

We go now to show that the last possibility stated in Theorem 4.1, namely $S_0 = S$, $R_f \not\subset R_g$, cannot arise at a minimizing point $\bar{r}$. To do this we consider the operator

$$H(r) = (F(\bar{r} + r), G(\bar{r} + r) - G(\bar{r})).$$

It is defined and of class C'' for r near to $\bar{r}$ and has values in $S \times T$, where T is the space of reals; also $H(0) = 0$. Now the set $dH(0; R)$ cannot be the entire space $S \times T$. For if it were, the equation

$$H(r) = (s, a)$$

would be solvable for all (s, a) near to $(0, 0)$ [4, p. 112]. Then in particular for $s = 0$ and $a < 0$, there would be a solution. But then one would have

$$F(\bar{r} + r) = 0, \qquad G(\bar{r} + r) < G(\bar{r}),$$

which is a contradiction of the minimizing properties of $\bar{r}$.[2] Thus $dH(0; R)$ is not the entire space.

[2] Graves used this technique to establish the multiplier rule in the case S_0 is closed. This result is contained in the as yet unpublished manuscript mentioned in footnote 1.

Consider now the set $dH(0; R)$ in our case: $S_0 = S$, $R_{fg} = R$, $g \not\equiv 0$. Suppose there is an r' in R such that $f(r') = 0$ and $g(r') \neq 0$. Then for an arbitrary real number b, there is an r'' in R such that

$$f(r'') = 0 \quad \text{and} \quad g(r'') = b;$$

this r'' can be chosen as $r'b/g(r')$. Consider now any point (s, a) in the product of S and T. Then since $f(R) = S_0 = S$, there is an r such that $f(r) = s$. Now by what has just preceded, we can also find an r'' such that

$$f(r'') = 0, \qquad g(r'') = a - g(r)$$

provided $a \neq g(r)$. Then $s = f(r + r'')$, $a = g(r + r'')$. If $a = g(r)$, we have $f(r) = s$, $g(r) = a$. Thus if there were an r' in R such that

$$f(r') = 0, \qquad g(r') \neq 0,$$

then $dH(0; R) = (dF(\tilde{r}; R), dG(\tilde{r}; R)) = (f(R), g(R))$ would be the entire space $S \times T$, which is a contradiction. Thus for every r such that $f(r) = 0$, we must have $g(r) = 0$. But this is equivalent to $R_f \subset R_g$, which contradicts $R_{fg} = R$, $g \not\equiv 0$. Thus

THEOREM 4.2. *At a minimizing point the case* $S_0 = S$, $R_{fg} = R$, $g \not\equiv 0$ *cannot occur. Thus every minimizing point satisfies the multiplier rule.*

This follows at once from Theorem 4.1 since the only exceptional case has been shown to lead to a contradiction.

5. Sufficient conditions

Here we formulate conditions on an admissible point $\tilde{r}$ that are sufficient to ensure that it is a minimizing point. To this end we have

THEOREM 5.1. *Let* $\tilde{r}$ *be an admissible point satisfying the multiplier rule with* $l = 1$, L *continuous and having the lower bound of*

$$d^2H(\tilde{r}; r, r) = d^2G(\tilde{r}; r, r) + Ld^2F(\tilde{r}; r, r),$$

for all r *on the unit sphere for which* $dF(\tilde{r}; r) = 0$, *positive. Then there is a neighborhood of* $\tilde{r}$ *such that for every admissible* r *in this neighborhood*

$$G(r) > G(\tilde{r}).$$

To establish this result we proceed with an expansion type proof. The function $d^2G(r'; r, r) + Ld^2F(r'; r, r)$ is continuous in r' at $r' = \tilde{r}$ uniformly for r in the unit sphere. Since its lower bound at $r' = \tilde{r}$ for r in the unit sphere and satisfying $dF(\tilde{r}; r) = 0$ is positive, there is a neighborhood of $\tilde{r}$ in which this bound stays positive, i.e. for r' near to $\tilde{r}$

$$d^2G(r'; r, r) + Ld^2F(r'; r, r) > 0 \tag{5.1}$$

for r in the unit sphere and $dF(\tilde{r}; r) = 0$.

By Taylor's theorem [3, p. 650] we have for r near to $\tilde{r}$

$$
\begin{aligned}
G(r) - G(\tilde{r}) &= dG(\tilde{r}; r - \tilde{r}) \\
&\qquad + \int_0^1 d^2G(\tilde{r} + t(r - \tilde{r}); r - \tilde{r}, r - \tilde{r})(1 - t)\, dt, \\
F(r) - F(\tilde{r}) &= dF(\tilde{r}; r - \tilde{r}) \\
&\qquad + \int_0^1 d^2F(\tilde{r} + t(r - \tilde{r}); r - \tilde{r}), r - \tilde{r})(1 - t)\, dt.
\end{aligned} \tag{5.2}
$$

But since r and $\tilde{r}$ are admissible, the second relation becomes

$$0 = dF(\tilde{r}; r - \tilde{r}) + \int_0^1 d^2F(\tilde{r} + t(r - \tilde{r}); r - \tilde{r}, r - \tilde{r})(1 - t)\, dt. \tag{5.3}$$

Combining (5.2) and (5.3), we find

$$
\begin{aligned}
G(r) - G(\tilde{r}) &= dH(\tilde{r}; r - \tilde{r}) \\
&\qquad + \int_0^1 d^2H(\tilde{r} + t(r - \tilde{r}); r - \tilde{r}, r - \tilde{r})(1 - t)\, dt.
\end{aligned} \tag{5.4}
$$

But since $\tilde{r}$ satisfies the multiplier rule, the first term on the right-hand side of (5.4) vanishes, and finally

$$G(r) - G(\tilde{r}) = \int_0^1 d^2H(\tilde{r} + t(r - \tilde{r}); r - \tilde{r}, r - \tilde{r})(1 - t)\, dt.$$

But by (5.1) the integrand is positive for r near to $\tilde{r}$. Thus $G(r) > G(\tilde{r})$.

6. Normal points

In what follows we shall need to add a restriction to the problem as formulated in Section 1. We shall assume that if $\tilde{r}$ is the admissible point being examined, then we have

I. There is a decomposition of R into the cartesian product of S and another Banach space U such that:

(a) the region in which F and G are defined and of class C'' is the product of a region in S and one in U with $\tilde{r} = (\tilde{s}, \tilde{u})$ having $\tilde{s}$ in the former and $\tilde{u}$ in the latter region;

(b) the partial differential $d_s F(\tilde{r}; s)$, which is defined on S with values in the same space, has an inverse.

LEMMA 6.1. *Under the assumption just made, every normal point* $\tilde{r} = (\tilde{s}, \tilde{u})$ *is a limit point of admissible points. Further, if* r *is an arbitrary point in* R_f, *there is a one-parameter family* $r(a)$ *of admissible points for* a *near* 0 *of class* C'' *such that*

$$r(0) = \tilde{r}, \qquad r'(0) = r. \tag{6.1}$$

Notice that the first part of the lemma is a consequence of the second. To establish that, let $r = (s, u)$, and set

$$u(a) = \tilde{u} + au. \tag{6.2}$$

Then the equation

$$H(s, a) = F(s, u(a)) = 0 \tag{6.3}$$

has an initial solution $(s, a) = (\tilde{s}, 0)$ at which the differential of H with respect to s, $d_s H(\tilde{s}, 0; s) = d_s F(\tilde{r}; s)$, has an inverse. Thus it is known [5, p. 100] from implicit function theory that the equation has a solution $s = s(a)$ for a near 0, passing through $(\tilde{s}, 0)$, and that this solution is of class C'' since F is. Substitute this solution into (5.3), and differentiate. This yields at $a = 0$ the result

$$0 = d_s F(\tilde{r}; s'(0)) + d_u F(\tilde{r}; u'(0)) = d_s F(\tilde{r}; s'(0)) + d_u F(\tilde{r}; u);$$

but for $r = (s, u)$ in R_f we have

$$0 = f(r) = dF(\tilde{r}; r) = d_s F(\tilde{r}; s) + d_u F(\tilde{r}; u).$$

Subtracting one of these from the other, we find

$$0 = d_s F(\tilde{r}; s - s'(0)),$$

which is equivalent to

$$s = s'(0),$$

since $d_s F$ vanishes only at the origin. Thus we have $s(0) = \tilde{s}$, $u(0) = \tilde{u}$, $s'(0) = s$, $u'(0) = u$, as was to be proved.

THEOREM 6.1. *Let $\tilde{r}$ be a normal point which minimizes G. Then there is a unique linear and continuous functional L defined on S such that the function*

$$H(r) = G(r) + LF(r)$$

has the following two properties: the functional $dH(\tilde{r}; r)$ vanishes at every r; and the lower bound of $d^2H(\tilde{r}; r, r)$, on the set of all r on the unit sphere for which $dF(\tilde{r}; r) = 0$, is nonnegative.

To prove this result we notice first that as a consequence of assumption I(b), the operator $f(r) = dF(\tilde{r}; r) = d_s F(\tilde{r}; s) + d_u F(\tilde{r}; u)$ is such that

$$f(R) = S_0 = S.$$

Thus S_0 is closed. Next the function $K(a) = G(r(a))$ has a minimum at $a = 0$, and $F(r(a)) \equiv 0$ where $r(a)$ is the function whose existence is asserted in Lemma 6.1; both these are consequences of Lemma 5.1. Thus

$$0 = K'(0) = dG(\tilde{r}; r'(0)) = g(r) = 0$$

for all r in R_f, i.e. $R_f \subset R_g$. Thus by Theorem 2.1 there exists a linear, continuous L on S such that $g(r) + Lf(r) = 0$ for all r. Then the function

$$H(r) = G(r) + LF(r)$$

is defined and of class C''. Further, since $dH(\tilde{r}; r) = g(r) + Lf(r)$, we have

$$dH(\tilde{r}; r) = 0$$

for r in R_f. Also since K is a minimum,

$$0 \leqq K''(0) = \frac{d^2}{da^2}[G(r(a))]^{a=0} = \frac{d^2}{da^2}[H(r(a))]^{a=0} = d^2H(\tilde{r}; r, r),$$

since $F(r(a)) = 0$, for r in R_f. Finally we recall that r in R_f means that

$$dF(\tilde{r}; r) = 0.$$

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