REPARTITIONS ON ABELIAN VARIETIES

BY

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Introduction

The concept of repartition, systematically used by Chevalley in his exposition of the algebraic geometry on a curve (see [10] of the bibliography at the end of this paper), has proved particularly fruitful in that subject: besides offering a direct approach to the theorem of Riemann-Roch (Weil's proof), it throws light on the nature of the differentials on a curve; a differential can be considered as such, that is, as a mapping of the set of the derivations into the field of the rational functions on the curve, or as a "differential mapping," namely a mapping of the set of the repartitions into the field of constants. The main advantage of the second interpretation lies in the possibility of defining the trace of a differential from a field to its p^{th} power, if $p \neq 0$ is the characteristic of the field of constants. The lack of such trace is the basic reason for which the earliest abstract treatment of algebraic correspondences between curves [11] was limited to the consideration of "separable" correspondences.

For these and other reasons, it is desirable to extend the concept of repartition to higher dimensional varieties; in the present paper we propose a definition, not of the repartitions, but of certain classes of repartitions, and proceed to show that they are reasonably adequate for abelian varieties; they do, for instance, provide a tool for the completion of the main result of [11] mentioned above (see results 4.2 and 4.3 of this paper), although this is more in the nature of a by-product. More interesting results are those which relate the repartitions on a curve to the invariant derivations on its jacobian, and the repartition classes on an abelian variety to the differentials of the second kind; also, results 3.3 and 3.4, which relate Severi's base number ρ, for an abelian variety, to Picard's number ρ, connected to the logarithmic singularities of the differentials of the third kind. The differential class dX defined in section 3 provides a matrix, modulo p, which appears to be the first approximation to a p-adic matrix attached to the divisor X; the p-adic matrix would complete the set of the $E_q(X)$ of [13] and the $E_p(X)$ of [9].

Although no knowledge of the theory of sheaves is assumed on the part of the reader, the classes of repartitions studied here fit naturally in that theory (for a brief description of it one can consult the report by Zariski in the Bull. Amer. Math. Soc., vol. 62 (1956), pp. 117–141): if V is a locally normal irreducible variety over the algebraically closed field k, let us denote by $\mathfrak{O}$ and $\mathfrak{K}$ the algebraic sheaves on V defined by setting, respectively, $\mathfrak{O}_P = Q(P/V)$ and $\mathfrak{K}_P = k(V)$ for any $P \in V$ (here $\mathfrak{O}_P$ and $\mathfrak{K}_P$ denote stalks). The

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sequence $0 \to \mathfrak{O} \to \mathfrak{K} \to \mathfrak{K}/\mathfrak{O} \to 0$ is exact, and the associated exact cohomology sequence is $0 \to H^0(V, \mathfrak{O}) \to H^0(V, \mathfrak{K}) \to H^0(V, \mathfrak{K}/\mathfrak{O}) \to H^1(V, \mathfrak{O}) \to H^1(V, \mathfrak{K}) \to \cdots$. Since $H^0(V, \mathfrak{O}) \cong k$, it is easily verified that $\mathcal{B}(V)$, as defined in section 1, is isomorphic to $H^0(V, \mathfrak{K}/\mathfrak{O})$, and that in this isomorphism $\mathcal{B}_e(V)$ maps onto $H^0(V, \mathfrak{K})/k$; since, on the other hand, $H^1(V, \mathfrak{K}) = 0$, we conclude that $\mathcal{B}(V)/\mathcal{B}_e(V) \cong H^1(V, \mathfrak{O})$; and the study of $\mathcal{B}(V)/\mathcal{B}_e(V)$ forms the main subject of this paper.

Actually, the contents of the present paper can be considered as the case $r = 1$ in the investigation of the structure of the group of factor sets $\Gamma(A, V)/\Gamma_0(A, V)$, A and V being, respectively, an abelian variety and an r-dimensional periodic group-variety of period p^r (see [4]); it is the case which presents almost all the main obstacles, but not all the results. The case $r > 1$ can be studied by means of *hyperclasses of height* r, these being defined as almost everywhere finite mappings of the set of the irreducible divisors on A into additive Witt vectors of length r with components in $k(A)$, modulo everywhere finite mappings;[1] when this is done, the operation $\omega \to \omega^{1/p}$ introduced in section 4 is susceptible of a very "natural" interpretation; also, for $r \to \infty$, one can replace the differential class dX, or the matrix modulo p related to it, by a linear mapping of certain "vectors" $(D_1, D_p, D_{p^2}, \cdots)$ of invariant hyperderivations on A into hyperclasses of height ∞, or respectively by a p-adic matrix; this is the matrix to which we alluded before.

In establishing properties of the classes of repartitions on an abelian variety we have made use of previously acquired results on the differentials of the second kind, for the case of characteristic zero (see [6]), or on the invariant hyperderivations, for the case of positive characteristic [8]; the process could be reversed, up to a point, thus making the consideration of the differentials of the second kind unnecessary also in the case of characteristic zero, as it is in the case of positive characteristic.

Since the word "variety" has acquired a number of related but inequivalent meanings, it is perhaps not out of place to remind the reader that the word "variety", as used in the present work, means "algebraic subvariety of a projective space", that is, the set of all the points of a projective space over a field k, whose coordinates satisfy a finite number of algebraic homogeneous equations with coefficients in k.

1. Classes of repartitions

Let V be a locally normal irreducible n-dimensional variety over the field k, and assume k to be algebraically closed in $k(V)$; by locally normal we mean that $Q(X/V)$ is integrally closed for each irreducible subvariety X of V. For each $(n-1)$-dimensional irreducible subvariety X of V, let $x(X)$ be an element of $k(V)$; the mapping $\mathfrak{b}$ such that $\mathfrak{b}X = x(X) + Q(X/V)$ is called

[1] Witt vectors are used extensively in J-P. Serre's *Sur la topologie des variétés algébriques en caractéristique p* (mimeographed address, Symposium on Algebraic Topology, Mexico, Summer 1956).

a *class of repartitions*, or simply a *class*, *on* V, if $x(X) \in Q(X/V)$ for all, but finitely many, X's; the X's for which $x(X) \notin Q(X/V)$ are called the *poles of* $\mathfrak{b}$. If $x \in k(V)$, there exists a unique class $\mathfrak{b}$ such that $\mathfrak{b}X = x + Q(X/V)$ for each X; such $\mathfrak{b}$ is called the *class of* x *on* V, and denoted by $\mathrm{cl}_V\, x$, or simply $\mathrm{cl}\, x$ if no confusion may arise. The classes of the type $\mathrm{cl}\, x$ are called *exact classes*. If $\mathfrak{a}$, $\mathfrak{b}$ are classes on V, and $a \in k$, the classes $\mathfrak{a} + \mathfrak{b}$, $a\mathfrak{b} = \mathfrak{b}a$ are defined by $(\mathfrak{a} + \mathfrak{b})X = \mathfrak{a}X + \mathfrak{b}X$, $(a\mathfrak{b})X = a(\mathfrak{b}X) + Q(X/V)$; it is understood that the sums and products are to be interpreted elementwise. The set of the classes on V thus becomes a k-module, whose zero is $0 = \mathrm{cl}\, 0$; this is the only class without poles; it is easily seen that $\mathrm{cl}\, x = 0$ if and only if $x \in k$. Two classes $\mathfrak{a}$, $\mathfrak{b}$ are (*linearly*) *equivalent*, in symbols $\mathfrak{a} \sim \mathfrak{b}$, if $\mathfrak{a} - \mathfrak{b}$ is exact. We shall occasionally need the following definition: if y is an element of $k(V)$, none of whose poles is a pole of the class $\mathfrak{b}$, the class $y\mathfrak{b} = \mathfrak{b}y$ is defined by $(y\mathfrak{b})X = Q(X/V)$ if X is not a pole of $\mathfrak{b}$, and $(y\mathfrak{b})X = yx + Q(X/V)$ if X is a pole of $\mathfrak{b}$ and $\mathfrak{b}X = x + Q(X/V)$; if $y\mathfrak{a}$ and $y\mathfrak{b}$ exist, then $y(\mathfrak{a} + \mathfrak{b})$ exists and equals $y\mathfrak{a} + y\mathfrak{b}$; if $y\mathfrak{b}$ and $z\mathfrak{b}$ exist, then $(y + z)\mathfrak{b}$ exists, and equals $y\mathfrak{b} + z\mathfrak{b}$. If $\mathfrak{b}$ is a class on V, and k has positive characteristic p, the class $\mathfrak{b}^p$ is defined by $\mathfrak{b}^pX = x^p + Q(X/V)$ if $\mathfrak{b}X = x + Q(X/V)$.

Let W be an irreducible subvariety of V, and let $\mathfrak{b}$ be a class on V; we shall say that $\mathfrak{b}$ is (*locally*) *closed at* W if there exists a class $\mathfrak{a} \sim \mathfrak{b}$ none of whose poles contains W, or, equivalently, if there exists an $x \in k(V)$ such that $\mathfrak{b}X = x + Q(X/V)$ for each X containing W; any such x is called a *representative of* $\mathfrak{b}$ *at* W; any two representatives of $\mathfrak{b}$ at W differ by an element of $Q(W/V)$. If $\mathfrak{b}$ is closed at W, and U is an irreducible subvariety of V containing W, $\mathfrak{b}$ is also closed at U; if $\dim W = n - 1$, each class is closed at W; if W is not a subvariety of a pole of $\mathfrak{b}$, $\mathfrak{b}$ is closed at W. We shall say that the class $\mathfrak{b}$ is *closed* if it is closed at each $P \in V$, hence at each nonempty irreducible subvariety of V; every exact class is closed, and every class is closed if $n = 1$. The k-module of the closed classes on V will be denoted by $\mathfrak{B}(V)$, and the k-module of the exact classes on V will be denoted by $\mathfrak{B}_e(V)$; if $n = 1$, $\mathfrak{B}(V)$ is isomorphic to $\mathfrak{R}/\mathfrak{R}'$, where $\mathfrak{R}$, $\mathfrak{R}'$ are the k-modules of, respectively, the repartitions on V, and the repartitions without poles on V (see [10]); consequently, in this case, $\mathfrak{B}(V)/\mathfrak{B}_e(V)$ is a free k-module of order equal to the genus of V.

Quite in general, let $\mathfrak{b}$ be a class on V, and let W be a locally normal m-dimensional irreducible subvariety of V; assume that none of the poles of $\mathfrak{b}$ contains W; for each $(m - 1)$-dimensional irreducible subvariety U of W, let $\mathfrak{b}$ be closed at U, and let $x(U)$ be a representative of $\mathfrak{b}$ at U; then $x(U) \in Q(W/V)$, so that, if π denotes the homomorphic mapping of $Q(W/V)$ onto $k(W)$ whose kernel is $\mathfrak{P}(W/V)$, the expression $\pi x(U)$ denotes an element of $k(W)$. The set $\pi x(U) + \pi Q(U/V) = \pi x(U) + Q(U/W)$ is independent of the choice of $x(U)$, and consequently there exists a class $\mathfrak{b}'$ on W such that, for each U, $\mathfrak{b}'U = \pi x(U) + Q(U/W)$; the class $\mathfrak{b}'$ is denoted by $\mathfrak{b} \cap W$ or $W \cap \mathfrak{b}$, and called the *intersection of* W *and* $\mathfrak{b}$. If $\mathfrak{b}$ is closed, so is $\mathfrak{b} \cap W$;

if $\mathfrak{b}$ is exact, so is $\mathfrak{b} \cap W$; if Y is an irreducible subvariety of W, and $\mathfrak{b}$ is closed at Y, $\mathfrak{b} \cap W$ is also closed at Y; moreover, if $Y \cap \mathfrak{b}$ and $W \cap \mathfrak{b}$ exist, then $Y \cap (W \cap \mathfrak{b})$ exists and equals $Y \cap \mathfrak{b}$. If W is as before, for each closed class $\mathfrak{b}$ on V there exists an $\mathfrak{a} \sim \mathfrak{b}$ such that $\mathfrak{a} \cap W$ exists; hence, the mapping $\mathfrak{b} \rightarrow W \cap \mathfrak{b}$ induces a homomorphism of $\mathcal{B}(V)/\mathcal{B}_e(V)$ into $\mathcal{B}(W)/\mathcal{B}_e(W)$.

Let F be another locally normal irreducible variety over k, of dimension r, and assume $F \times V$ to be locally normal (this being the case, for instance, if k is algebraically closed). If $\mathfrak{b}$ is a class on V, the class $\mathfrak{b} \times F = F \times \mathfrak{b}$ on $V \times F$ is defined by $(\mathfrak{b} \times F)(X \times F) = \mathfrak{b}X + Q(X \times F/V \times F)$ for each $(n - 1)$-dimensional irreducible subvariety X of V, and $(\mathfrak{b} \times F)Y = Q(Y/V \times F)$ for each $(r + n - 1)$-dimensional irreducible subvariety Y of $V \times F$, not of the type $X \times F$. The mapping $\mathfrak{b} \rightarrow \mathfrak{b} \times F$ is an isomorphism of k-modules; also, $\mathfrak{b} \times F$ is closed, or exact, if and only if $\mathfrak{b}$ is, respectively, closed or exact.

If k is algebraically closed, and $\mathfrak{b}$ is a class on V, the *extension* $\mathfrak{b}_K$ *of* $\mathfrak{b}$ *over* the extension K of k is defined by $\mathfrak{b}_K X_K = \mathfrak{b}X + Q(X_K/V_K)$ for each $(n - 1)$-dimensional irreducible subvariety X of V, and $\mathfrak{b}_K Y = Q(Y/V_K)$ for each $(n - 1)$-dimensional irreducible subvariety Y of V_K not of the type X_K; again, the mapping $\mathfrak{b} \rightarrow \mathfrak{b}_K$ is an isomorphism of k-modules; also, $\mathfrak{b}_K$ is closed, or exact, if and only if $\mathfrak{b}$ is, respectively, closed or exact.

If F has the previous meaning, let $\mathfrak{b}$ be a class on $V \times F$; the *V-component of* $\mathfrak{b}$ is the class $\mathfrak{a}$ on $V \times F$ such that $\mathfrak{a}Y = \mathfrak{b}Y$ if Y is not of the type $V \times X$ with $X \subset F$, while $\mathfrak{a}Y = Q(Y/V \times F)$ otherwise; on the other hand, if $K = k(F)$, $\mathfrak{b}\{F\}$ shall denote the class $\mathfrak{B}$ on V_K such that $\mathfrak{B}(Y\{F\}) = \mathfrak{b}Y$ for each Y not of the type $V \times X$; here, $Y\{F\}$ has a meaning since Y is an algebraic correspondence between F and V (see section 1 of [1]); we thus have $\mathfrak{b}\{F\} = \mathfrak{a}\{F\}$. If $\mathfrak{b}$ is closed or exact, so is $\mathfrak{b}\{F\}$, although the converse is false; also, given a class $\mathfrak{B}$ on V_K, there is a unique class $\mathfrak{b}$ on $V \times F$ which coincides with its own V-component, and is such that $\mathfrak{B} = \mathfrak{b}\{F\}$. If $\mathfrak{B}$ is exact, $\mathfrak{b}$ differs from an exact class by a class whose poles are all of the type $V \times X$, with $X \subset F$.

Let k be arbitrary again, and let U be another locally normal irreducible variety over k, of dimension m; let λ be a rational mapping of U onto V, without fundamental points on U. If $\mathfrak{b}$ is a class on V, its *extension* or *cotrace on* U, in symbols $T_\lambda^{-1}\mathfrak{b}$, is the class $\mathfrak{B}$ on U defined, when possible, in the following manner: for each $(m - 1)$-dimensional irreducible subvariety Y of U, set $X = \lambda Y$, and let x be a representative of $\mathfrak{b}$ at X; set then $\mathfrak{B}Y = x + Q(Y/U)$. Since $Q(X/V) \subseteq Q(Y/U)$, $\mathfrak{B}$ is defined if and only if $\mathfrak{b}$ is closed at each component of the fundamental locus of λ on V; in particular, it is defined when $\mathfrak{b}$ is closed, in which case $\mathfrak{B}$ is also closed; if $\mathfrak{b}$ is exact, so is $\mathfrak{B}$ (but the converse is generally false). An equivalent definition, if $U \times V$ is locally normal, is the following: if the rational mapping λ of U onto V is considered as a subvariety of $U \times V$, $\mathfrak{B}$ is the unique class on U for which $(\mathfrak{B} \times V) \cap \lambda = (\mathfrak{b} \times U) \cap \lambda$; this will be taken as definition of $T_\lambda^{-1}\mathfrak{b}$ when

λ is into V rather than onto V. We may remark that, in this case, $T_\lambda^{-1}\mathfrak{b} = T_\lambda^{-1}(\mathfrak{b} \cap \lambda U)$ when λU is locally normal.

Assume now λ to be onto V, and $m = n$, so that $k(U)$ is a finite extension of $k(V)$; if $\mathfrak{B}$ is a class on U, its *trace* $T_\lambda \mathfrak{B}$ *on* V is the class $\mathfrak{b}$ on V defined as follows: for each $(n - 1)$-dimensional irreducible subvariety X of V, let $Y_1, \cdots, Y_r$ be the distinct irreducible subvarieties of U on which $\lambda\{X\}$ operates; then each Y_i has dimension $n - 1$, and $X = \lambda Y_i$; let x be a representative of $\mathfrak{B}$ at each Y_i; such x exists by Krull's theorem on the independence of valuations. Set then $\mathfrak{b}X = Tx + Q(X/V)$, where T denotes the usual trace from $k(U)$ to $k(V)$; it is clear that $\mathfrak{b}$ does not depend on the choice of x. If $\mathfrak{B}$ is exact, so is $\mathfrak{b}$; if $\mathfrak{B}$ has the property that for any finite set S of points of U, there exists an $\mathfrak{A} \sim \mathfrak{B}$ none of whose poles contains any point of S, and if in addition no pole of $\mathfrak{b}$ intersects the fundamental locus of λ on V, $\mathfrak{b}$ is closed. Finally, if $\mathfrak{a}$ is a class on V and $T_\lambda^{-1}\mathfrak{a}$ exists, we have $T_\lambda T_\lambda^{-1}\mathfrak{a} = [k(U):k(V)]\mathfrak{a}$. These definitions are in agreement with those given in [10] for the repartitions.

If V has the usual meaning, and k is algebraically closed, we shall say that a set $\{\mathfrak{b}_1, \mathfrak{b}_2, \cdots\}$ of elements of $\mathfrak{G}(V)$ is a *set of representatives of an absolute basis of* $\mathfrak{G}(V)/\mathfrak{G}_e(V)$ if the $\mathfrak{b}_i$ are linearly independent over k, mod $\mathfrak{G}_e(V)$, and if, in addition, for each extension K of k, each element of $\mathfrak{G}(V_K)$ is equivalent to a linear combination, with coefficients in K, of (finitely many) elements $(\mathfrak{b}_i)_K$.

1.1. THEOREM. *Let U, V be locally normal irreducible varieties over the algebraically closed field k, of dimensions* 1 *and n respectively; assume that there exists a set of representatives of an absolute basis of $\mathfrak{G}(V)/\mathfrak{G}_e(V)$. If $F = U \times V$, there exists a set of representatives of an absolute basis of $\mathfrak{G}(F)/\mathfrak{G}_e(F)$; moreover, for each $\mathfrak{b} \in \mathfrak{G}(F)$, there exist elements $\mathfrak{c}$, $\mathfrak{d}$ of $\mathfrak{G}(U)$, $\mathfrak{G}(V)$ respectively, unique* mod $\mathfrak{G}_e(U)$ *and* mod $\mathfrak{G}_e(V)$ *respectively, such that* $\mathfrak{b} \sim \mathfrak{c} \times V + U \times \mathfrak{d}$.

Proof. Set $H = k(U)$, $K = k(V)$; it is well known [10] that if $\sum_{i=1}^{g} X_i$ is a nonspecial cycle on U (where the X_i are distinct points of U), a set of representatives of an absolute basis of $\mathfrak{G}(U)/\mathfrak{G}_e(U)$ is given by the set $\{\mathfrak{u}_1, \cdots, \mathfrak{u}_g\}$, $\mathfrak{u}_i$ being the class on U whose only pole is X_i, and such that $\mathfrak{u}_i X_i = x_i^{-1} + Q(X_i/U)$, for a given regular parameter x_i of $Q(X_i/U)$. Let then $\{\mathfrak{v}_1, \mathfrak{v}_2, \cdots\}$ be a set of representatives of an absolute basis of $\mathfrak{G}(V)/\mathfrak{G}_e(V)$; for a $\mathfrak{b} \in \mathfrak{G}(F)$, set $\mathfrak{b}^* = \mathfrak{b}\{V\}$; then, for a suitable $t \in k(F)$, we have $\mathfrak{b}^* - \mathrm{cl}^* t = \sum_i z_i(\mathfrak{u}_i)_K$, where $z_i \in K$, and cl^* means class on U_K. But then, $\mathfrak{b}' = \mathfrak{b} - \mathrm{cl}_F\, t = \sum_i z_i(\mathfrak{u}_i \times V) + \mathfrak{a}$, where $\mathfrak{a}$ is a class on F whose poles are all of the type $U \times Y$, with $Y \subset V$; as a consequence,

$$\mathfrak{b}'\{U\} = \mathfrak{a}\{U\} = \textstyle\sum_i z_i'(\mathfrak{v}_i)_H + \mathrm{cl}'\, t',$$

where $z_i' \in H$, cl' means class on V_H, $t' \in k(F)$, and the summation is extended over finitely many i's. Since t' has, on V_H, only poles of the type Y_H, with $Y \subset V$, it belongs to the direct product $H \times K$ over k; it follows that each pole of $\mathfrak{b}'$ is either of the type $X_i \times V$, in which case we shall say that it is of the first type, or of the type $U \times Y$, with $Y \subset V$, in which case we shall say that it is of the second type; and it also follows that a representative of $\mathfrak{b}'$ at $X_i \times V$ is $x_i^{-1}z_i$, while at a pole of the second type $\mathfrak{b}'$ can be represented by an element of $H \times K$. We intend to prove that $z_i \in k$ for each i, and that a representative of $\mathfrak{b}'$ at a pole of the second type can be selected in K.

Quite in general, let X be a point of U, Y an $(n-1)$-dimensional irreducible subvariety of V, and let x, y be regular parameters of, respectively, $Q(X/U)$, $Q(Y/V)$; let ab, $c = \sum_{i=1}^h a_i b_i$, with $a, a_i \in H$, $b, b_i \in K$, be representatives of $\mathfrak{b}'$ at, respectively, $X \times U$ and $V \times Y$; assume also that h is the smallest integer for which such a representative exists. Set

$$\mathfrak{O} = Q(X \times Y/F), \qquad \mathfrak{R} = Q(X/U), \qquad \mathfrak{S} = Q(Y/V),$$

and let $\mathfrak{O}^*$, $\mathfrak{R}^*$, $\mathfrak{S}^*$ be their completions, so that $\mathfrak{O}^* = k(Y)\{x, y]$, $\mathfrak{R}^* = k\{x]$, $\mathfrak{S}^* = k(Y)\{y]$; let $\mathfrak{L}$, $\mathfrak{H}$, $\mathfrak{K}$ be the fields of quotients of, respectively, $\mathfrak{O}^*$, $\mathfrak{R}^*$, $\mathfrak{S}^*$; we shall identify $k(F)$ with a subfield of $\mathfrak{L}$; notice that the smallest subring of $\mathfrak{L}$ containing $\mathfrak{H}$ and $\mathfrak{K}$ is their direct product $\mathfrak{H} \times \mathfrak{K}$ over k. We shall also denote by u, v the normalized valuations of, respectively, H, K over k, whose centers on U, V are respectively X, Y; u and v shall also denote the extensions of u, v to $k(F)$ (or to $\mathfrak{L}$) over, respectively, K, H (or, respectively, $\mathfrak{K}$, $\mathfrak{H}$).

Consider the case in which $-r = \min[u(a), u(a_1), \cdots, u(a_h)] < 0$ and $-s = \min\,[v(b), v(b_1), \cdots, v(b_h)] < 0$. Since $\mathfrak{b}'$ is closed, there exists an element $\varphi \in k(F)$, with no other poles containing $X \times Y$ but, at most, $X \times V$ and $U \times Y$, and such that

$$f = \varphi - ab \in Q(X \times V/F), \qquad g = \varphi - c \in Q(U \times Y/F);$$

we have $u(\varphi) \geqq -r$, $v(\varphi) \geqq -s$, $v(f) \geqq -s$, $u(g) \geqq -r$, so that

$$x^r y^s f \in x^r \mathfrak{O}, \qquad x^r y^s g \in y^s \mathfrak{O}.$$

The previous two relations give then

$$(x^r a)(y^s b) - \textstyle\sum_i (x^r a_i)(y^s b_i) \in x^r \mathfrak{O} + y^s \mathfrak{O}.$$

When the elements a and a_i of $\mathfrak{H}$ are written as power series in x, let a', a_i' be the "principal parts" of such series, that is, the parts involving only negative exponents; let b', b_i' be similarly related to $b, b_i \in \mathfrak{K}$; then the previous relation implies $x^r y^s(a'b' - \sum_i a_i' b_i') = 0$, or $a'b' = \sum_i a_i' b_i'$. The following two statements are consequences of this equality:

1.2. If Y is a pole of b, so that $b' \neq 0$, and if $X \times V$ is a pole of $\mathfrak{b}'$, so that $a' \neq 0$, then $a_i' b_i' \neq 0$ for some i; hence, by the minimal property of h, $\mathfrak{b}'$ has a pole at $U \times Y$, and some a_i has a pole at X.

1.3. If $U \times Y$ is a pole of $\mathfrak{b}'$, so that, by the minimal property of h, the b_i' are linearly independent over $\mathfrak{H}$, and if X is a pole of some a_i, say a_1, so that $a_1' \neq 0$, then $a' \neq 0$ and $b' \neq 0$, so that $\mathfrak{b}'$ has a pole at $X \times V$, and b a pole at Y; moreover, b' is linearly dependent on $b_1', \cdots, b_h'$ over $\mathfrak{H}$, hence over k, so that there are elements $\gamma_1, \cdots, \gamma_h \in k$ such that $b' = \sum_i \gamma_i b_i'$; but then, by the linear independence of the b_i', we have $a_i' = \gamma_i a'$ for each i, $a_1' = \gamma_1 a'$, $\gamma_1 \neq 0$, so that a has, at X, a pole of the same order as a_1.

We can now apply these results to the case at hand; assume, for instance, z_1 not to belong to k; if we take $X = X_1$, $Y =$ *a pole of* z_1 *on* V, 1.2 says that $U \times Y$ is a pole of $\mathfrak{b}'$; also, if a representative of $\mathfrak{b}'$ at $U \times Y$ is $\sum_{i=1}^h a_i b_i$, where h has the minimal property, some a_i has a pole at X_1; but, by 1.3, any pole of a_i is an X_j, and each a_i has at each X_j a pole of order at most 1. This contradicts the fact that $\sum_j X_j$ is a nonspecial divisor; we conclude that z_1, hence each z_i, belongs to k; but then 1.3 implies that either $\mathfrak{b}'$ has no pole of the second type, or that if it does, at each one of them it is possible to select a representative of $\mathfrak{b}'$ which belongs to K. We have thus proved our contention, an immediate consequence of which is that

$$\mathfrak{b}' = \mathfrak{c} \times V + U \times \mathfrak{d},$$

where $\mathfrak{c}$, $\mathfrak{d}$ are classes on U, V respectively; hence, $\mathfrak{b} \sim \mathfrak{c} \times V + U \times \mathfrak{d}$. The classes $\mathfrak{c}$ and $\mathfrak{d}$ are certainly closed since, for instance,

$$\mathfrak{c} \times P = \mathfrak{b}' \cap (U \times P)$$

for a generic $P \in V$; this also proves that $\mathfrak{c}$ and $\mathfrak{d}$ are unique mod $\mathcal{B}_e(U)$ or $\mathcal{B}_e(V)$ respectively, Q.E.D.

1.4. Lemma. *Let U, V be locally normal irreducible varieties over k, of the same dimension, and let λ be a rational mapping of U onto V, without fundamental points on U; assume $k(V) \subsetneq k(U)$ as prescribed by λ. If there exists a class $\mathfrak{b}$ on V which is not exact, but such that $T_\lambda^{-1}\mathfrak{b}$ is exact, k has characteristic $p \neq 0$, and $[k(U):k(V)]$ is divisible by p. If, in addition, $k(U)$ is a normal separable extension of $k(V)$, with Galois group G, there exists an invariant proper subgroup N of G such that G/N is a direct product of cyclic groups of order p.*

Proof. We have $T_\lambda T_\lambda^{-1}\mathfrak{b} = [k(U):k(V)]\mathfrak{b}$; thus, $[k(U):k(V)]\mathfrak{b}$ is exact, hence $= 0$, since $\mathfrak{b}$ is not exact; this proves that $[k(U):k(V)]$, as an element of k, is zero, and also proves the first two statements. Assume now $k(U)$ to be separable normal over $k(V)$, and set $T_\lambda^{-1}\mathfrak{b} = \mathrm{cl}_U\, x$. For each $g \in G$ we must have $\mathrm{cl}_U\, x = \mathrm{cl}_U\, gx$, so that $gx = x + h(g)$, $h(g) \in k' =$ *algebraic closure of k in $k(U)$*. The mapping $g \to h(g)$ is an additive character of G into k', and is not principal since $x \notin k(V)$; the existence of N follows immediately, Q.E.D.

2. Classes on abelian varieties

From now on, k will denote an algebraically closed field, of characteristic $p \geqq 0$.

Let A be a nonsingular abelian variety over k, and let $\mathfrak{b}$ be a class on A; for $P \in A$, $\sigma_P \mathfrak{b}$ is defined by $(\sigma_P \mathfrak{b})X = \sigma_P [\mathfrak{b}\sigma_P^{-1}X]$ (see [3] for the definition of σ_P); $\mathfrak{b}$ is invariant if $\sigma_P \mathfrak{b} = \mathfrak{b}$ for each P, and *semi-invariant* if $\sigma_P \mathfrak{b} \sim \mathfrak{b}$ for each P; the only invariant class is the class 0; any semi-invariant class is closed. If D is an invariant hyperderivation on A (see [8]), the class $D\mathfrak{b}$ is defined by $(D\mathfrak{b})X = Dx + Q(X/A)$ if $\mathfrak{b}X = x + Q(X/A)$; since D applies $Q(X/A)$ into itself (this was shown in the course of the proof of 2.1 of [6] if D is a derivation, and can be shown in the same manner, as a consequence of 1.5 of [8], if D is a hyperderivation), this definition is independent of the choice of x. If $\mathfrak{b}$ is, respectively, closed, exact, or semi-invariant, $D\mathfrak{b}$ has the same property.

2.1. LEMMA. *Let C be an irreducible curve over k, without singularities, of genus n; let J be the jacobian, supposed nonsingular, of C. Then the k-module $\mathfrak{B}(J)/\mathfrak{B}_e(J)$ has order $\leqq n$ if $p \neq 2$ or $n < 2$, and order $\leqq n + 1$ if $p = 2$ and $n \geqq 2$.*

Proof. For $n = 0$, the result is trivially true; for $n = 1$, it is known (see [10]) that $\mathfrak{B}(C)/\mathfrak{B}_e(C)$ has order 1. Suppose then $n \geqq 2$, and let $C_1 = C, C_2, \cdots, C_n$ be copies of C; set $V = C_1 \times \cdots \times C_n$: then [13] $k(J)$ is the set of the elements of $k(V)$ which are invariant under each element of the group G of the $n!$ permutations among the C_i's. Let Φ be the rational mapping of V onto J generated by this embedding; then Φ has no fundamental point on V, by Theorem 6 of [13]. Let $\mathfrak{b} \in \mathfrak{B}(J)$, and set $\mathfrak{B} = T_\Phi^{-1}\mathfrak{b}$, so that $\mathfrak{B} \in \mathfrak{B}(V)$; by 1.1, $\mathfrak{B}$ is equivalent to a class of the type

$$\textstyle\sum_i C_1 \times \cdots \times C_{i-1} \times \mathfrak{a}_i \times C_{i+1} \times \cdots \times C_n ,$$

where $\mathfrak{a}_i \in \mathfrak{B}(C_i)$ is unique but for equivalence. Since $\mathfrak{B}$ is invariant for each element of G, the $\mathfrak{a}_i$ can be selected to be the copies, on C_i, of a same class $\mathfrak{a}$ on C; the correspondence $\mathfrak{b} \to \mathfrak{a}$ establishes a homomorphism φ^* of $\mathfrak{B}(J)/\mathfrak{B}_e(J)$ into $\mathfrak{B}(C)/\mathfrak{B}_e(C)$.

Let $\mathfrak{b}$ be such that $\mathfrak{b} + \mathfrak{B}_e(J)$ belongs to the kernel of φ^*; this is so if and only if $\mathfrak{B}$ is exact; 1.4 implies then that, if $p = 0$, $\mathfrak{b}$ itself must be exact. If $p \neq 0$, $\mathfrak{b}$ may not be exact, again by 1.4, only if G contains an invariant subgroup N of index p; now, the only proper subgroup N of G such that G/N is commutative is the alternating group on n elements, which has index 2; thus, unless $p = 2$, $\mathfrak{b}$ must be exact. If $p = 2$, each $\mathfrak{b}$ such that $\mathfrak{B}$ is exact defines, according to the proof of 1.4, an additive character $h = h_\mathfrak{b}$ of G, and $h_\mathfrak{b}$ is the principal character if and only if $\mathfrak{b} \in \mathfrak{B}_e(J)$; also, the kernel of $h_\mathfrak{b}$ is either G or N. For any two elements $\mathfrak{a} + \mathfrak{B}_e(J)$, $\mathfrak{b} + \mathfrak{B}_e(J)$ of the kernel of φ^*, and for a $g \in G - N$, set $h_\mathfrak{a}(g) = a$, $h_\mathfrak{b}(g) = b$, where $a, b \in k$; if $\mathfrak{c} = b\mathfrak{a} + a\mathfrak{b}$, we have $h_\mathfrak{c}(g) = 0$, so that $h_\mathfrak{c}$ is the principal character, and

$\mathfrak{c} \in \mathfrak{B}_e(J)$. This proves that in this case the kernel of φ^* has order at most 1; therefore, ord $\mathfrak{B}(J)/\mathfrak{B}_e(J) \leqq$ ord $\mathfrak{B}(C)/\mathfrak{B}_e(C)$ + ord (*kernel of* φ^*), and this equals n if $p \neq 2$, or not more than $n + 1$ if $p = 2$, Q.E.D.

Remark. If $\varphi = [\Phi; C_1 \times P_2 \times \cdots \times P_n, J]$ is the restriction of Φ to $C_1 \times P_2 \times \cdots \times P_n$, where $P_i \in C_i$, it is easily verified that

$$T_\varphi^{-1}\mathfrak{b} \in \varphi^*(\mathfrak{b} + \mathfrak{B}_e(J)) \times P_2 \times \cdots \times P_n,$$

when this exists.

2.2. Lemma. *Let A be a nonsingular abelian variety over k; if there exists an integer N such that, for any algebraically closed extension K of k, $\mathfrak{B}(A_K)/\mathfrak{B}_e(A_K)$ is a K-module of order $\leqq N$, each closed class on A is semi-invariant. In particular, each closed class on a nonsingular jacobian variety is semi-invariant.*

Proof. The last statement follows from the main statement and from 2.1. In order to prove the main statement, we shall remark first that, for some K, the order of $\mathfrak{B}(A_K)/\mathfrak{B}_e(A_K)$ will reach a maximum, which we may assume to be N; if we then prove that each closed class on A_K is semi-invariant, the same will be true of the closed classes on A. We shall therefore assume that the maximum N is already reached on A, so that there exists a set of representatives $\{\mathfrak{b}_1, \cdots, \mathfrak{b}_N\}$ of an absolute basis of $\mathfrak{B}(A)/\mathfrak{B}_e(A)$. Let A_1 be a copy of A; if $\{x\}$ is a n.h.g.p. (nonhomogeneous general point) of A, let $\{y\}$ be its copy in $k(A_1)$; set $H = k(A_1)$, and let X be the rational point of A_H at which the x_i acquire the values y_i. Then $\sigma_X(\mathfrak{b}_i)_H \sim \sum_j z_{ij} (\mathfrak{b}_j)_H$, where the z_{ij} are uniquely determined elements of H. Denote by Z the matrix (z_{ij}), and by $\mathfrak{B}$ the one-column matrix of the $\mathfrak{b}_i$; the previous relation can be written $\sigma_X \mathfrak{B}_H \sim Z\mathfrak{B}_H$.

For a generic $P \in A$, let $Z(P)$ be obtained from Z by replacing each z_{ij} by the value it acquires at the copy P_1 of P on A_1; then $\sigma_P \mathfrak{B} \sim Z(P)\mathfrak{B}$. If Q is another generic point of A, we have $\sigma_{PQ} \mathfrak{B} \sim Z(PQ)\mathfrak{B}$, but also

$$\sigma_{PQ} \mathfrak{B} = \sigma_Q \sigma_P \mathfrak{B} \sim \sigma_Q Z(P)\mathfrak{B} \sim Z(P)Z(Q)\mathfrak{B},$$

so that $Z(PQ) = Z(P)Z(Q)$. Hence, and by Lemma 2.3 of [4], the embedding $k(z) \subseteq k(A_1)$ establishes a homomorphism of A onto the Vessiot variety V whose n.h.g.p. is $\{z\}$, and whose law of composition is the multiplication of matrices; since A is abelian, V must reduce to a point, or $z_{ij} = \delta_{ij}$ (Kronecker's delta), or finally $\sigma_P \mathfrak{B} \sim \mathfrak{B}$ for each $P \in A$, Q.E.D.

We now introduce the k-module $X(A, \alpha)$ already used in section 3 of [8]: if α is a homomorphism of the nonsingular abelian variety A onto an abelian variety B, $X(A, \alpha)$ is the set of the $x \in k(A)$ such that $\sigma_P x - x \in k(B)$ for each $P \in A$; notice that now we do not require α to have positive degree. For the meaning of $\mathfrak{D}_r(A)$, $\mathfrak{D}(A)$, which we shall need in the following proof, see section 2 of [8]. We have:

2.3. Theorem. *Let A, B be nonsingular abelian varieties over k, and let α be a homomorphism of A onto B; write $\alpha = \beta\gamma$, where β has positive degree, and γ is such that its inseparability is 1, and its kernel is irreducible; set $C = \gamma A$, and assume $k(B) \subsetneqq k(C) \subsetneqq k(A)$ as prescribed by β and γ, so that $k(C)$ is the algebraic closure of $k(B)$ in $k(A)$; select C to be nonsingular. Then $X(A, \alpha) = X(C, \beta)$; also, $x \in X(A, \alpha)$ if and only if $\sigma_P x - x \in k$ for each P of the kernel of α, and $Dx \in k$ for each D of the nucleus of α (see section 2 of* [8]). *The free k-module $X(A, \alpha)/k(B)$ has order 0 if $p = 0$, and order $\leqq \nu$ if $p \neq 0$ and p^ν is the largest power of p which divides the degree of β. For each $x \in X(A, \alpha)$ there is a semi-invariant $\mathfrak{b} \in \mathfrak{B}(B)$ such that $T_\alpha^{-1}\mathfrak{b} = \mathrm{cl}_A x$; conversely, if $\mathfrak{b} \in \mathfrak{B}(B)$ and $T_\alpha^{-1}\mathfrak{b} = \mathrm{cl}_A y$ for a $y \in k(A)$, then $y \in X(A, \alpha)$. The correspondence $x \to \mathfrak{b}$ generates an isomorphism of $X(A, \alpha)/k(B)$ into $\mathfrak{B}(B)/\mathfrak{B}_e(B)$.*

Proof. If $x \in X(A, \alpha)$, for each $P \in A$ we have $\sigma_P x - x = h(P) \in k(B)$; if also $Q \in A$, we have $h(PQ) = \sigma_{PQ} x - x = \sigma_P(\sigma_Q x - x) + \sigma_P x - x = \sigma_P h(Q) + h(P)$. If P belongs to the kernel L of α, this gives $h(PQ) = h(Q) + h(P)$; on the other hand, $h(PQ) = h(QP) = \sigma_Q h(P) + h(Q)$, so that $\sigma_Q h(P) = h(P)$, which implies that $h(P) \in k$, as claimed. If now also $Q \in L$, we see that the mapping $P \to h(P)$ is a homomorphism of the group of the points of L into the additive group k; if P is restricted to the component L_0 of the identity in L, it is easily verified that the mapping h is also a homomorphism of L_0 into a 1-dimensional vector variety over k; this proves that $h(P) = 0$ for each $P \in L_0$, so that $x \in k(C)$. It is thus proved that

$$X(A, \alpha) = X(C, \beta).$$

If D is an invariant hyperderivation on A belonging to the nucleus N of α, we have $\sigma_P Dx - Dx = 0$ for each $P \in A$, so that $Dx \in k$, as claimed. Conversely, assume $\sigma_P x - x = h(P) \in k$ for each $P \in L$, and $Dx \in k$ for each $D \in N$; for each $Q \in A$ set $h(Q) = \sigma_Q x - x$. Then, for $P \in L$, $\sigma_P h(Q) = \sigma_{PQ} x - \sigma_P x = \sigma_{PQ} x - x - (\sigma_P x - x) = h(PQ) - h(P) = h(QP) - \sigma_Q h(P) = \sigma_{QP} x - x - \sigma_Q(\sigma_P x - x) = \sigma_Q x - x = h(Q)$, so that, by Lemma 2.1 of [4], $h(Q)$ is purely inseparable over $k(B)$. But we also have $Dh(Q) = 0$ if $D \in N$, so that, by 2.3 of [8], $h(Q) \in k(B)$, and $x \in X(A, \alpha)$, as claimed.

Now, let L' be the kernel of β, and for each $x \in X(A, \alpha) = X(C, \beta)$, and each $P \in L'$, set $l(P, x) = \sigma_P x - x \in k$; for a fixed x, l is a homomorphic mapping of L' into k; since L' is finite, this implies that, if $p = 0$, $l(P, x) = 0$, or $x \in k(B)$; hence, in this case, $X(A, \alpha)/k(B)$ is a k-module of order 0. If $p \neq 0$, the kernel L' is the direct product of a group L_1, whose order is prime to p, and a group L_2 whose order is a power of p; accordingly, we can write $\beta = \beta_2\beta_1$, where the kernel of β_1 is L_1 (and β_1 is separable), while the kernel of β_2 is $\beta_1 L_2 \cong L_2$. Set $F = \beta_1 C$; if $x \in X(C, \beta)$ and $P \in L_1$, we see again that $l(P, x) = 0$, so that $x \in X(F, \beta_2)$. Now, if $P_1, \cdots, P_s$ are the generators of $\beta_1 L_2$, for each $x \in X(F, \beta_2)$ set $l_i(x) = \sigma_{P_i} x - x \in k$; then x is purely inseparable over $k(B)$ if and only if $l_i(x) = 0$ for each i. Write

$$\beta_2 = \zeta\zeta_0\,\zeta_1 \cdots \zeta_r,$$

where ζ is separable, each ζ_i is purely inseparable, and $[k(F)]^p \subsetneq k(\zeta_r F)$, $[k(\zeta_i\,\zeta_{i+1} \cdots \zeta_r F)]^p \subseteqq k(\zeta_{i-1}\,\zeta_i \cdots \zeta_r F)$ for each $i \geqq 1$ and $\leqq r$; for $i = 0, 1, \cdots, r - 1$, let $\{D_{i1}, \cdots, D_{is_i}\}$ be a set of elements of $N \cap \mathfrak{D}(F)$ which induce in $k(\zeta_{i+1} \cdots \zeta_r F)$ a k-independent basis of $N_i \cap \mathfrak{D}_0\,(\zeta_{i+1} \cdots \zeta_r F)$, N_i being the nucleus of ζ_i; let also $\{D_{r1}, \cdots, D_{rs_r}\}$ be a k-independent basis of $N_r \cap \mathfrak{D}_0\,(F)$. Then the mapping

$$x \rightarrow \{l_1(x), \cdots, l_s(x); D_{01}\,x, \cdots, D_{0s_0}\,x; \cdots; D_{r1}\,x, \cdots, D_{rs_r}\,x\}$$

induces an isomorphism of $X(F, \beta_2)/k(B)$ into a k-module of order $s + s_0 + s_1 + \cdots + s_r$; this sum, by 2.3 of [8], is $\leqq \nu$, as claimed.

Given a $\mathfrak{b} \in \mathfrak{B}(B)$ such that $T_\alpha^{-1}\mathfrak{b} = \mathrm{cl}_A\, x$, we have, for $P \in L$,

$$\mathrm{cl}_A\,(\sigma_P\, x - x) = T_\alpha^{-1}(\sigma_P\,\mathfrak{b} - \mathfrak{b}) = 0,$$

or $\sigma_P\, x - x \in k$; also, $\mathrm{cl}_A\, Dx = T_\alpha^{-1}D\mathfrak{b} = 0$ if $D \in N$; thus, $x \in X(A, \alpha)$. Conversely, if $x \in X(A, \alpha)$, for each irreducible subvariety Y of B, of dimension equal to $\dim B - 1$, set $\mathfrak{b}Y = x - \sigma_P\, x + Q(Y/B)$, where $P \in A$ is selected in such a way that no component of $\alpha^{-1}Y$ is a pole of $\sigma_P\, x$. Then $\mathfrak{b} \in \mathfrak{B}(B)$, and $T_\alpha^{-1}\mathfrak{b} = \mathrm{cl}_A\, x$; we have $\mathfrak{b} \in \mathfrak{B}_e(B)$ if and only if $x \in k(B)$. Finally, for any such $\mathfrak{b}$, and for any $Q \in A$, we have, after setting $P = \alpha Q$:

$$T_\alpha^{-1}(\sigma_P\,\mathfrak{b} - \mathfrak{b}) = \mathrm{cl}_A\,(\sigma_Q\, x - x);$$

as $\sigma_Q\, x - x = y \in k(B)$, this implies $\sigma_P\,\mathfrak{b} - \mathfrak{b} = \mathrm{cl}_B\, y$, so that $\mathfrak{b}$ is semi-invariant, Q.E.D.

2.4. COROLLARY. *Let A be a nonsingular abelian variety over k; then:* (1) *$\mathfrak{B}(A)/\mathfrak{B}_e(A)$ is a finite free k-module, and there exists an integer N such that $\mathfrak{B}(A_K)/\mathfrak{B}_e(A_K)$ has order $\leqq N$ for each algebraically closed extension K of k;* (2) *a class on A is closed if and only if it is semi-invariant.*

Proof. It is known that A is the homomorphic image, in a homomorphism α, of a jacobian J; we shall use the letter α also to denote the extension α_K of α to a homomorphism of J_K onto A_K. If $\mathfrak{K}_K$ denotes the K-module of the $\mathfrak{b} \in \mathfrak{B}(A_K)$ such that $T_\alpha^{-1}\mathfrak{b} \in \mathfrak{B}_e(J_K)$, by 2.3 there exists an integer ν, independent of K, such that $\mathrm{ord}\,\mathfrak{K}_K/\mathfrak{B}_e(A_K) \leqq \nu$; on the other hand, T_α^{-1} induces a homomorphism of $\mathfrak{B}(A_K)/\mathfrak{B}_e(A_K)$ into $\mathfrak{B}(J_K)/\mathfrak{B}_e(J_K)$, whose kernel is $\mathfrak{K}_K/\mathfrak{B}_e(A_K)$. Since, by 2.1, $\mathrm{ord}\,\mathfrak{B}(J_K)/\mathfrak{B}_e(J_K) \leqq 1 + \dim J$, we conclude that $\mathrm{ord}\,\mathfrak{B}(A_K)/\mathfrak{B}_e(A_K) \leqq 1 + \nu + \dim J$, and this proves statement (1). Statement (2) is a consequence of this, and of 2.2, Q.E.D.

2.5 LEMMA. *Let A be a nonsingular abelian variety over k, and let A_1, A_2, A_3 be copies of A; let D be the rational mapping of $A_1 \times A_2$ onto A_3 which gives* *the law of composition on A, and assume $k(A_3) \subsetneq k(A_1 \times A_2)$ as prescribed by D. For a closed class $\mathfrak{b}$ on A, let $\mathfrak{b}_i$ be its copy on A_i. Then*

$$T_D^{-1}\mathfrak{b}_3 \sim \mathfrak{b}_1 \times A_2 + A_1 \times \mathfrak{b}_2 .$$

Proof. Let $\{x\}$ be a n.h.g.p. of A_1, $\{y\}$ the copy of $\{x\}$ in $k(A_2)$; set $K = k(A_2)$, and let X be the point of $(A_1)_K$ at which each x_i acquires the value y_i; then $\sigma_X^{-1}x_i$ is the copy of x_i in $k(A_3)$. Therefore, after setting $\mathfrak{B} = T_D^{-1}\mathfrak{b}_3$, we have $\mathfrak{B}\{A_2\} = \sigma_X^{-1}(\mathfrak{b}_1)_K$, so that, by 2.4, $\mathfrak{B}\{A_2\} = (\mathfrak{b}_1)_K + \mathrm{cl}^* t$, where $t \in k(A_1 \times A_2)$, and cl* means class on $(A_1)_K$; notice here that 2.4 only assures that t belongs to $L((A_1)_L)$ for some finite extension L of K, but that a simple argument, based on the consideration of a K-basis of L, shows that t will also belong to $K((A_1)_K) = k(A_1 \times A_2)$. Set

$$\mathfrak{B}' = \mathfrak{B} - \mathfrak{b}_1 \times A_2 - \mathrm{cl}\, t,$$

where cl means class on $A_1 \times A_2$; then $\mathfrak{B}'\{A_2\} = 0$, so that $\mathfrak{B}'$ has only poles of the type $A_1 \times Y$, with $Y \subset A_2$. Moreover, $\mathfrak{B}'\{A_1\} = (\mathfrak{b}_2)_H + \mathrm{cl}' t'$, where $H = k(A_1)$, $t' \in k(A_1 \times A_2)$, and cl′ means class on $(A_2)_H$; thus, t' has only, on $A_1 \times A_2$, poles of the type $A_1 \times Y$ with $Y \subset A_2$, or $Y \times A_2$ with $Y \subset A_1$, and consequently it belongs to the direct product $H \times K$ over k. If then $A_1 \times Y$ is a pole of $\mathfrak{B}'$, a representative of $\mathfrak{B}'$ at $A_1 \times Y$ can be written in the form $\sum_{i=1}^{h} a_i b_i$, where $a_i \in H$ and $b_i \in K$; if h is the smallest integer for which this is true, an argument similar to the one used in the proof of 1.1 shows that $\mathfrak{B}'$ must have a pole at each $Y \times A_2$ such that Y is a pole of some a_i on A_1. As $\mathfrak{B}'$ has no pole of this kind, we conclude that $a_i \in k$ for each i, so that $\mathfrak{B}' = A_1 \times \mathfrak{c}$ for a $\mathfrak{c} \in \mathfrak{B}(A_2)$; but then we can select t' in K, hence $\mathfrak{B}' = A_1 \times \mathfrak{b}_2 + \mathrm{cl}\, t'$, or finally $\mathfrak{B} = \mathfrak{b}_1 \times A_2 + A_1 \times \mathfrak{b}_2 + \mathrm{cl}\,(t + t')$, Q.E.D.

2.6. Theorem. *Let V be an irreducible variety over k, without singularities, and let A be a nonsingular abelian variety over k; let λ, μ be rational mappings of V into A, and set $\nu = \lambda + \mu$. For a $\mathfrak{b} \in \mathfrak{B}(A)$ we have*

$$T_\nu^{-1}\mathfrak{b} \sim T_\lambda^{-1}\mathfrak{b} + T_\mu^{-1}\mathfrak{b}.$$

Proof. Let A_1, A_2, A_3 be copies of A, and let D be the rational mapping of $A_1 \times A_2$ onto A_3 which gives the law of composition on A; thus, D is an irreducible subvariety of $A_1 \times A_2 \times A_3$. Let λ_1, μ_2, ν_3 be the copies of, respectively, λ, μ, ν which apply V into, respectively, A_1, A_2, A_3. Result 2.5 can now be written:

$$(\mathfrak{b}_1 \times A_2 \times A_3 + \mathfrak{b}_2 \times A_1 \times A_3 - \mathfrak{b}_3 \times A_1 \times A_2) \cap D \sim 0;$$

moreover, $W = (\lambda_1 \times A_2 \times A_3) \cap (\mu_2 \times A_1 \times A_3) \cap (\nu_3 \times A_1 \times A_2)$ is an irreducible subvariety of $D \times V$ which, considered as an algebraic correspondence between V and D, is a rational mapping of V into D. Thus:

$$
\begin{aligned}
[(T_\lambda^{-1}\mathfrak{b} &+ T_\mu^{-1}\mathfrak{b} - T_\nu^{-1}\mathfrak{b}) \times A_1 \times A_2 \times A_3] \cap W \\
&= \{[(T_\lambda^{-1}\mathfrak{b} \times A_1) \cap \lambda_1] \times A_2 \times A_3\} \cap W \\
&\qquad + \{[(T_\mu^{-1}\mathfrak{b} \times A_2) \cap \mu_2] \times A_1 \times A_3\} \cap W \\
&\qquad - \{[(T_\nu^{-1}\mathfrak{b} \times A_3) \cap \nu_3] \times A_1 \times A_2\} \cap W \\
&= \{[(V \times \mathfrak{b}_1) \cap \lambda_1] \times A_2 \times A_3\} \cap W \\
&\qquad + \{[(V \times \mathfrak{b}_2) \cap \mu_2] \times A_1 \times A_3\} \cap W \\
&\qquad - \{[(V \times \mathfrak{b}_3) \cap \nu_3] \times A_1 \times A_2\} \cap W \\
&= [V \times (\mathfrak{b}_1 \times A_2 \times A_3 + \mathfrak{b}_2 \times A_1 \times A_3 - \mathfrak{b}_3 \times A_1 \times A_2)] \cap W \\
&= [V \times (\cdots)] \cap (V \times D) \cap W = \{V \times [(\cdots) \cap D]\} \cap W \sim 0.
\end{aligned}
$$

Since W operates on the whole V, and is birationally regularly equivalent to V, this means $T_\lambda^{-1}\mathfrak{b} + T_\mu^{-1}\mathfrak{b} - T_\nu^{-1}\mathfrak{b} \sim 0$, Q.E.D.

2.7. Theorem. *Let A be an n-dimensional nonsingular abelian variety over k; then $\mathfrak{B}(A)/\mathfrak{B}_e(A)$ is a free k-module of order n, and there exists a set of representatives of an absolute basis of it.*

2.8. Corollary. *Result* 1.1 *remains true when U, V are nonsingular abelian varieties over k, of arbitrary dimensions.*

Proof. The corollary is an immediate consequence of the theorem, and the second statement of the theorem is an immediate consequence of the first. We shall therefore concentrate on proving the first statement.

Assume first $p \neq 0$; iterated application of 2.6 to the identity homomorphism δ_A of A onto A shows that, if $\alpha = p\delta_A$, $T_\alpha^{-1}\mathfrak{b} \sim 0$ for each $\mathfrak{b} \in \mathfrak{B}(A)$; 2.3 implies then that $\mathfrak{B}(A)/\mathfrak{B}_e(A) \cong X(A, \alpha)/k(\alpha A)$; and this is a k-module of order n, by 3.3 of [8].

Assume now $p = 0$, and let $\mathfrak{D}_i(A)$, $\mathfrak{D}_e(A)$ denote, as usual, the k-modules of, respectively, the closed differentials of the i^{th} kind on A, for $i = 1, 2$, and the exact differentials on A. If $\omega \in \mathfrak{D}_2(A)$, for each $(n - 1)$-dimensional irreducible subvariety X of A there exists an $x \in k(A)$ such that $\omega - d_A x$ does not have a pole at X; the class $\mathfrak{b}$ such that $\mathfrak{b}X = x + Q(X/A)$ depends only on ω, by 1.4 of [6]; also, $\mathfrak{b} \in \mathfrak{B}(A)$, since for each $P \in A$ there exists an $x \in k(A)$ such that no pole of $\omega - d_A x$ contains P. Consequently, the mapping $\omega \longrightarrow \mathfrak{b}$ establishes an isomorphism of $\mathfrak{D}_2(A)/\mathfrak{D}_1(A)$ into $\mathfrak{B}(A)$, and also of $\mathfrak{D}_2(A)/\mathfrak{D}_1(A) + \mathfrak{D}_e(A)$ into $\mathfrak{B}(A)/\mathfrak{B}_e(A)$; hence by 2.8 of [6], $\mathfrak{B}(A)/\mathfrak{B}_e(A)$ has order $\geqq n$. Now, A is a homomorphic image of some nonsingular jacobian variety J over k, and therefore J is isogenous to $A \times B$, for a suitable m-dimensional nonsingular abelian variety B over k; $\mathfrak{B}(J)/\mathfrak{B}_e(J)$ contains a submodule isomorphic to the complementary sum of $\mathfrak{B}(A)/\mathfrak{B}_e(A)$ and $\mathfrak{B}(B)/\mathfrak{B}_e(B)$, which have orders $\geqq n$ and $\geqq m$ respectively; but, by 2.1, $\mathfrak{B}(J)/\mathfrak{B}_e(J)$ has order $\leqq m + n$, so that $\mathfrak{B}(A)/\mathfrak{B}_e(A)$ has order n, Q.E.D.

3. The differential classes

Let A be an n-dimensional nonsingular abelian variety over k; the k-modules $\mathfrak{D}_i(A)$, $\mathfrak{D}_e(A)$ have been defined in the course of the proof of 2.7. We shall define a *differential class on* A to be a linear mapping of $\mathfrak{D}_0(A)$ ($=$ k-module of the invariant derivations on A) into $\mathfrak{B}(A)$; in particular, we have proved at the beginning of section 2 that if $\mathfrak{b} \in \mathfrak{B}(A)$, there exists a differential class, to be denoted by $d\mathfrak{b}$, such that $(d\mathfrak{b})D = D\mathfrak{b}$; differential classes of this type are called *exact*; a differential class Ω is *closed* if $D\Omega\Delta = \Delta\Omega D$ for any D, $\Delta \in \mathfrak{D}_0(A)$. If ω is a differential on A, there exists a differential class, to be denoted by $\mathrm{cl}\,\omega$, such that $(\mathrm{cl}\,\omega)D = \mathrm{cl}_A\,(\omega D)$ for each $D \in \mathfrak{D}_0(A)$; differential classes of this type are said to be *principal*; if ω is closed, so is $\mathrm{cl}\,\omega$. Finally, a differential class Ω is principal if and only if ΩD is exact for each $D \in \mathfrak{D}_0(A)$; and the ω such that $\Omega = \mathrm{cl}\,\omega$ is unique but for an additive differential of the first kind.

The structure of the k-modules $\mathfrak{D}_1(A)$, $\mathfrak{D}_2(A)$ has considerable importance if $p = 0$, as shown in [6]; not so if $p \neq 0$; concerning these two cases, the following results have been proved in [6]: (1) a differential on A is of the first kind if and only if it is invariant; (2) a differential of the first kind on A is necessarily closed; (3) any semi-invariant differential on A is of the second kind; (4) if $p = 0$, any closed differential of the second kind on A is semi-invariant; (5) $\mathfrak{D}_1(A)$ has order n; (6) $\mathfrak{D}_2(A)/\mathfrak{D}_e(A)$ has order $2n$ if $p = 0$. The following result gives a link between closed classes on A and differentials on A, and incidentally brings the following improvement to results (1) $\cdots$ (6): (7) if $p = 0$, any differential of the second kind on A is closed, hence semi-invariant. We do not pursue the investigation of the still open questions concerning the case $p \neq 0$.

Before proceeding, we shall introduce a new operation in $\mathfrak{D}(A)$, when $p \neq 0$, in the following manner: if $D \in \mathfrak{D}(A)$, $D^{(1/p)}$ will be the element Δ of $\mathfrak{D}(A)$ such that, for any $x \in k(A)$, $(\Delta x)^p = D(x^p)$; we have

$$(D^{(1/p)})^p = (D^p)^{(1/p)};$$

also, $D^p = 0$ if and only if D belongs to the nucleus of the homomorphism α such that $p\delta_A = \delta_{1,\alpha A}\,\alpha$ (see section 1 and result 2.5 of [8]); dually, $D^{(1/p)} = 0$ if and only if D belongs to the nucleus of $\delta_{1,A}$; and $(D^p)^{(1/p)} = 0$ if and only if D belongs to the nucleus of $p\delta_A$. According to 3.2 of [8] and its proof, and in the present notations, the subset X' of $X(A, \alpha)$, consisting of the elements x which are purely inseparable over $k(\alpha A)$, is such that $X'/k(\alpha A)$ is dual to the k-module R/S, where R is the set of the $D \in \mathfrak{D}(A)$ with $D^p = 0$, and S is the subset of R consisting of the D for which $Dx = 0$ for each $x \in X'$. The operations $D \to D^{(1/p)}$, $x \to x^p$ induce dual operations in R/S, $X'/k(\alpha A)$ respectively; this implies, in particular, that if $x \in X'$ and $Dx = 0$ for each $D \in \mathfrak{D}_0(A)$ such that $D^p = 0$, then $x - y^p \in k(\alpha A)$ for a suitable $y \in X'$.

3.1. THEOREM. *Let A be an n-dimensional nonsingular abelian variety over k; for each exact differential class $d\mathfrak{b}$ $(\mathfrak{b} \in \mathcal{B}(A))$ on A, there exists a semi-invariant $\omega \in \mathfrak{D}_2(A)$, unique but for an additive element of $\mathfrak{D}_1(A)$, such that $d\mathfrak{b} = \mathrm{cl}\,\omega$; the mapping $\mathfrak{b} \rightarrow \omega$ induces a homomorphism of the k-module $\mathcal{B}(A)$ into the k-module $\mathfrak{D}_2(A)/\mathfrak{D}_1(A)$, which is onto if $p = 0$, and whose kernel is 0 if $p = 0$, or $[\mathcal{B}(A)]^p$ if $p \neq 0$; also, $\omega \in \mathfrak{D}_1(A) + \mathfrak{D}_e(A)$ if and only if $\mathfrak{b} \sim 0$ when $p = 0$, or $\mathfrak{b} \sim \mathfrak{a}^p$ for some $\mathfrak{a} \in \mathcal{B}(A)$ when $p \neq 0$. If $p = 0$, each differential of the second kind on A is closed.*

Proof. Assume first $p = 0$, and denote temporarily by $\mathfrak{D}'$ the k-module of the (not necessarily closed) differentials of the second kind on A. The reasoning used in the proof of 2.7 can be applied to any $\omega \in \mathfrak{D}'$; the $\mathfrak{b} \in \mathcal{B}(A)$ which corresponds to a given ω has the property that $d\mathfrak{b} = \mathrm{cl}\,\omega$; the mapping $\omega \rightarrow \mathfrak{b}$ establishes an isomorphism of $\mathfrak{D}'/\mathfrak{D}_1(A) + \mathfrak{D}_e(A)$ into $\mathcal{B}(A)/\mathcal{B}_e(A)$, so that, by 2.7, the former has order $\leqq n$; but it contains

$$\mathfrak{D}_2(A)/\mathfrak{D}_1(A) + \mathfrak{D}_e(A),$$

which has order n by Theorems 2.8 and 2.1 of [6]. This proves, at the same time, that $\mathfrak{D}' = \mathfrak{D}_2(A)$, and that for any $\mathfrak{b} \in \mathcal{B}(A)$, $d\mathfrak{b} = \mathrm{cl}\,\omega$ for some $\omega \in \mathfrak{D}_2(A)$. This concludes the case $p = 0$.

Assume now $p \neq 0$, and let B be a copy of A such that $k(A) \subsetneqq k(B)$ as prescribed by $\alpha = p\delta_B$. Result 2.7 and its proof show that for any $\mathfrak{b} \in \mathcal{B}(A)$ there exists an $x \in X(B, \alpha)$ such that $T_\alpha^{-1}\mathfrak{b} = \mathrm{cl}_B\, x$; for any element D_i of a k-independent basis of $\mathfrak{D}_0(A)$, let D_i' be an element of $\mathfrak{D}(B)$ which induces D_i in $k(A)$; such D_i' exists by 2.2 and 2.3 of [8]. If Δ belongs to the nucleus of α, we have $\Delta D_i' x = D_i' \Delta x = 0$, since $\Delta x \in k$ by 2.3; also, if P belongs to the kernel of α, we have $\sigma_P D_i' x - D_i' x = D_i' (\sigma_P x - x) = 0$, again by 2.3; thus $D_i' x \in k(A)$, and there exists a differential ω on A such that

$$\omega D_i = D_i' x$$

for each i; ω is obviously closed, and also semi-invariant, since, for $P \in A$, $(\sigma_P\, \omega)D_i - \omega D_i = D_i' (\sigma_Q\, x - x)$, if $Q \in \alpha^{-1}P$; as $\sigma_Q\, x - x = y \in k(A)$, we have $\sigma_P\, \omega - \omega = d_A\, y$. Thus, ω is semi-invariant and belongs to $\mathfrak{D}_2(A)$, and $d\mathfrak{b} = \mathrm{cl}\,\omega$; if also $d\mathfrak{b} = \mathrm{cl}\,\omega'$, then $\mathrm{cl}\,(\omega - \omega') = 0$, or $\omega - \omega' \in \mathfrak{D}_1(A)$.

We must now investigate the kernel of the mapping $\mathfrak{b} \rightarrow \omega + \mathfrak{D}_1(A)$; if $\omega \in \mathfrak{D}_1(A)$, that is, if $d\mathfrak{b} = 0$, in the previous notation we have $D_i' x \in k$ for each i, or also $\Delta x \in k$ for each $\Delta \in \mathfrak{D}(B^p)$ such that $\Delta^p \in \mathfrak{D}_0(B^p)$, since, by [8], Δ induces a derivation in $k(A)$ if and only if Δ^p is a derivation. But then, since each element of $\mathfrak{D}_0(B^p)$ is the p^{th} power of some element of $\mathfrak{D}(B^p)$, we have $Dx = 0$ for each $D \in \mathfrak{D}_0(B^p)$. Now, according to the proof of 3.3 of [8], we can write $x = x' + x''$, where x' and x'' are elements of $X(B, \alpha)$, respectively purely inseparable and separable over $k(A)$; we have $Dx'' = 0$ for each $D \in \mathfrak{D}_0(B^p)$ for which $D^p = 0$; hence $Dx' = 0$ for any such D. But then, as we saw at the beginning of this section, $x' - y^p \in k(A)$ for a suitable $y \in X(B, \alpha)$; since any x'' of $X(B, \alpha)$, separable over $k(A)$, satisfies a similar relation, x itself must satisfy a relation of this kind. Therefore, $\mathfrak{b} = \mathfrak{a}^p + \mathrm{cl}_A z$ for some $z \in k(A)$; but then $d\mathfrak{b} = 0$ implies $\mathrm{cl}\, d_A z = 0$, or $d_A z \in \mathfrak{D}_1(A)$; we claim that this, in turn, implies that $\mathrm{cl}_A z \in [\mathfrak{B}(A)]^p$. And in fact, after setting $\beta = \delta_{1,A}$, we see, by 2.3, that there exists a $\mathfrak{c}' \in \mathfrak{B}(A^p)$ such that $T_\beta^{-1}\mathfrak{c}' = \mathrm{cl}_A z$; and this is the same as saying that $\mathrm{cl}_A z = \mathfrak{c}^p$ for a $\mathfrak{c} \in \mathfrak{B}(A)$. Thus, $d\mathfrak{b} = 0$ implies $\mathfrak{b} = (\mathfrak{a} + \mathfrak{c})^p$, as claimed.

Finally, if $\mathfrak{b}$ is such that the corresponding ω belongs to $\mathfrak{D}_1(A) + \mathfrak{D}_e(A)$, that is, if $d\mathfrak{b} = \mathrm{cl}\, d_A x$ for some $x \in k(A)$, we have $d(\mathfrak{b} - \mathrm{cl}_A x) = 0$, or $\mathfrak{b} \sim \mathfrak{a}^p$ for some $\mathfrak{a} \in \mathfrak{B}(A)$, Q.E.D.

3.2. THEOREM. *Let A be an n-dimensional nonsingular abelian variety over k, and let X be an $(n - 1)$-dimensional unmixed cycle on A. Let $X = \sum_i a_i X_i$ be a minimal representation of X; if v_i is the normalized valuation of $k(A)$ over k whose center on A is X_i, let $x \in k(A)$ be such that $v_i(x) = a_i$ for each i. For each invariant derivation D on A, define the class $\mathfrak{d} = \mathfrak{d}(D)$ by setting $\mathfrak{d}X_i = x^{-1}Dx + Q(X_i/A)$ for each i, and $\mathfrak{d}X = Q(X/A)$ if X is not one of the X_i's. Then*: (1) *$\mathfrak{d}$ is a closed class, and the differential class dX such that, for each D, $(dX)D = \mathfrak{d}(D)$, is independent of the choice of x*; (2) *$d(X + Y) = dX + dY$*; (3) *dX is a closed differential class*; (4) *if $p \neq 0$, $(dX)(D^p) \sim [(dX)D]^p$ for any D.*

Proof. Let z be another element of $k(A)$ such that $v_i(z) = a_i$ for each i; then $z = xu$, where $v_i(u) = 0$ for each i; hence, $z^{-1}Dz - x^{-1}Dx = u^{-1}Du$, and $v_i(u^{-1}Du) \geqq 0$ for each i. This proves that $\mathfrak{d}$ is independent of the choice of x.

Now, let $P \in \mathrm{rad}\, X$, and let, for instance, $X_1, \cdots, X_r$ be those, among the X_i's, which contain P; let x_i be a generator of the ideal $\mathfrak{P}_{v_i} \cap Q(P/A)$, for $i = 1, \cdots, r$, and set $y = \prod_i x_i^{a_i}$; then $y^{-1}Dy$ represents $\mathfrak{d}$ at P, and this proves that $\mathfrak{d}$ is closed, hence that dX is a differential class, independent of the choice of x. If on the other hand, for each i, x_i is such that $v_j(x_i) = \delta_{ij}$ (Kronecker symbol), and if $y = \prod_i x_i^{a_i}$, then $y^{-1}Dy$ represents $\mathfrak{d}$ at each X_i, while $x_i^{-1}Dx_i$ represents $(dX_i)D$ at each X_j. Since

$$y^{-1}Dy = \textstyle\sum_i a_i x_i^{-1}Dx_i,$$

we conclude that $dX = \sum_i a_i\, dX_i$, from which (2) follows immediately.

If $D, \Delta \in \mathfrak{D}_0(A)$, $\Delta(dX)D$ is represented at each X_i by

$$\Delta(x^{-1}Dx) = x^{-1}\Delta Dx - x^{-2}(Dx)(\Delta x);$$

this also represents $D(dX)\Delta$, and this equality proves (3).

We shall now assume $p \neq 0$, and proceed to prove (4). If x has the same meaning as in the definition of $\mathfrak{d}$, we want to compute

$$D^{p-1}(x^{-1}Dx) = x^{-p}D^{p-1}(x^{p-1}Dx);$$

Leibnitz formula (the classical case) gives

$$D^{p-1}(x^{-1}Dx) = -x^{-p} \sum_{i+i_p=p} \frac{i_p}{i!\, i_p!} (D^i x^{p-1})(D^{i_p} x) = \cdots$$

$$= -x^{-p} \sum \frac{i_p}{i_1!\, i_2! \cdots i_p!} (D^{i_1}x)(D^{i_2}x) \cdots (D^{i_p}x),$$

where $\sum$ is extended to all the nonnegative values of the i_r for which $\sum_r i_r = p$. In this summation, the coefficient of $x^{c_0}(Dx)^{c_1} \cdots (D^p x)^{c_p}$ (where $c_r \geqq 0, \sum_r rc_r = \sum_r c_r = p$) turns out to be

$$\frac{p!}{c_0!\, c_1! \cdots c_p!(2!)^{c_2}(3!)^{c_3} \cdots (p!)^{c_p}},$$

where it is understood that the result must be interpreted mod p, after all possible simplifications. But then this coefficient is $\neq 0$ only in the cases $c_0 = p - 1, c_1 = \cdots = c_{p-1} = 0, c_p = 1$, in which case the coefficient is -1, and $c_0 = 0, c_1 = p, c_2 = \cdots = c_p = 0$, in which case it is $+1$. We conclude that $D^{p-1}(x^{-1}Dx) = -x^{-p}[-x^{p-1}D^p x + (Dx)^p] = x^{-1}D^p x - (x^{-1}Dx)^p$. It follows that $(dX)(D^p) - [(dX)D]^p = D^{p-1}(dX)\, D \in \mathcal{B}_e(A)$, by 3.1, Q.E.D.

In the following result, $\equiv$ will denote the type of equivalence, between divisors on an abelian variety, which was introduced in [13]; by (16) of [5] (see also the Appendix at the end of this paper), it coincides with arithmetical equivalence; and by (19) of [5] and 4.3 of [8], it also coincides with algebraic equivalence. We have the following two results, which are special cases of famous results due to Picard and Severi in the classical case.

3.3 THEOREM. *Let A be a nonsingular n-dimensional abelian variety over k; let X be an $(n - 1)$-dimensional cycle on A. If $X \equiv 0$, there exists a closed differential ω on A such that $dX = \mathrm{cl}\ \omega$. If $p = 0$, the converse is true in the following form: if dX is a principal differential class, then $X \equiv 0$.*

Proof. Let A_1, A_2 be copies of A over k, and assume $k(A_2) \subsetneqq k(A \times A_1)$ as prescribed by the rational mapping D of $A \times A_1$ onto A_2 which gives the law of composition on A. Let X_1, X_2 be the copies of X on A_1, A_2 respectively, and set $X' = N_D^{-1}X_2$ (see section 3 of [9] for the definition of N^{-1}). According to Theorem 4.1 of [4], and its proof, if $X \equiv 0$ there is an element $t \in k(A \times A_1)$ such that the divisor of t on $A \times A_1$ is $X \times A_1 + A \times X_1 - X'$; the method of section 7 of [4] shows then that there exists a closed differential ω on A such that, if ω_i is its copy on A_i, and ω' is the extension of ω_2 on $A \times A_1$, $t^{-1}\, dt = \omega \times A_1 + A \times \omega_1 - \omega'$, if d means construction of differential on $A \times A_1$. Now, t is related to $X \times A_1 + A \times X_1 - X'$ as x is to X in 3.2; therefore, $dX \times A_1 + A \times dX_1 - dX' = \mathrm{cl}\ t^{-1}\, dt = \mathrm{cl}\ \omega \times A_1 + A \times \mathrm{cl}\ \omega_1 - \mathrm{cl}\ \omega'$; this gives $dX = \mathrm{cl}\ \omega$, as claimed.

Suppose now $p = 0$, and $dX = \mathrm{cl}\ \omega$ for some differential ω on A. Let C be an irreducible curve on A such that, for each irreducible component variety X_i of X, (1) the intersection $(C \cap X_i, A)$ exists and, if not empty, has only components with multiplicity 1, (2) $C \cap X_i$ and $C \cap X_j$ have no points in common if $i \neq j$, and (3) each component of each $C \cap X_i$ is simple on C; if a preassigned C does not have these properties, $\sigma_P\, C$ will, for a suitable $P \in A$. By 1.3 of [6], $\omega \cap C$ exists, since the poles of ω are among the X_i; if $P \in X_i \cap C$, let x be an element of $Q(P/A)$ which generates the ideal $\mathfrak{P}(X_i/A) \cap Q(P/A)$; if π denotes reduction mod $\mathfrak{P}(C/A)$, πx is a regular parameter of $Q(P/C)$, by Theorem 5.8 of [2]. If $a_i \in k$ is the multiplicity of X_i in X, the fact that $dX = \mathrm{cl}\ \omega$ implies that $a_i\, x^{-1}\, d_A x - \omega$ does not have a pole at X_i; it does not have a pole at P either, since no other pole of x, or x^{-1}, or ω contains P; therefore $a_i\, (\pi x)^{-1}\, d_C\, \pi x - \omega \cap C$ does not have a pole at P, by 1.5 of [6], so that $a_i = i(P, C \cap X, A)$ is the residue of $\omega \cap C$ at P. Since the sum of the residues of $\omega \cap C$ is 0, by the corollary to Theorem 3, Chapter III of [10], we conclude that $(C \cap X, A)$ has order zero; but then (16) of [5] states that $X \equiv 0$, Q.E.D.

The previous result, and (15) of [5], imply:

3.4. Corollary. *Let A be as in* 3.3, *and assume $p = 0$; let ρ be the base number of A, that is, the number of generators of the group of the $(n - 1)$-dimensional cycles on A, modulo the group of those arithmetically equivalent to zero. Then $\rho + 1$ is the smallest integer m having the following property: for any m $(n - 1)$-dimensional (integral virtual) cycles $X_1, \cdots, X_m$ of A, there exist m integers $a_1, \cdots, a_m$, not all zero, such that $\sum_i a_i\, dX_i$ is a principal differential class.*

The last two results indicate the importance of the additive group $\mathcal{C}/\mathcal{P}$, where $\mathcal{C}$ denotes the additive group of the dX, and $\mathcal{P}$ the additive group of those dX which are principal; if $p = 0$, $\mathcal{C}/\mathcal{P}$ is isomorphic to the additive group of the $(n - 1)$-dimensional cycles on A; if $p \neq 0$, it is only a homomorphic image of it. The elements of $\mathcal{C}/\mathcal{P}$ can be interpreted as homomorphisms of the k-module $\mathfrak{D}_0(A)$ into the k-module $\mathcal{B}(A)/\mathcal{B}_e(A)$, and can therefore be represented by square matrices of order n with elements in k. We shall prove the following result:

3.5. Theorem. *Let J be the jacobian, assumed nonsingular, of a curve C of genus n, without singularities, over k, and let Θ be the $(n - 1)$-dimensional cycle on J defined in section* 41 *of* [13]; *then $d\Theta$ maps any k-independent basis of $\mathfrak{D}_0(J)$ onto a set of representatives of an absolute basis of $\mathcal{B}(J)/\mathcal{B}_e(J)$.*

Proof. Let V, Φ be related to J and C as in the proof of 2.1; let φ be a canonical rational mapping of C into J; then Θ is the set of the $\prod_i \varphi P(i)$, for points $P(1), \cdots, P(n - 1)$ of C. According to section 37 of [13], φ and Φ are related by $\Phi[Q_1(1) \times \cdots \times Q_n(n)] = Q \prod_i \varphi Q(i)$, for any set $\{Q(1), \cdots, Q(n)\}$ of points of C, Q being a fixed point of J, and $Q_j(i)$ being the copy of $Q(i)$ on C_j; we shall select φ in such a manner that $Q = E_J =$ *identity of* J. Then, for a $Q \in C$, $\sigma_{\varphi Q}\,\Theta = \Phi[Q_1 \times C_2 \times \cdots \times C_n] = \Phi[C_1 \times \cdots \times C_{i-1} \times Q_i \times C_{i+1} \times \cdots \times C_n]$ for any i. Let R be a point of J such that $R\left(\prod_i \varphi K_i\right)^{-1}$ does not belong to W'_{n-2}, in the sense of Proposition 16 of [13], $\sum_i K_i$ being a canonical divisor on C; this means that the points $Q(i)$ of C $(i = 1, \cdots, n)$ such that $R = \prod_i \varphi Q(i)$ are uniquely determined; we may also, and do, require R to be such that the $Q(i)$ are mutually distinct; then, R belongs to the intersection of the $\sigma_{\varphi Q(i)}\,\Theta = \Theta_i$, but not to any $\sigma_{\varphi Q}\,\Theta$ if Q is not one of the $Q(i)$.

By Proposition 18 of [13], R is simple on each Θ_i; let x_i be an element of $Q(R/J)$ such that $x_i\,Q(R/J) = Q(R/J) \cap \mathfrak{P}(\Theta_i/J)$. The conditions on the $Q(i)$ imply easily that R is a component variety of the intersection of the Θ_i, so that $\{x_1, \cdots, x_n\}$ is a set of parameters of $Q(R/J)$. Let $Q_j(i)$ denote the copy of $Q(i)$ on C_j, and set $S(i_1, \cdots, i_n) = Q_1(i_1) \times \cdots \times Q_n(i_n)$, for any permutation $\{i_1, \cdots, i_n\}$ of $\{1, \cdots, n\}$; then $R = \Phi S(i_1, \cdots, i_n)$, and the $S(i_1, \cdots, i_n)$ are all the component varieties of $\Phi[R]$, since R is not fundamental for Φ; thus, by the ramification theorem for geometric domains (Lemma 2.2 of [1]), we have

$$\sum e\big(Q(S(i_1, \cdots, i_n)/V);x\big) = n!\,e\big(Q(R/J);x\big),$$

if $\sum$ is extended to all the permutations. But also, for the same reason, and by Proposition 15 of [13]:

$$(n-1)!\sum_{1}^{n}{}_{j}\, e\big(Q(\Theta_{ij}/V);x_i\big) = n!\,e\big(Q(\Theta_i/J);x_i\big) = n!,$$

where $\Theta_{ij} = C_1 \times \cdots \times C_{j-1} \times Q_j(i) \times C_{j+1} \times \cdots \times C_n$; thus, x_i is a regular parameter of each $Q(\Theta_{ij}/V)$. Now, $S(i_1, \cdots, i_n)$ is a component variety of the intersection $(\Theta_{i_1 1} \cap \cdots \cap \Theta_{i_n n}, V)$, and appears in it with multiplicity 1, so that, by Theorem 5.8 of [2] and Theorem 2.1 of [1],

$$e\big(Q(S(i_1, \cdots, i_n)/V);\big)x = 1;$$

the previous relation gives then $e\big(Q(R/J);x\big) = 1$.

Now, let $0 \neq D \in \mathfrak{D}_0(J)$, and assume $(d\Theta)D \in \mathfrak{B}_e(J)$, say $(d\Theta)D = \mathrm{cl}_J\,y$; if $y \in k$, this implies $(d\Theta_i)D = 0$ for each i, or $x_i^{-1}Dx_i \in Q(\Theta_i/J)$, $Dx_i \in \mathfrak{P}(R/J)$ for each i. As this contradicts Lemma 5.2 of [4], we conclude that $y \notin k$, so that the divisor of y on J is of the form $\Theta' - \Theta$, Θ' being an effective cycle on A, no component of which is Θ. For a generic $P \in J$ we have then $\sigma_P\,\Theta' \cap \varphi C \sim \sigma_P\,\Theta \cap \varphi C$ on φC; but, for a generic P, $\sigma_P\,\Theta' \cap \varphi C$ and $\sigma_P\,\Theta \cap \varphi C$ do not have points in common, and the latter is a nonspecial divisor on φC, by Theorem 20 of [13]. Thus, for any $D \in \mathfrak{D}_0(J)$, not zero, $(d\Theta)D$ is not exact, Q.E.D.

Now, let A be an n-dimensional nonsingular abelian variety over k, and let B be an m-dimensional abelian subvariety of A; then B is also nonsingular, by Lemma 1.3 of [4]. Let $x_1, \cdots, x_m$ be elements of $Q(E_A/A)$ whose images mod $\mathfrak{P}(B/A)$ form a regular set of parameters of $Q(E_B/B)$, and let $\{x_{m+1}, \cdots, x_n\}$ be a regular set of parameters of $Q(E_C/C)$, if $C = A/B$; then $\{x_1, \cdots, x_n\}$ is a regular set of parameters of $Q(E_A/A)$, and $\{x_{m+1}, \cdots, x_n\}$ of $Q(B/A)$. Let D_i $(i = 1, \cdots, n)$ be the elements of $\mathfrak{D}_0(A)$ related to $x_1, \cdots, x_n$ by the relation $D_i x_j \equiv \delta_{ij} \pmod{\mathfrak{P}(E_A/A)}$; then $D_1, \cdots, D_m$ are regular at B (see section 1 of [6]), and induce on B a k-basis $\{\Delta_1, \cdots, \Delta_m\}$ of $\mathfrak{D}_0(B)$. Now, if Ω is a differential class on A, by $\Omega \cap B = B \cap \Omega$ we shall denote the differential class ω on B, when it exists, such that $\omega\Delta_i = B \cap \Omega D_i$. Let A' be a nonsingular abelian variety over k, and let μ be a homomorphism of A' into A; the cotrace $T_\mu^{-1}\Omega$ is defined, when it exists, as the differential class on A' such that $(T_\mu^{-1}\Omega)D = [(\mu A') \cap \Omega]D'$, if D' is the invariant derivation induced by D in $k(\mu A')$, when $k(\mu A') \subsetneqq k(A')$ as prescribed by μ. We have the following result, whose easy proof we omit:

3.6. Lemma. *Let μ be a homomorphism of the nonsingular abelian variety B over k into the nonsingular abelian variety A over k; if X is a cycle on A, we have $T_\mu^{-1}(dX) = dN_\mu^{-1}X$, the conorm N_μ^{-1} being defined as in section* 3 *of* [9], *provided these expressions are meaningful.*

4. Jacobian varieties

We shall proceed to establish a few elementary properties of curves, and to see how they are reflected on their jacobians. If ω is a differential on a curve C over k, without singularities, the corresponding *differential mapping* (called a differential in [10]) will still be denoted by ω. If μ is a rational mapping of C into another curve C' without singularities, the trace T_μ of a differential mapping on C, or of a repartition on C, and the cotrace T_μ^{-1} of a differential mapping on C', or of a repartition on C', are defined in [10]. If $p \neq 0$, and ω is a differential mapping on C, we can consider the differential mapping $\omega^{1/p}$ defined as follows: for any repartition β on C, $\omega^{1/p}\beta = (\omega\beta^p)^{1/p}$. If π is the rational mapping of C onto C^p generated by the embedding $k(C^p) = [k(C)]^p$, $\omega^{1/p}$ can be interpreted in the following manner: set first $\omega' = T_\pi\, \omega$; then, if $\omega' = y^p\, d_{\pi C}\, x^p$, set $\omega^{1/p} = y d_C\, x$. The mapping $\omega \to \omega^{1/p}$ satisfies the relations $(a\omega)^{1/p} = a^{1/p}\omega^{1/p}$ if $a \in [k(C)]^p$, and $(\omega + \psi)^{1/p} = \omega^{1/p} + \psi^{1/p}$; the previous interpretation of $\omega^{1/p}$ provides a formula for its computation: let $x \in k(C)$ be such that $k(C)$ is a finite separable extension of $k(x)$, and write $\omega = y d_C x$, $y \in k(C)$; the set $\{1, x, x^2, \cdots, x^{p-1}\}$ is a $k(C^p)$-independent basis of $k(C)$, so that $y = \sum_{i=0}^{p-1} a_i\, x^i$, $a_i \in k(C^p)$. But then, if $z = \sum_{i=0}^{p-2} (i+1)^{-1} a_i\, x^{i+1}$, we also have $\omega = d_C\, z + a_{p-1}\, x^{p-1}\, d_C\, x$; from the definition of $T_\pi\, \omega$ follows readily that $T_\pi\, d_C\, z = 0$, and that

$$T_\pi\, a_{p-1}\, x^{p-1}\, d_C\, x = a_{p-1}\, x^p T_\pi\, x^{-1}\, d_C\, x = a_{p-1}\, x^p x^{-p}\, d_{\pi C}\, x^p,$$

so that $T_\pi\, \omega = a_{p-1}\, d_{\pi C}\, x^p$, and[2] $\omega^{1/p} = a_{p-1}^{1/p}\, d_C\, x$.

[2] This definition is the one used in the work quoted in footnote 1; the mapping $\omega \to \omega^{1/p}$ is there called "operation of Cartier and Tate", in recognition of the use made of it by these authors.

We have $\omega^{1/p} = 0$ if and only if ω is exact, and $\omega^{1/p} = \omega$ if and only if $\omega = y^{-1} d_C y$ for some $y \in k(C)$. Moreover, given ω, there exists a differential mapping ψ such that $\psi^{1/p} = \omega$: if $\omega = d_C z + ax^{-1} d_C x$, with $a \in k(C^p)$, all the ψ's with this property differ by an exact differential from $z^{p-1} d_C z + a^p x^{-1} d_C x$. If ω is of the first kind, so is $\omega^{1/p}$; if ω is of the second kind, $\omega^{1/p}$ is of the first kind; in fact, for any $P \in C$, let y be such that $\omega - d_C y$ is regular at P; then $(\omega - d_C y)^{1/p} = \omega^{1/p}$ is regular at P; conversely, if $\omega^{1/p}$ is of the first kind, ω is of the second kind: in fact, if P is a pole of ω, and x is a regular parameter of $Q(P/C)$, write $\omega = d_C z + ax^{-1} d_C x$, $a \in k(C^p)$; then $\omega^{1/p} = a^{1/p}x^{-1} d_C x$; as this does not have a pole at P, we must have $a^{1/p}x^{-1} \in Q(P/C)$, hence $a \in [\mathfrak{P}(P/C)]^p$, so that $ax^{-1} d_C x$ is regular at P; this proves that ω is of the second kind. If $\{\omega_1, \cdots, \omega_n\}$ is a k-independent basis for $\mathfrak{D}_1(C)$, and if ψ_i is such that $\psi_i^{1/p} = \omega_i$, then $\{\psi_1, \cdots, \psi_n\}$ is a set of representatives of a k-basis for $\mathfrak{D}_2(C)/\mathfrak{D}_e(C)$, so that this k-module has order n, as already proved in [12].

We shall now define a similar operation for differentials of the first kind on a nonsingular abelian variety A over k, again under the assumption $p \neq 0$. If ω is a differential of the first kind on A, $\omega^{1/p}$ shall be the differential of the first kind on A defined by $\omega^{1/p}D = (\omega D^p)^{1/p}$ for any $D \in \mathfrak{D}_0(A)$; the relations $(\omega + \psi)^{1/p} = \omega^{1/p} + \psi^{1/p}$, $(a\omega)^{1/p} = a^{1/p}\omega^{1/p}$, for $a \in k$, are satisfied; also, $\omega^{1/p} = 0$ if and only if $\omega D^p = 0$ for each D; and this is the case, by 4.4 of [8], if and only if ω is exact. On the other hand, $\omega^{1/p} = \omega$ if and only if $\omega D^p = (\omega D)^p$ for each $D \in \mathfrak{D}_0(A)$; this implies $\omega D = 0$ if $D^{p^i} = 0$ for some i, so that ω belongs to the k-module Ω of 4.3 of [8]; if $D_1, \cdots, D_f$ are the independent elements of $\mathfrak{D}_0(A)$ with the property $D_i^p = D_i$, the previous relation implies $\omega D_i = (\omega D_i)^p$, so that $\omega D_i \in k_0 =$ *prime field of* k, and $\omega = y^{-1} d_A y$ for some $y \in k(A)$; conversely, any ω of this type satisfies the relation $\omega^{1/p} = \omega$.

If, in particular, A has dimension 1, it is easily verified that the two definitions of $\omega^{1/p}$ coincide; in fact, in this case a differential of the first kind on A is either exact, in which case $\omega^{1/p} = 0$ in both senses, or of the type $ay^{-1} d_A y$, $a \in k$, in which case $\omega^{1/p} = a^{1/p}y^{-1} d_A y$ in both senses. We intend to investigate the relation between these two operations for any value of the genus.

4.1. LEMMA. *Let J be the jacobian, assumed nonsingular, of a curve C over k, without singularities, and let φ be a canonical rational mapping of C into J; then the mapping $\mathfrak{b} \to T_\varphi^{-1}\mathfrak{b}$, for $\mathfrak{b} \in \mathfrak{B}(J)$, induces an isomorphism φ^* of $\mathfrak{B}(J)/\mathfrak{B}_e(J)$ onto $\mathfrak{B}(C)/\mathfrak{B}_e(C)$.*

Proof. If $p \neq 2$, this is an immediate consequence of the proof of 2.1, and of 2.7. If $p = 2$, the result will be proved if we can prove that the kernel of φ^* is 0. According to the proof of 2.1, if this kernel is not 0, it has order 1; let then $\mathfrak{b}$ be an element of $\mathfrak{B}(J)$, not exact, but such that $T_\varphi^{-1}\mathfrak{b}$ is exact; we may assume that no pole of $\mathfrak{b}$ contains E_J. Then, in the notations of the proof of 2.1, $T_\Phi^{-1}\mathfrak{b} = \mathrm{cl}_V\, x$, for an element x of $k(V)$, but not of $k(J)$; we also have $T_\Phi^{-1}\mathfrak{b}^2 = \mathrm{cl}_V\, x^2$, and $\mathfrak{b}^2 \sim a\mathfrak{b}$ for a suitable $a \in k$. If $a = 0$, this gives $\mathfrak{b}^2 = \mathrm{cl}_J\, y$, for a $y \in k(J)$, so that $\mathrm{cl}_V\, y = \mathrm{cl}_V\, x^2$, $x^2 - y \in k$, $x^2 \in k(J)$, which contradicts the known fact that $k(V)$ is a separable extension of $k(J)$. If $a \neq 0$, from $\mathfrak{b}^2 + a\mathfrak{b} \in \mathfrak{G}_e(J)$ follows $x^2 + ax = z \in k(J)$. After replacing $\mathfrak{b}$ with $a^{-1}\mathfrak{b}$, we may assume $a = 1$, so that $z = x^2 + x$.

Let J_1, J_2 be copies of J, and assume $k(J_2) \subsetneqq k(J \times J_1)$ as prescribed by the rational mapping D of $J \times J_1$ onto J_2 which gives the law of composition on J. We have, by 2.5, $\mathfrak{b} \times J_1 + J \times \mathfrak{b}_1 + T_D^{-1}\mathfrak{b}_2 = \mathrm{cl}_{J\times J_1}\, t$ for a suitable $t \in k(J \times J_1)$; hence,

$$(\mathfrak{b}^2 + \mathfrak{b}) \times J_1 + J \times (\mathfrak{b}_1^2 + \mathfrak{b}_1) + T_D^{-1}(\mathfrak{b}_2^2 + \mathfrak{b}_2) = \mathrm{cl}_{J\times J_1}\,(t^2 + t);$$

but the same expression also equals $\mathrm{cl}_{J\times J_1}\,(z + z_1 + z_2)$, so that $t^2 + t + z + z_1 + z_2 \in k$. Since t is determined but for an additive element of k, it can be selected in such a manner that $t^2 + t = z + z_1 + z_2$. Now, set $K = k(J)(x)$, $K_1 = k(J_1)(x_1) =$ copy of K. The field of quotients of the direct product $K \times K_1$ over k contains $k(J \times J_1)$, hence $k(J_2)$, as prescribed by D; it also contains the element $x_2 = x + x_1 + t$. We have $x_2^2 + x_2 = x^2 + x + x_1^2 + x_1 + t^2 + t = z + z_1 + t^2 + t = z_2$, so that $K_2 = k(J_2)(x_2)$ is isomorphic to K. If F is a variety over k such that $k(F) = K$, the embedding $k(F_2) \subsetneqq k(F \times F_1)$ establishes a normal law of composition on F: to prove this, it suffices to prove the associativity, and in order to do that we must prove that, for a generic set of points $\{P, Q, R\}$ of J, we have (with obvious meaning of the symbols) $t(P \times Q_1) + t(PQ \times R_1) + t(Q \times R_1) + t(P \times Q_1 R_1) = 0$; now, this relation is true if t is replaced by $t^2 + t = z + z_1 + z_2$; consequently the above expression equals either 0 or 1; but it equals 0 if $P = E_J$, hence also for generic P. Thus, by the main result of [3], F may be selected to be a group-variety, hence an abelian variety; the embeddings $k(J) \subset k(F) \subsetneqq k(V)$ generate a rational mapping ψ of C into F, and a rational mapping β of F onto J, which will necessarily be of the form $\beta = P\alpha$, for a $P \in J$ and a homomorphism α of F onto J; we also have $\varphi = \beta\psi = P\alpha\psi$. But on the other hand, by Theorem 21 of [13], there exist a homomorphism γ of J into F, and a $Q \in F$, such that $\psi = Q\gamma\varphi$, from which it follows that $\varphi = P\alpha(Q\gamma\varphi)$, or that $\varphi R = P(\alpha Q)\,(\alpha\gamma\varphi R)$ for any $R \in C$. This implies $P(\alpha Q) = E_J$ and $\alpha\gamma = \delta_J$; as α has degree 2, this is impossible. Hence the class $\mathfrak{b}$ cannot exist, Q.E.D.

4.2. THEOREM. *Let J, C, φ be as in 4.1, and let Θ be as in 3.5; for any $\omega \in \mathfrak{D}_1(J)$, let ω^* be the element of $\mathfrak{D}_1(C)$ such that $(\omega^* \times J) \cap \varphi = (C \times \omega) \cap \varphi$. Let $R \in J$ be such that φC is not a subvariety of $\sigma_R\,\Theta = \Theta^*$. Then, for any $D \in \mathfrak{D}_0(J)$, we have*

$$\omega^* T_\varphi^{-1}(d\Theta^*)D = \omega D;$$

in particular, the mapping $\omega \to \omega^*$ is an isomorphism of $\mathfrak{D}_1(J)$ onto $\mathfrak{D}_1(C)$; and if $p \neq 0$, we have

$$(\omega^*)^{1/p} = (\omega^{1/p})^*.$$

In the previous statement, for any $\omega^* \epsilon \mathfrak{D}_1(C)$, and each class $\mathfrak{b}$ on C, $\omega^*\mathfrak{b}$ means $\omega^*\beta$, β being any repartition such that $\beta P + Q(P/C) = \mathfrak{b}P$ for all $P \epsilon C$.

Proof. We shall assume the genus n of C, which equals dim J, to be > 1, as otherwise the proof is trivial. We shall see first how the second and third statements follow from the first. The mapping $\omega \rightarrow \omega^*$ is a homomorphism of $\mathfrak{D}_1(J)$ into $\mathfrak{D}_1(C)$; if ω belongs to its kernel, that is, if $\omega^* = 0$, the first statement implies $\omega D = 0$ for each D, hence $\omega = 0$; the homomorphism is thus an isomorphism, and it must be onto, since $\mathfrak{D}_1(J)$ and $\mathfrak{D}_1(C)$ have the same order n. We also have, if $p \neq 0$, $(\omega^*)^{1/p}T_\varphi^{-1}(d\Theta^*)D = \{\omega^*[T_\varphi^{-1}(d\Theta^*)D]^p\}^{1/p} = \{\omega^*T_\varphi^{-1}[(d\Theta^*)D]^p\}^{1/p} = [\omega^*T_\varphi^{-1}(d\Theta^*)D^p]^{1/p} = (\omega D^p)^{1/p} = \omega^{1/p}D = (\omega^{1/p})^*T_\varphi^{-1}(d\Theta^*)D$ by 3.2; hence $(\omega^*)^{1/p} = (\omega^{1/p})^*$ by 3.5 and 4.1. We shall now proceed to prove the first statement.

In order to do so, we shall, first of all, select φ in such a manner that, if $\sum_i K_i$ is a canonical divisor on C, the relation $\prod_i \varphi K_i = E_J$ is satisfied; in the second place, we shall identify C with φC. Moreover, we shall select a nonspecial divisor $\sum_{i=1}^n Q(i)$ on C, with the $Q(i)$ mutually distinct, and take $R = \prod_i Q(i)$; this R satisfies the condition stated in the theorem, by Theorem 20 and Proposition 16 of [13]; the choice of R is immaterial, since the $(d\Theta^*)D$, for all admissible R's, are equivalent to each other. Theorem 20 of [13] also implies that $(\Theta^* \cap C,\ J) = \sum_i Q(i)$. In the course of the proof of 3.5 we found that R is a simple component of the intersection $(\sigma_{Q(1)}\,\Theta \cap \cdots \cap \sigma_{Q(n)}\,\Theta,\ J)$. Set $P(i) = R\big(Q(i)\big)^{-1} = \prod_{j\neq i} Q(j)$; it is easily verified that $(\bigcap_{j\neq i} \sigma_{Q(j)}\,\Theta,\ J) = \sigma_{P(i)}\,C$, so that $C = (\bigcap_{j\neq i} \sigma_{P(i)}^{-1}\,\sigma_{Q(j)}\,\Theta\,,\ J)$. By Theorem 20 of [13], the cycle $\sigma_P\,\Theta$ contains C if and only if $P \epsilon W'_{n-2}$; the previous result tells us that one can find $n - 1$ among these cycles in such a manner that their intersection is precisely C, with multiplicity 1; it also tells us that these $n - 1$ cycles can be found by first fixing an arbitrary i, and then taking for P all the points $P_{ir} = \prod_{j\neq i,r} \big(Q(j)\big)^{-1}$, when r ranges over the numbers $1, \cdots, i-1, i+1, \cdots, n$; we shall set $\Theta_{ir} = \sigma_{P_{ir}}\,\Theta$.

The homomorphism $-\delta_J$ is such that $-\delta_J\,P = P^{-1}$ for each $P \epsilon J$; for any set A of points of J, A' will denote $-\delta_J\,A$; also, $-\delta_J$ induces an automorphism of $k(J)$ over k; for any $x \epsilon k(J)$, x' will denote its correspondent in this automorphism. Our choice of φ implies, by Proposition 19 of [13], that $\Theta' = \Theta$. If $P \epsilon J$, and X is an $(n-1)$-dimensional irreducible subvariety of J containing P, we shall say that the element $x \epsilon k(J)$ *represents* X *at* P if $xQ(P/J) = Q(P/J) \cap \mathfrak{P}(X/J)$. It is possible to select a $t \epsilon k(J)$ which represents Θ at all the $\big(P(i)\big)^{-1}$, and all the $P_{ir}\,\big(Q(j)\big)^{-1}$, for mutually distinct j, i, r. Then $x = \sigma_R\,t$ represents Θ^* at all the $\big(P(i)\big)^{-1}R = Q(i)$; also, $x_{ir} = \sigma_{P_{ir}}\,t'$ represents Θ_{ir} at all the $Q(j)$. By what precedes, for each i the set $\{x_{i1}\,, \cdots, x_{ii-1}\,, x_{ii+1}\,, \cdots, x_{in}\,, x\}$ is a regular set of parameters of each $Q\big(Q(j)/J\big)$; if π is the homomorphic mapping of $Q(C/J)$ onto $k(C)$ whose kernel is $\mathfrak{P}(C/J)$, πx is a regular parameter of each $Q\big(Q(j)/C\big)$. For a $D \epsilon \mathfrak{D}_0(J)$, a representative of $T_\varphi^{-1}(d\Theta^*)D = C \cap (d\Theta^*)D$ at each of its possible poles, that is, at each $Q(j)$, is $(\pi x)^{-1}\pi Dx$; let $h_i(D)$ be the element of k to which πDx is congruent mod $\mathfrak{P}(Q(i)/C)$, or also to which Dx is congruent mod $\mathfrak{P}(Q(i)/J)$. The mapping $D \to \{h_1(D), \cdots, h_n(D)\}$ is a homomorphism of k-modules; if $h_i(D) = 0$ for each i, it is known that $C \cap (d\Theta^*)D = 0$; in this case, by 4.1, $(d\Theta^*)D \sim 0$, and finally, by 3.5, $D = 0$; thus, the previous homomorphism is an isomorphism onto.

We claim that if $h_j(D) = 0$ for all values of j except one, say $j = m$, then $Dx_{mr} \in \mathfrak{P}(Q(m)/J)$ for each $r \neq m$. And in fact, the condition implies that $D\sigma_R\, t \in \mathfrak{P}(Q(j)/J)$ for $j \neq m$, or that $Dt \in \mathfrak{P}((P(j))^{-1}/J)$ for $j \neq m$, so that $Dt' = -(Dt)' \in \mathfrak{P}(P(j)/J)$ for $j \neq m$, or finally $Dx_{mj} \in \mathfrak{P}(Q(m)/J)$ if $j \neq m$, as claimed.

Now, given $\omega \in \mathfrak{D}_1(J)$, we have $\omega^* = \omega \cap C$; given the previous D, write $\omega = \sum_{j \neq m} y_j\, d_J x_{mj} + y_m\, d_J x$; then $y_i \in Q(Q(j)/J)$ for any j, $\omega^* = (\pi y_m)\, d_C\, \pi x$, and $\omega^*[C \cap (d\Theta^*)D] =$ *sum of the elements of* k *to which* $(\pi y_m)\pi Dx$ *is congruent*, mod $\mathfrak{P}(Q(j)/C)$, *for* $j = 1, \cdots, n$; this is also the sum of the elements of k to which $y_m\, Dx$ is congruent mod $\mathfrak{P}(Q(j)/J)$. Since $h_j(D) = 0$ for $j \neq m$, that sum reduces to $h_m(D)\pi_m\, y_m$, if π_m denotes reduction of $Q(Q(m)/J)$ mod $\mathfrak{P}(Q(m)/J)$. Thus, $\omega^* T_\varphi^{-1}(d\Theta^*)D = h_m(D)\pi_m\, y_m$. On the other hand, ωD, being an element of k, equals $\sum_{j \neq m} (\pi_m\, y_j)\pi_m\, Dx_{mj} + h_m(D)\pi_m y_m$; but we have proved that $\pi_m\, Dx_{mj} = 0$, so that $\omega D = h_m(D)\pi_m y_m = \omega^* T^{-1}(d\Theta^*)D$. The first statement of the theorem is thus proved for any D of the previous type. Since we have seen that $\mathfrak{D}_0(J)$ has a k-basis consisting of such D's, the statement is proved for any D, Q.E.D.

Remark. If A, X', α, R, S have the same meanings as in the paragraph preceding 3.1, and if A is assumed to be the jacobian of a curve, by 3.5, 3.2, and 2.3, $X'/k(\alpha A)$ is isomorphic to the k-module of the $D \in \mathfrak{D}_0(A^{1/p})$ such that $D^{p^i} = 0$ for some positive integer i; hence R/S is isomorphic to the k-module of the $\omega \in \mathfrak{D}_1(A^{1/p})$ such that $\omega^{1/p^i} = 0$ for some positive integer i. Moreover, this isomorphism is such that if $D + S$ and ω correspond to each other, so do $D^{(1/p)} + S$ and $\omega^{1/p}$.

Result 4.2 allows us to reëstablish in general a classical formula which shows the effect on the differentials of an algebraic correspondence between curves; the formula was given in [11] for the differentials of the first kind, and for "separable" algebraic correspondences; it was reëstablished in [6] (formula 2.10) for differentials of the second kind,[3] in the case of characteristic zero; we shall now extend it to the differentials of the first kind, for any algebraic correspondence.

If μ is a rational mapping of a curve C over k, without singularities, into a nonsingular abelian variety A, for any $\omega \in \mathfrak{D}_1(A)$ we shall denote by $T_\mu^{-1}\omega$ the element $\omega^* \in \mathfrak{D}_1(C)$ such that $(\omega^* \times A) \cap \mu = \mu \cap (C \times \omega)$; thus, the ω^* of 4.2 would be denoted by $T_\varphi^{-1}\omega$. On the other hand, let M be an al-

[3] In formulae 2.9 and 2.10 of [6], the symbol $=$ should be replaced by $\sim$; the same applies to the formulae found in lines 19 and 20 on the same page 102. The π_i found on line 12 should be a Π_i.

gebraic correspondence between two curves C, C_1 over k, without singularities, every component of which has dimension 1, and write

$$M = \sum_i a_i M_i,$$

where each M_i is irreducible, and the a_i are rational integers. Each M_i is an algebraic correspondence between C and C_1; if it operates on the whole C and the whole C_1, we may assume $k(C)$, $k(C_1)$ to be subfields of $k(M_i)$ as prescribed by M_i; in this case, for any $\omega \in \mathfrak{D}_1(C_1)$, let ω' be the cotrace of ω from $k(C_1)$ to $k(M_i)$, and denote by ω_i the trace of ω' from $k(M_i)$ to $k(C)$; if M_i does not operate on the whole C and the whole C_1, set $\omega_i = 0$; we shall define then $M^*(\omega)$ to be $\sum_i a_i \omega_i$. We have the following result:

4.3. THEOREM. *Let C, C_1 be irreducible curves over k, without singular points; let J, J_1 be their respective jacobians, assumed nonsingular, and let φ, φ_1 be canonical rational mappings of C, C_1 into J, J_1 respectively. Let M be an algebraic correspondence between C and C_1, every component of which has dimension* 1, *and let μ be the homomorphism of J into J_1 related to M as in Theorem* 22 *of* [13]. *Then, for any $\omega \in \mathfrak{D}_1(J_1)$, we have $T_{\mu\varphi}^{-1}\omega = M^*(T_{\varphi_1}^{-1}\omega)$.*

Proof. If $M = \sum_i a_i M_i$ as before, and if μ_i is related to M_i as μ is to M, we have $\mu = \sum_i a_i \mu_i$ in the additive notation; the reasoning which precedes 2.5 of [6] can be repeated, and shows that $T_{\mu\varphi}^{-1}\omega = \sum_i a_i T_{\mu_i\varphi}^{-1}\omega$; on the other hand, $M^*(T_{\varphi_1}^{-1}\omega) = \sum_i a_i M_i^*(T_{\varphi_1}^{-1}\omega)$ by the definition of M^*. Thus, the result will be true if it is true for each M_i. We can accordingly assume M to be irreducible. If M does not operate on the whole C and the whole C_1, we have $\mu = 0$, $T_{\mu\varphi}^{-1}\omega = 0$, and $M^*(T_{\varphi_1}^{-1}\omega) = 0$, so that the result is true in this case. We now claim that the result is true for an irreducible M which operates on the whole C and the whole C_1, if it is true in the special cases $k(M) = k(C)$ and $k(M) = k(C_1)$. And in fact, let R, S be the rational mappings of M onto, respectively, C and C_1, generated by the embeddings $k(C) \subsetneq k(M)$ and $k(C_1) \subsetneq k(M)$ respectively, prescribed by M; let ρ, σ be related to R (from C to M), S (from M to C_1) as μ is to M, and let A, ψ be respectively the jacobian of M, assumed nonsingular, and a canonical rational mapping of M into A. Then $\mu = \sigma\rho$, $T_{\mu\varphi}^{-1}\omega = T_{\sigma(\rho\varphi)}^{-1}\omega = T_{\rho\varphi}^{-1}\omega'$, if ω' is the extension of $\omega \cap \sigma A$ to $k(A)$ when $k(\sigma A) \subsetneq k(A)$ as prescribed by σ; since the result is assumed to be true for R and S, we also have $T_{\rho\varphi}^{-1}\omega' = R^*(T_{\psi}^{-1}\omega') = R^*(T_{\sigma\psi}^{-1}\omega) = R^*S^*(T_{\varphi_1}^{-1}\omega) = M^*(T_{\varphi_1}^{-1}\omega)$, as claimed. We have therefore reduced the proof to proving the theorem in each of the following two cases:

Case 1. M is generated by an embedding $k(C_1) \subsetneq k(C)$. In this case, $T_{\mu\varphi}^{-1}\omega = T_{\varphi}^{-1}\omega'$, if ω' is the extension of $\omega \cap \mu J$ to $k(J)$ when $k(\mu J) \subsetneq k(J)$ as prescribed by μ; and it is readily verified that $M^*(T_{\varphi_1}^{-1}\omega)$ is the extension of $T_{\varphi_1}^{-1}\omega$ to $k(C)$ (when $T_{\varphi_1}^{-1}\omega$ is considered as a differential, not a differential mapping), and that consequently $T_{\mu\varphi}^{-1}\omega = M^*(T_{\varphi_1}^{-1}\omega)$.

Case 2. M is generated by an embedding $k(C) \subsetneq k(C_1)$. If $p \neq 0$, we can then write $k(C) \subseteq k(C_1^{p^r}) \subseteq k(C_1)$ for a suitable nonnegative integer r, the first extension being separable; then, by the same argument used before, the result is true in Case 2 if it is true in each of following two subcases:

Subcase 1. $k(C) = k(C_1^{p^r})$; again, the result is true for any r if it is true for $r = 1$; assume then $r = 1$. In this case, any entity α_1 related to C_1 has a "replica" α related to C, which is obtained by the substitution $x \to x^p$ for any $x \in k(C_1)$. The homomorphism μ of J onto J_1 is such that $\delta_{1,J_1}\mu = p\delta_J$, and we also have $J = J_1^p$; we shall extend to J and J_1 the notion of replica. If ω is now denoted by ω_1, since it is related to J_1, let Ω be the extension of ω_1 to $k(J)$, when $k(J_1) \subseteq k(J)$ as prescribed by μ. If $D \in \mathfrak{D}_0(J)$, D induces D_1^p in $k(J_1)$, by [8]; hence, $\Omega D = \omega_1 D_1^p$, so that $\Omega = \omega^{1/p}$, ω being the replica of ω_1; therefore, $T_{\mu\varphi}^{-1}\omega_1 = T_\varphi^{-1}\omega^{1/p}$. On the other hand, $M^*(T_{\varphi_1}^{-1}\omega_1)$ is the trace, from $k(C_1)$ to $k(C)$, of $T_{\varphi_1}^{-1}\omega_1$, and this coincides with $\psi^{1/p}$, if ψ is the replica of $T_{\varphi_1}^{-1}\omega_1$; now, ψ is necessarily $T_\varphi^{-1}\omega$, and 4.2 asserts that $(T_\varphi^{-1}\omega)^{1/p} = T_\varphi^{-1}\omega^{1/p}$, as claimed.

Subcase 2. $r = 0$, that is, $k(C_1)$ is a separable extension of $k(C)$. Let K be the smallest normal separable extension of $k(C)$ which contains a subfield, over $k(C)$, isomorphic to $k(C_1)$ over $k(C)$, and let B be a model of K over k, without singularities. Then $M\{C\}$ is a point Δ of $(C_1)_{k(C)}$, whose extension Δ_K over K is the sum of finitely many distinct rational points $\Delta_1, \cdots, \Delta_s$ of $(C_1)_K$; let M_i be the rational mapping of B onto C_1 such that $M_i\{B\} = \Delta_i$, and set $\psi = \sum_i \varphi_1 M_i$, so that ψ is a rational mapping of B into J_1. Then $T_\psi^{-1}\omega = \sum_i T_{\varphi_1 M_i}^{-1}\omega = \sum_i T_i^{-1}T_{\varphi_1}^{-1}\omega$, where T_i^{-1} indicates cotrace from $k(C_1)$ to $K = k(B)$ when $k(C_1) \subseteq K$ as prescribed by M_i. Now, $\psi\{B\} = [(\mu\varphi)\{C\}]_K$, so that $\psi = \mu\varphi D$, if D is the rational mapping of B onto C generated by the embedding $k(C) \subseteq K$; therefore, $T_\psi^{-1}\omega$ is the extension to K of $T_{\mu\varphi}^{-1}\omega$, while $\sum_i T_i^{-1}T_{\varphi_1}^{-1}\omega$ is the extension to K of $M^*(T_{\varphi_1}^{-1}\omega)$. Since these extensions coincide, and K is separable over $k(C)$, we must have $T_{\mu\varphi}^{-1}\omega = M^*(T_{\varphi_1}^{-1}\omega)$, Q.E.D.

Results 4.2 and 4.3 could be the starting point for extensive developments of this section; we shall only mention the possibility of defining a "conucleus" of each purely inseparable homomorphism, constructed with the closed classes as the nucleus is with the invariant derivations; one could show that, on a jacobian variety, the interchange of the nucleus with the conucleus corresponds to the Rosati involution.

Appendix

Result (16) of [5] reads:

THEOREM. *Let A be a nonsingular n-dimensional abelian variety over the algebraically closed field k, and let X be an $(n-1)$-dimensional cycle on A; then $X \equiv 0$ if and only if* ord $(X \cap C, A) = 0$ *for each irreducible curve C on A for which $(X \cap C, A)$ exists.*

This means also that the equivalence $\equiv$, on an abelian variety, coincides with arithmetical equivalence. The proof given in [5] utilizes the fact that $\rho(X) = 0$ implies $\lambda_X' = 0$, in the notation of section 72 of [13]; as this result is valid only when A is simple, the proof given in [5] applies only to this case; it was remarked in footnote 6 of [7] that a standard method shows that if the result is true when A is simple, it is true for any A. Actually, it is simpler to give a direct proof valid for any A, and we shall proceed to do so now. The condition $X \equiv 0$ implies $\lambda_X' = 0$ for any curve Γ on k, without singularities, and any homomorphism λ of the jacobian J_Γ of Γ into A, by Corollary 1 to Theorem 30 of [13]; $\lambda_X' = 0$ implies, in turn, $T(\lambda_X'\lambda) = 0$, if T (replacing the σ of [13]) denotes construction of trace in the semisimple algebra of the endomorphisms of J_Γ, according to a certain faithful representation. From this, and by Theorem 31 of [13], follows $d(\lambda, X) = 0$, or also

$$\text{ord}\,(C \cap X, A) = 0,$$

if Γ is taken to be birationally equivalent to a given C, and λ is the linear extension, from J_Γ to A, of the birational mapping of Γ onto C.

Conversely, assume $(C \cap X, A)$ to have order zero whenever it exists; let A be the homomorphic image, in a homomorphism λ, of the jacobian J of a curve Γ on k, without singularities, and let φ be a canonical rational mapping of Γ into J; φ can be selected in such a manner that $(\lambda\varphi\Gamma \cap X, A)$ exists, in which case it must have order zero; this implies $d(\lambda, X) = 0$, hence $T(\lambda_X'\lambda) = 0$ by Theorem 31 of [13]. If α is any endomorphism of J, set $\mu = \lambda\alpha$; we must have again $d(\mu, X) = d(\lambda + \mu, X) = 0$, so that, by Proposition 20 of [4], $T(\lambda_X'\mu) = 0$, or $T(\lambda_X'\lambda\alpha) = 0$. Thus, λ_X' is an element of a semisimple algebra, whose product on the right by any other element has zero trace, according to a certain faithful representation; by a wellknown result, this implies $\lambda_X'\lambda = 0$, or $\lambda_X'\lambda J = E_J$; since $\lambda J = A$, this means $\lambda_X' A = E_J$, or $\lambda_X' = 0$. Theorem 30 of [13] shows then that $X \equiv 0$, Q.E.D.

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