A UNIQUENESS THEOREM FOR MINKOWSKI'S PROBLEM FOR CONVEX SURFACES WITH BOUNDARY

BY
CHUAN-CHIH HSIUNG

1. Introduction

By a convex surface with boundary imbedded in a three-dimensional Euclidean space E^3 we mean a two-dimensional compact subset of the boundary of a convex region in the space E^3. The purpose of this paper is to establish the

THEOREM. *Let S and S^* be two orientable convex surfaces of class C^2 with boundaries C and C^*, respectively, and positive Gaussian curvatures imbedded in a three-dimensional Euclidean space E^3. Suppose that there is a differentiable homeomorphism H of the surface S onto the surface S^* such that at corresponding points the two surfaces S and S^* have the same unit inner normal vectors and equal Gaussian curvatures. If the homeomorphism H restricted to the boundary C is a translation carrying the boundary C onto the boundary C^*, then the homeomorphism H is a translation carrying the whole surface S onto the whole surface S^*.*

When the two surfaces S and S^* are closed, the above theorem becomes the well-known uniqueness theorem for Minkowski's problem, which was first established by Minkowski [5] and several decades later proved by Lewy [4] for analytic surfaces S and S^*, by Miranda [6] for surfaces S and S^* having derivatives of 5th order satisfying Hölder's conditions of positive exponent, by Stoker [7] for surfaces S and S^* of class C^3 and by Chern [1] for surfaces S and S^* of class C^2. The methods used by these authors are all different. Minkowski applied Brun-Minkowski's theory concerning the mixed volumes of convex bodies; Lewy, Miranda, and Stoker used different results about elliptic differential equations; and Chern modified the proof of Herglotz [2] for Cohn-Vossen's theorem on isometries of closed convex surfaces by deriving some integral formulas.

The method used in this paper is essentially the same as that used by Chern. It should also be remarked that the uniqueness theorem for Christoffel's problem can also be extended to the same form as that of the above theorem (for this, see [3]).

2. Preliminaries

In a three-dimensional Euclidean space E^3, let us consider a fixed right-handed orthogonal frame $Oe_1 e_2 e_3$, where e_1, e_2, and e_3 form an ordered triple set of mutually orthogonal unit vectors at a point O. Then the position vector of a point P of a surface S of class C^2 in the space E^3 with respect to

Received March 8, 1957.

the orthogonal frame $Oe_1 e_2 e_3$ can be given by the vector equation

$$X = F(u^1, u^2), \tag{2.1}$$

where the two parameters u^1 and u^2 take values in a simply connected domain D of the two-dimensional real number space, the components of the vector $F(u^1, u^2)$ are of the second class, and the Jacobian matrix $\| \partial X/\partial u^\alpha \|$ is of rank two at all points of the domain D. Throughout this paper all Latin indices take the values 1, 2, 3; Greek indices the values 1, 2 unless stated otherwise; and the partial derivative of any differentiable vector function V of u^1 and u^2 with respect to u^α is denoted by V_α. We shall also follow the convention that repeated indices imply summation.

Let N be the unit normal vector, $\| g_{\alpha\beta} \|$ the matrix of the positive definite metric of the surface S at the point P, and $\| g^{\alpha\beta} \|$ the inverse matrix of $\| g_{\alpha\beta} \|$. Then we have

$$N_\alpha = -b_{\alpha\beta}\, g^{\beta\gamma} X_\gamma, \tag{2.2}$$

where $b_{\alpha\beta} = b_{\beta\alpha}$ are the coefficients of the second fundamental form of the surface S at the point P. From equation (2.2) it follows immediately that

$$f_{\alpha\beta} \equiv N_\alpha \cdot N_\beta = b_{\alpha\rho}\, b_{\beta\sigma}\, g^{\rho\sigma}, \tag{2.3}$$

and therefore that

$$g^{\alpha\beta} = f_{\rho\sigma}\, b^{\alpha\rho}\, b^{\beta\sigma}, \tag{2.4}$$

where the dot denotes the scalar product of the two vectors N_α and N_β, and $\| b^{\alpha\beta} \|$ is the inverse matrix of $\| b_{\alpha\beta} \|$. Substituting equation (2.4) in equation (2.2) we obtain

$$N_\alpha = -f_{\alpha\rho}\, b^{\beta\rho} X_\beta . \tag{2.5}$$

The Gaussian curvature K of the surface S at the point P is given by

$$K = b/g = f/b, \tag{2.6}$$

where g denotes the determinant $| g_{\alpha\beta} | > 0$, $b = | b_{\alpha\beta} |$, and $f = | f_{\alpha\beta} |$. From equations (2.6) it follows that

$$f = K^2 g > 0. \tag{2.7}$$

The element of area of the surface S at the point P is

$$dA = g^{1/2}\, du^1\, du^2. \tag{2.8}$$

If the surface S is orientable, we can choose N to be the unit inner normal vector so that the second fundamental form of the surface S at the point P is positive definite. For this inner normal vector N, there is a unique pair of values of α_1 and α_2 such that $\alpha_1, \alpha_2 = 1, 2$, $\alpha_1 \neq \alpha_2$, and $PX_{\alpha_1} X_{\alpha_2} N$ is a right-handed frame at the point P, from which it follows that

$$X_{\alpha_1} \times X_{\alpha_2} = g^{1/2} N, \tag{2.9}$$

$$| X_{\alpha_1}, X_{\alpha_2}, N | = g^{1/2}, \tag{2.10}$$

where the cross $\times$ denotes the vector product of the two vectors X_{α_1} and X_{α_2}, and the left side of equation (2.10) is a determinant indicated by writing only a typical row. Let d be the exterior differentiation, and $\otimes$ the combined operator of the vector product $\times$ and the exterior product $\wedge$. By means of equations (2.8) and (2.9) we can easily obtain

$$dX \otimes dX = 2(X_{\alpha_1} \times X_{\alpha_2})\, du^{\alpha_1} \wedge du^{\alpha_2} = 2N\, dA. \tag{2.11}$$

3. Proof of the theorem

Without loss of generality we may assume that the corresponding points of the two surfaces S and S^* have the same local coordinates u^1 and u^2. Then §2 can be applied to the surface S, and for the corresponding quantities and equations for the surface S^* we shall use the same symbols and numbers with a star respectively.

Since by the assumption of the theorem $N^* = N$ and $K^* = K$, from equations (2.3), (2.6), (2.8), (2.3)*, (2.6)*, and (2.8)* it follows that

$$\overset{*}{f}_{\alpha\beta} = f_{\alpha\beta}, \tag{3.1}$$

$$b^* = b > 0, \tag{3.2}$$

$$dA^* = dA. \tag{3.3}$$

Equating the right side of the two equations (2.5) and (2.5)*, we have, in consequence of equation (3.1),

$$f_{\alpha\rho}\, b^{\beta\rho} X_\beta = f_{\alpha\sigma}\, b^{*\gamma\sigma} \overset{*}{X}_\gamma . \tag{3.4}$$

Let $\| f^{\alpha\beta} \|$ be the inverse matrix of $\| f_{\alpha\beta} \|$. First multiplying equation (3.4) by $f^{\alpha\lambda}$ and summing for α and then multiplying the resulting equation by $\overset{*}{b}_{\lambda\mu}$ and summing for λ, we can easily obtain

$$\overset{*}{X}_\alpha = \overset{*}{b}_{\alpha\beta}\, b^{\beta\gamma} X_\gamma . \tag{3.5}$$

By means of equations (2.8), (2.9), and (3.5) a simple calculation suffices to demonstrate that

$$dX^* \otimes dX = \frac{1}{b} (\overset{*}{b}_{11} b_{22} - \overset{*}{b}_{12} b_{21} - \overset{*}{b}_{21} b_{12} + \overset{*}{b}_{22} b_{11}) N\, dA. \tag{3.6}$$

From equations (2.11), (3.2), and (3.6) it follows that

$$\begin{aligned} d\,| X^*, X, dX | &= X^* \cdot (dX \otimes dX) - X \cdot (dX^* \otimes dX) \\ &= 2p^*\, dA - \frac{p}{b} (\overset{*}{b}_{11} b_{22} - \overset{*}{b}_{12} b_{21} - \overset{*}{b}_{21} b_{12} + \overset{*}{b}_{22} b_{11})\, dA \\ &= 2(p^* - p)\, dA + \frac{p}{b} \begin{vmatrix} \overset{*}{b}_{11} - b_{11} & \overset{*}{b}_{12} - b_{12} \\ \overset{*}{b}_{21} - b_{21} & \overset{*}{b}_{22} - b_{22} \end{vmatrix} dA, \end{aligned} \tag{3.7}$$

where we have placed

$$p = X \cdot N, \qquad p^* = X^* \cdot N. \tag{3.8}$$

Interchanging the roles of the two surfaces S and S^* in equation (3.7) and making use of equations (3.2) and (3.3), we thus obtain

$$d\,|X, X^*, dX^*| = 2(p - p^*)\,dA + \frac{p^*}{b}\begin{vmatrix} b^*_{11} - b_{11} & b^*_{12} - b_{12} \\ b^*_{21} - b_{21} & b^*_{22} - b_{22} \end{vmatrix} dA. \tag{3.9}$$

Addition of equations (3.7) and (3.9) gives immediately

$$\begin{aligned} d(|X^*, X, dX| &+ |X, X^*, dX^*|) \\ &= \frac{1}{b}\begin{vmatrix} b^*_{11} - b_{11} & b^*_{12} - b_{12} \\ b^*_{21} - b_{21} & b^*_{22} - b_{22} \end{vmatrix} (p + p^*)\,dA. \end{aligned} \tag{3.10}$$

From the assumption that the given homeomorphism H restricted to the boundary C is a translation T carrying the boundary C onto the boundary C^*, it follows that along the boundary C we have $dX^* = dX$, and therefore $|X^*, X, dX| + |X, X^*, dX^*| = 0$. Integrating equation (3.10) over the surface S and applying Stokes' theorem to the left side of the equation, we thus arrive at the integral formula

$$\iint_S \frac{1}{b}\begin{vmatrix} b^*_{11} - b_{11} & b^*_{12} - b_{12} \\ b^*_{21} - b_{21} & b^*_{22} - b_{22} \end{vmatrix} (p + p^*)\,dA = 0. \tag{3.11}$$

Using the given translation T carrying the boundary C onto the boundary C^* if necessary, without loss of generality we may assume that the intersection $\mathfrak{D}$ of the convex hulls of the two surfaces S and S^* contains interior points. Then we can choose the origin O of the fixed orthogonal frame $Oe_1\,e_2\,e_3$ in the space E^3 to be in the region $\mathfrak{D}$ such that $p > 0$ and $p^* > 0$. On the other hand, due to equation (3.2) and the fact that the second fundamental forms of the surfaces S and S^* at the corresponding points P and P^* are positive definite, it is well-known that

$$\begin{vmatrix} b^*_{11} - b_{11} & b^*_{12} - b_{12} \\ b^*_{21} - b_{21} & b^*_{22} - b_{22} \end{vmatrix} \leqq 0, \tag{3.12}$$

where the equality holds when and only when $b^*_{\alpha\beta} = b_{\alpha\beta}$ for $\alpha, \beta = 1, 2$. Thus the integrand on the left side of equation (3.11) is nonpositive, and equation (3.11) holds when and only when the equality in (3.12) holds. Therefore we have

$$b^*_{\alpha\beta} = b_{\alpha\beta} \qquad (\alpha, \beta = 1, 2). \tag{3.13}$$

From equations (3.5) and (3.13) it follows immediately that over the whole surface S

$$X^*_\alpha = X_\alpha \qquad (\alpha = 1, 2), \tag{3.14}$$

which completes the proof of the theorem.

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LEHIGH UNIVERSITY
BETHLEHEM, PENNSYLVANIA